

IRREDUCIBILITY OF THE (-1) -CLASSES ON SMOOTH RATIONAL SURFACES

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This research is dedicated to my mother, Hnia Hamami

ABSTRACT. We give a characterization for a (-1) -divisor D on a smooth rational surface X to be irreducible under the assumption that an anticanonical divisor $-K_X$ of X is nef. Here $-K_X$ is nef means $K_X.C \leq 0$ for every effective divisor C on X , and a (-1) -divisor D is a divisor such that the two numerical conditions $D^2 = -1 = D.K_X$ hold.

As an application we give explicit examples of blowing up the projective plane at nine points infinitely near such that the obtained surface has an infinite number of (-1) -curves. A (-1) -curve is a smooth rational curve of self-intersection -1 .

1. INTRODUCTION

Given a smooth rational surface X for which an anticanonical divisor $-K_X$ is nef, we would like to know if a divisor E on X satisfying the two numerical conditions $E^2 = -1$ and $E.K_X = -1$ (E will be called a (-1) -divisor on X) is irreducible (hence reduced) or not. Such a divisor E is always effective, that is, the group of sections of the invertible sheaf $\mathcal{O}_X(E)$ associated to E has a nontrivial section (see Lemma 2.2 below). Unfortunately, it may be neither reduced nor irreducible. One of our motivations to consider such a question is to have a criterion to check if X has an infinite number of (-1) -curves or not. For example, when the self-intersection of an anticanonical divisor of X is zero, it is not difficult to find an infinite family of divisors on X such that each member E of the family is a (-1) -divisor.

Nagata ([6]) proved that each (-1) -divisor on the surface obtained by blowing up 9 points in general position of the projective plane is irreducible, and then it is a (-1) -curve, that is, a smooth rational curve of self-intersection -1 . We are interested in giving a criterion to check whether a (-1) -divisor E on X is irreducible reduced or not.

The plan of this note is as follows: in section 2, we give some standard facts about smooth rational surfaces and fix our notation. Section 3 contains our main

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result, which gives a necessary and sufficient condition for a (-1) -divisor to be reduced and irreducible. In section 4, there are some applications: the surface obtained by blowing up 9 infinitely near points of the projective plane such that the configuration of the set of (-2) -curves is of type A_8 (respectively D_8 , E_8 and A_1A_7) has an infinite number of (-1) -curves. A (-2) -curve is a smooth rational curve of self-intersection -2 .

2. PRELIMINARIES

From now on X will be a smooth rational surface such that $-K_X$ is nef, where K_X is a canonical divisor on X . Here the condition $-K_X$ is nef means that $K_X.C \leq 0$ for every effective divisor C on X , where $.$ denotes the intersection form on the group of divisors on X .

In this context, the integer K_X^2 is greater than or equal to zero (see e.g. [3, Proposition 4, page 729]). Also, it is easy to see that the self-intersection of every reduced irreducible curve on X is greater than or equal to -2 . In particular, define a (-2) -curve on X as a smooth rational curve of self-intersection -2 and a (-1) -curve as one with self-intersection -1 .

Definition 2.1. Let E be a divisor on X . E is a (-1) -divisor if and only if $E^2 = -1 = E.K_X$. The class of a (-1) -divisor in the Néron-Severi group of X will be called a (-1) -class, or simply a (-1) -divisor.

The following lemma shows that a (-1) -divisor on X is always effective.

Lemma 2.2. *Every (-1) -divisor on X is effective.*

Proof. Let E be a (-1) -divisor on X . The Riemann-Roch theorem applied to the invertible sheaf $\mathcal{O}_X(E)$ gives the following inequality:

$$h^0(X, \mathcal{O}_X(E)) + h^0(X, \mathcal{O}_X(K_X - E)) \geq 1.$$

Since $K_X.(K_X - E) = K_X^2 + 1 > 0$, the divisor $K_X - E$ is not effective. So we are done. \square

Remark 2.3. As the proof suggests, the result of Lemma 2.2 holds for any smooth rational surface having an anticanonical divisor of self-intersection greater than or equal to zero.

Remark 2.4. In [2], there are examples which prove that a (-1) -divisor may not be effective if $K_X^2 < 0$.

We will need the following well-known result (see for instance [4]):

Lemma 2.5. *Let X be a smooth projective rational surface such that the self-intersection of its canonical divisor K_X is zero. Then the intersection form is negative semi-definite on K_X^\perp ; and for every element $x \in K_X^\perp$, the following equivalence holds:*

$$x^2 = 0 \text{ if and only if } qx = pK_X \text{ for some integers } p \text{ and } q \neq 0.$$

K_X^\perp is the set of all divisors D on X which satisfy $D.K_X = 0$.

We give here an elementary proof. Let x be an element of K_X^\perp . We can suppose that x is the divisor class of a divisor D on X . If D^2 is larger than zero, then

by using the Hodge Index Theorem (see e.g. [5], [1]), K_X would be either of self-intersection less than zero or numerically trivial. These are impossible since X is rational and $K_X^2 = 0$.

Let A be an ample divisor on X , and assume furthermore that $D^2 = 0$. Consider the divisor E defined as $E = (A.K_X)D - (A.D)K_X$. We have $E^2 = 0 = A.E$. Hence, by the Hodge Index Theorem, the divisor $(A.K_X)D$ is numerically equivalent to $(A.D)K_X$. Since X is rational and $-K_X$ is an effective divisor, we obtain that

$$D = \frac{A.D}{A.K_X}K_X.$$

3. THE CRITERION

Let $\{C_1, \dots, C_r\}$ be the set of (-2) -curves on X (r is a nonnegative integer which can be zero if X has no (-2) -curves). The following is our main result.

Theorem 3.1. *Let X be a smooth rational surface having an anticanonical divisor being nef and let $\{C_1, \dots, C_r\}$ be the set of (-2) -curves on X , where r is a non-negative integer. Then, the following assertions are equivalent for a (-1) -divisor E on X :*

1. E is a (-1) -curve.
2. $E.C_i \geq 0$ for every $i = 1, \dots, r$.

Proof. It is obvious that if E is a (-1) -curve on X , then the intersection number of E with any (-2) -curve on X is greater than or equal to zero.

Conversely, let E be a (-1) -divisor such that the numerical conditions $E.C_i \geq 0$ for every $i = 1, \dots, r$ hold. By Lemma 2.2, E is effective; hence E is linearly equivalent to $E_0 + \sum_{i=1}^{i=r} b_i C_i$, where E_0 is an effective divisor having no (-2) -curve as a component, and the b_i are nonnegative integers for every $i = 1, \dots, r$. The condition $E^2 = -1$ gives the following equality:

$$-1 = E_0^2 + \sum_{i=1}^{i=r} b_i(E_0.C_i + E.C_i).$$

Hence, $E_0^2 \leq -1$ (since $C_i.E_0 \geq 0$ and $C_i.E \geq 0$ for all $i = 1, \dots, r$).

The condition $E.(-K_X) = 1$ implies the equality: $E_0.(-K_X) = 1$. If $\Gamma_0, \Gamma_1, \dots, \Gamma_\sigma$ are the components of the effective divisor E_0 , i.e., $E_0 = \sum_{i=0}^{i=\sigma} a_i \Gamma_i$, then

$$1 = a_0 \Gamma_0.(-K_X) + \dots + a_\sigma \Gamma_\sigma.(-K_X).$$

Since the divisor $(-K_X)$ is nef, we can suppose that $a_0 \Gamma_0.(-K_X) = 1$; so we have $a_0 = \Gamma_0.(-K_X) = 1$ and $a_i \Gamma_i.(-K_X) = 0$ for every $i = 1, \dots, \sigma$.

We distinguish between two cases:

Case where $K_X^2 > 0$: Let i be any element of the set $\{1, 2, \dots, \sigma\}$. From $a_i \Gamma_i.(-K_X) = 0$, we will prove that $a_i = 0$. To do so, if $\Gamma_i.(-K_X) = 0$, then by the Hodge Index Theorem the only case which can occur is $\Gamma_i^2 < 0$, that is, by the adjunction formula Γ_i is a (-2) -curve which is impossible. Hence $a_i = 0$. In conclusion, $E_0 = \Gamma_0$. But $-1 \leq \Gamma_0^2$ and $E_0^2 \leq -1$; hence E_0 is a (-1) -curve.

Case where $K_X^2 = 0$: In this case for every $i = 1, \dots, \sigma$, we have $-2 \leq \Gamma_i^2 \leq 0$. The only case which can occur is $\Gamma_i^2 \geq 0$. Then there exist some natural integers p_i and $q_i, i = 1, \dots, \sigma$ ($q_i \neq 0$ for all $i = 1, \dots, \sigma$), such that $q_i \Gamma_i = p_i(-K_X)$ for all $i = 1, \dots, \sigma$. Consequently, there exist two natural

integers $N(N = q_1 \dots q_\sigma)$, and a such that: $NE_0 = N\Gamma_0 + a(-K_X)$. Hence by using the adjunction formula applied to Γ_0 , we obtain that Γ_0 is a (-1) -curve and $a = 0$. In other words, E_0 is a (-1) -curve on X .

So in both cases, E_0 is a (-1) -curve. Hence we have the equalities:

$$E \cdot \sum_{i=1}^{i=r} b_i C_i = E_0 \cdot \sum_{i=1}^{i=r} b_i C_i = 0.$$

Now define C as $C = \sum_{i=1}^{i=r} b_i C_i$. Then $C^2 = 0$. It follows immediately that there exist some natural integers p and q ($q \neq 0$) such that $qC = p(-K_X)$; hence $qE = qE_0 + p(-K_X)$.

The facts that E is a (-1) -divisor, E_0 is a (-1) -curve and $q \neq 0$ prove that the complete linear system $|E|$ associated to E is equal to the singleton $\{E_0\}$ having the (-1) -curve E_0 as its unique element. □

4. SOME APPLICATIONS

In this section, we apply our criterion to prove that certain blowings-up of the projective plane at nine points have an infinite number of (-1) -curves. The method consists in giving an explicit infinite family of (-1) -curves.

Since X is a blow-up of the projective plane at nine points (may be infinitely near), the Néron-Severi group $NS(X)$ of X has a basis $(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_9)$ defined by:

- \mathcal{E}_0 is the class of lines on the projective plane which do not pass through any of the nine points in consideration;
- \mathcal{E}_i is the class of the exceptional divisor corresponding to the i th point being blown up for every $i = 1, \dots, 9$.

The class of a divisor on X will be represented by the 10-tuple (d, m_1, \dots, m_9) ; in particular, if $m_i = m_{i+1} = \dots = m_j = m$, $1 \leq i < j \leq 9$, we write this class as $(d, m_1, \dots, m_{i-1}, m^{j-i+1}, m_{j+1}, \dots, m_9)$.

The intersection form on $NS(X)$ is given by:

- $\mathcal{E}_0^2 = 1 = -\mathcal{E}_i^2$ for every $i = 1, \dots, 9$;
- $\mathcal{E}_i \cdot \mathcal{E}_j = 0$ for every $i, j = 0, 1, \dots, 9$, with $i \neq j$.

Example 4.1. Taking the first point in the projective plane, the second point will be a general point on the exceptional divisor, the third point will be a general point on the exceptional divisor of the blowing-up of the second point, and so on. Then the classes of (-2) -curves are : $(0, -1, 1, 0^7)$, $(0, 0, -1, 1, 0^6)$, $(0, 0^2, -1, 1, 0^5)$, $(0, 0^3, -1, 1, 0^4)$, $(0, 0^4, -1, 1, 0^3)$, $(0, 0^5, -1, 1, 0^2)$, $(0, 0^6, -1, 1, 0)$, $(0, 0^7, -1, 1)$.

For each positive integer d , the class $(3d^2, (d^2 + d), (d^2)^6, (d^2 - 1), (d^2 - d))$ is the class of a (-1) -curve on X (hence X has an infinite number of (-1) -curves). Indeed, we have:

- $(3d^2, (d^2 + d), (d^2)^6, (d^2 - 1), (d^2 - d)) \cdot (0, -1, 1, 0^7) = d$,
- the intersection number of $(3d^2, (d^2 + d), (d^2)^6, (d^2 - 1), (d^2 - d))$ with any $(0, 0, -1, 1, 0^6)$, $(0, 0^2, -1, 1, 0^5)$, $(0, 0^3, -1, 1, 0^4)$, $(0, 0^4, -1, 1, 0^3)$, $(0, 0^5, -1, 1, 0^2)$ is equal to zero,
- $(3d^2, (d^2 + d), (d^2)^6, (d^2 - 1), (d^2 - d)) \cdot (0, 0^6, -1, 1, 0) = 1$,
- $(3d^2, (d^2 + d), (d^2)^6, (d^2 - 1), (d^2 - d)) \cdot (0, 0^7, -1, 1) = d - 1$.

Example 4.2. Taking the first point in the projective plane, the second point will be a general point on the exceptional divisor, the third point will be a general point on the exceptional divisor of the second point blown-up, and so on until the eighth point. The ninth point will be an ordinary point in the projective plane such that it belongs to the line determined by the first point and the second point (which is an infinitely near point of the first point). Then the classes of (-2)-curves are: $(0, -1, 1, 0^7)$, $(0, 0, -1, 1, 0^6)$, $(0, 0^2, -1, 1, 0^5)$, $(0, 0^3, -1, 1, 0^4)$, $(0, 0^4, -1, 1, 0^3)$, $(0, 0^5, -1, 1, 0^2)$, $(0, 0^6, -1, 1, 0)$, $(1, 1, 1, 0^6, 1)$.

For integers $p \geq 0$ and $q \geq 0$, the class $(2^p \times 3^{q+1} \times (2^p \times 3^q - 1), (4^p \times 9^q - 1), (2^p \times 3^q \times (2^p \times 3^q - 1))^7, (2^p \times 3^q \times (2^p \times 3^q - 1))^7, (2^{p+1} \times 3^q \times (2^{p-1} \times 3^q - 1)))$ is the class of a (-1)-curve on X (hence X has an infinite number of (-1)-curves). Indeed, we have:

- $(2^p \times 3^{q+1} \times (2^p \times 3^q - 1), (4^p \times 9^q - 1), (2^p \times 3^q \times (2^p \times 3^q - 1))^7, (2^{p+1} \times 3^q \times (2^{p-1} \times 3^q - 1))) \cdot (0, -1, 1, 0^7) = (2^p \times 3^q - 1)$,
- the intersection number of $(2^p \times 3^{q+1} \times (2^p \times 3^q - 1), (4^p \times 9^q - 1), (2^p \times 3^q \times (2^p \times 3^q - 1))^7, (2^{p+1} \times 3^q \times (2^{p-1} \times 3^q - 1)))$ with any $(0, 0, -1, 1, 0^6)$, $(0, 0^2, -1, 1, 0^5)$, $(0, 0^3, -1, 1, 0^4)$, $(0, 0^4, -1, 1, 0^3)$, $(0, 0^5, -1, 1, 0^2)$, $(0, 0^6, -1, 1, 0)$ is equal to zero.
- $(2^p \times 3^{q+1} \times (2^p \times 3^q - 1), (4^p \times 9^q - 1), (2^p \times 3^q \times (2^p \times 3^q - 1))^7, (2^{p+1} \times 3^q \times (2^{p-1} \times 3^q - 1))) \cdot (1, 1, 1, 0^6, 1) = 1$.

Example 4.3. Taking the first point in the projective plane, the second point will be a general point on the exceptional divisor, the third point will be the point belonging to the exceptional divisor of the blowing-up of the second point and the strict transform of the line determined by the first ordinary point and the second point. The fourth point will be a general point on the third exceptional divisor and so on until the eighth point. The ninth point will be an ordinary point such that it does not belong to the line determined by the first point and the second point (which is an infinitely near point of the first point). Then the classes of (-2)-curves are: $(0, -1, 1, 0^7)$, $(0, 0, -1, 1, 0^6)$, $(0, 0^2, -1, 1, 0^5)$, $(0, 0^3, -1, 1, 0^4)$, $(0, 0^4, -1, 1, 0^3)$, $(0, 0^5, -1, 1, 0^2)$, $(0, 0^6, -1, 1, 0)$, $(1, 1, 1, 1, 0^6)$.

For each integer p , the class $(6 \times (p^2 + 1) \times (2 \times p^2 + 1), (2 \times (p^2 + 1) \times (2 \times p^2 + 1))^7, 4 \times p^2 \times (p^2 + 1), 4 \times p^2 \times (p^2 + 2) + 3)$ is the class of a (-1)-curve on X (hence X has an infinite number of (-1)-curves). Indeed, we have:

- The intersection number of $(6 \times (p^2 + 1) \times (2 \times p^2 + 1), (2 \times (p^2 + 1) \times (2 \times p^2 + 1))^7, 4 \times p^2 \times (p^2 + 1), 4 \times p^2 \times (p^2 + 2) + 3)$ with any $(0, -1, 1, 0^7)$, $(0, 0, -1, 1, 0^6)$, $(0, 0^2, -1, 1, 0^5)$, $(0, 0^3, -1, 1, 0^4)$, $(0, 0^4, -1, 1, 0^3)$, $(0, 0^5, -1, 1, 0^2)$, $(1, 1, 1, 1, 0^6)$ is equal to zero,
- $(6 \times (p^2 + 1) \times (2 \times p^2 + 1), (2 \times (p^2 + 1) \times (2 \times p^2 + 1))^7, 4 \times p^2 \times (p^2 + 1), 4 \times p^2 \times (p^2 + 2) + 3) \cdot (0, 0^6, -1, 1, 0) = 2 \times (p^2 + 1)$.

Example 4.4. Taking the first point in the projective plane, the second point will be a general point on the exceptional divisor, and so on until the seventh point. The eighth point will be an ordinary point, and the ninth point will be an infinitely near point of the eighth point and will belong to the line determined by the first point and the eighth point. Then the classes of (-2)-curves are: $(0, -1, 1, 0^7)$, $(0, 0, -1, 1, 0^6)$, $(0, 0^2, -1, 1, 0^5)$, $(0, 0^3, -1, 1, 0^4)$, $(0, 0^4, -1, 1, 0^3)$, $(0, 0^5, -1, 1, 0^2)$, $(0, 0^7, -1, 1)$ and $(1, 1, 0^6, 1, 1)$.

The class $(3^{d+1} \times (3^d + 1), 3^d \times (3^d + 2), (3^d \times (3^d + 1))^7, 3^{2d} - 1)$ is irreducible for each positive integer d (hence X has an infinite number of (-1) -curves). Indeed, we have:

- $(3^{d+1} \times (3^d + 1), 3^d \times (3^d + 2), (3^d \times (3^d + 1))^7, 3^{2d} - 1) \cdot (0, -1, 1, 0^7) = 3^d$,
- the intersection number of

$$(3^{d+1} \times (3^d + 1), 3^d \times (3^d + 2), (3^d \times (3^d + 1))^7, 3^{2d} - 1)$$
 with any $(0, 0, -1, 1, 0^6)$, $(0, 0^2, -1, 1, 0^5)$, $(0, 0^3, -1, 1, 0^4)$, $(0, 0^4, -1, 1, 0^3)$, $(0, 0^5, -1, 1, 0^2)$ is equal to zero.
- $(3^{d+1} \times (3^d + 1), 3^d \times (3^d + 2), (3^d \times (3^d + 1))^7, 3^{2d} - 1) \cdot (0, 0^7, -1, 1) = 3^d + 1$,
- $(3^{d+1} \times (3^d + 1), 3^d \times (3^d + 2), (3^d \times (3^d + 1))^7, 3^{2d} - 1) \cdot (1, 1, 0^6, 1, 1) = 1$.

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