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ALMOST AUTOMORPHIC SOLUTIONS OF SEMILINEAR EVOLUTION EQUATIONS

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ABSTRACT. We are concerned with the semilinear differential equation in a Banach space $\mathbb{X}$,

$$x'(t) = Ax(t) + F(t, x(t)), \quad t \in \mathbb{R},$$

where A generates an exponentially stable C_0-semigroup and $F(t, x) : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a function of the form $F(t, x) = P(t)Q(x)$. Under appropriate conditions on P and Q, and using the Schauder fixed point theorem, we prove the existence of an almost automorphic mild solution to the above equation.

1. Introduction

Consider in a Banach space $(\mathbb{X}, \|\cdot\|)$ the semilinear differential equation

$$x'(t) = Ax(t) + F(t, x(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where the linear operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ generates an exponentially stable C_0-semigroup $\mathcal{T} = (T(t))_{t \geq 0}$; that is, $\mathcal{T}$ satisfies the estimate

$$\|T(t)\| \leq Me^{-\epsilon t}, \tag{1.2}$$

for some constants $M > 0, \epsilon > 0$ and all $t \geq 0$. Let $F : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be jointly continuous. A *mild solution* to (1.1) is a function $x \in C(\mathbb{R}, \mathbb{X})$ satisfying the integral equation

$$x(t) = T(t-a)x(a) + \int_a^t T(t-s)F(s, x(s))ds \tag{1.3}$$

for every $a \in \mathbb{R}$ and every $t \geq a$.

A fundamental problem is the existence of almost automorphic mild solutions to (1.1). Recently, G. M. N'Guérékata [5] showed, using the Banach fixed point theorem, that if

i) F is Lipschitzian in $x \in \mathbb{X}$, uniformly in $t \in \mathbb{R}$, that is,

$$\|F(t, x) - F(t, y)\| \leq L\|x - y\| \tag{1.4}$$

for all $x, y \in \mathbb{X}$, and $t \geq 0$, and L is sufficiently small, namely $L < \dfrac{\epsilon}{M}$, where ϵ and M are as in (1.2), and

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ii) $F(t,x)$ is almost automorphic in $t \in \mathbb{R}$ for each $x \in \mathbb{X}$,

then problem (1.1) has a unique almost automorphic mild solution.

In this paper, we are going to prove the existence of almost automorphic mild solutions to (1.1), F being not necessarily Lipschitzian. But first, let us recall some definitions.

Definition 1.1. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be *almost automorphic* if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that

$$g(t) = \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$.

It is well known that the range $\mathcal{R}_f = \{f(t) | t \in \mathbb{R}\}$ of an almost automorphic function f is relatively compact in $\mathbb{X}$, thus bounded in norm (see [6], Theorem 2.13). The function g in the definition is also bounded and strongly measurable. Also, the set $AA(\mathbb{X})$ of all almost automorphic functions $f : \mathbb{R} \to \mathbb{X}$ equipped with the sup-norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|,$$

is a Banach space (see [6], page 20).

Also, given two Banach spaces $(\mathbb{X}_1, \|\cdot\|_1)$ and $(\mathbb{X}_2, \|\cdot\|_2)$, $B(\mathbb{X}_1,\mathbb{X}_2)$ will denote the Banach space of bounded linear operators $L : \mathbb{X}_1 \to \mathbb{X}_2$, $BC(\mathbb{R},\mathbb{X}_1)$ is the Banach space of all continuous and bounded functions $f : \mathbb{R} \to \mathbb{X}_1$, and $BUC(\mathbb{R},\mathbb{X}_1)$ is the Banach space of all bounded and uniformly continuous functions $f : \mathbb{R} \to \mathbb{X}_1$.

2. Preliminaries

In this paper $(\mathbb{Y}, |\cdot|)$ will denote a Banach space algebraically contained in $\mathbb{X}$ such that the canonical injection $\mathbb{Y} \to \mathbb{X}$ is compact. An example of such a space $\mathbb{Y}$ is an abstract Sobolev space that we construct as follows:

Let A be as in (1.1), (1.2). By (1.2), $0 \in \rho(A)$, so that the fractional powers $(-A)^\alpha$, $0 < \alpha < 1$, are well defined. Also, since $0 \in \rho(A)$, the norm

$$|f| = \|(-A)^\alpha f\| \tag{2.1}$$

is equivalent to the graph norm

$$\|f\|_\alpha = \|(-A)^\alpha f\| + \|f\|.$$

Now we take $\mathbb{X} = L^p(\Omega)$, where $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain in $\mathbb{R}^n$. Let A be a linear uniformly elliptic operator (with suitable boundary conditions), of order $2m$. Then let $\mathbb{Y}$ be the domain of $(-A)^\alpha$ with norm (2.1); we have

$$W_0^{2m\alpha,p}(\Omega) \subset \mathbb{Y} \subset W^{2m\alpha,p}(\Omega)$$

and the norm $|\cdot|$ in $\mathbb{Y}$ is equivalent to the usual norm in $W^{2m\alpha,p}(\Omega)$. Also, the injection $\mathbb{Y} \to \mathbb{X}$ is compact in this case, by Sobolev embedding.

3. Main results

Now let $\mathbb{Y} = D((-A)^\alpha)$, the domain of $(-A)^\alpha$, with norm

$$|y| = \|(-A)^\alpha y\|, \quad y \in D((-A)^\alpha),$$

where $0 < \alpha < 1$ is fixed. We get

$$|T(t)y| = \|T(t)(-A)^\alpha y\| \leq Me^{-\epsilon t}\|(-A)^\alpha y\| = Me^{-\epsilon t}|y| \tag{3.1}$$

for each $y \in \mathbb{Y}$ and every $t \geq 0$, by (1.2).

We also make the following assumptions:

$$F(t,x) = P(t)Q(x), \quad \text{for all} \quad t \in \mathbb{R}, x \in \mathbb{X}, \tag{3.2}$$

where $P(t) \in AA(\mathbb{Z})$ for each $t \in \mathbb{R}$ with $\mathbb{Z} = B(\mathbb{X},\mathbb{Y})$; P is continuous from $\mathbb{R}$ to $AA(\mathbb{Z})$, and $Q : BC(\mathbb{R},\mathbb{X}) \to BC(\mathbb{R},\mathbb{X})$ is continuous and satisfies the estimate

$$\|Q\varphi\|_\infty \leq \mathcal{M}(\|\varphi\|_\infty), \tag{3.3}$$

where $\|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\|$ and $\mathcal{M} \in C(\mathbb{R}^+,\mathbb{R}^+)$ satisfies

$$\lim_{r\to\infty} \frac{\mathcal{M}(r)}{r} = 0. \tag{3.4}$$

Note that $\mathcal{M}$ can be unbounded but must grow slower than a linear function. Let

$$[P] := \sup_{t\in\mathbb{R}} \|P(t)\|_{\mathbb{Z}} < \infty. \tag{3.5}$$

Define $G : BC(\mathbb{R},\mathbb{X}) \to BC(\mathbb{R},\mathbb{Y})$ by

$$(G\varphi)(t) = \int_{-\infty}^t T(t-s)F(s,\varphi(s))ds. \tag{3.6}$$

For $\varphi \in BC(\mathbb{R},\mathbb{X})$, this integral exists. Indeed, we have

$$\begin{aligned} |(G\varphi)(t)| &\leq \int_{-\infty}^t |T(t-s)||P(t)Q(\varphi(s))|ds \\ &\leq \int_{-\infty}^t Me^{-\epsilon(t-s)}[P]\mathcal{M}(\|\varphi\|_\infty)ds \end{aligned}$$

using (3.1), (3.3) and (3.5). Consequently

$$\begin{aligned} |G\varphi|_\infty &= \sup_{t\in\mathbb{R}} |(G\varphi)(t)| \\ &\leq M\epsilon^{-1}[P]\mathcal{M}(\|\varphi\|_\infty). \end{aligned} \tag{3.7}$$

Continuity of G is straightforward by virtue of continuity of both P and Q. Thus we have

$$G(BC(\mathbb{R},\mathbb{X})) \subset BC(\mathbb{R},\mathbb{Y}).$$

Finally, for $0 < \delta \leq 1$, let

$$BC^\delta(\mathbb{R},\mathbb{Y}) \equiv \{f \in BC(\mathbb{R},\mathbb{Y}) : |f|_{\delta,\mathbb{Y}} < \infty\},$$

where

$$|f|_{\delta,\mathbb{Y}} \equiv \sup_{t\in\mathbb{R}} |f(t)| + \delta \sup_{t,s\in\mathbb{R},t\neq s} \frac{|f(t)-f(s)|}{|t-s|^\delta}.$$

With the norm $|\cdot|_{\delta,\mathbb{Y}}$, $BC^\delta(\mathbb{R},\mathbb{Y})$ turns out to be a Banach space of all bounded Hölder continuous $\mathbb{Y}$-valued functions on $\mathbb{R}$ of Hölder exponent δ.

Proposition 3.1. *The function G defined above maps bounded sets of $BC(\mathbb{R},\mathbb{X})$ into bounded sets of $BC^{\delta}(\mathbb{R},\mathbb{Y})$ for any $\delta > 0$ satisfying $\delta < \alpha$, where $0 < \alpha < 1$ is the exponent defining $\mathbb{Y} = D(-A)^{-\alpha}$.*

Proof. The proof is basically a modification of the above remarks. Let $0 < \beta < \alpha$. Then

$$\begin{aligned} |(G\varphi)(t)| &= |\int_{-\infty}^{t} T(t-s)(-A)^{\beta}(-A)^{-\beta}F(s,\varphi(s))ds| \\ &\leq \int_{-\infty}^{t} |T(t-s)(-A)^{\beta}||(-A)^{-\beta}P(s)||Q(\varphi(s)|ds. \end{aligned} \tag{3.8}$$

Now, by semigroup theory (see for instance [4]), there exists a constant M_1 such that

$$\|T(r)(-A)^{\beta}\| \leq \frac{M_1 e^{-\epsilon r}}{r^{\beta}}$$

for all $r > 0$. Thus we obtain, as previously,

$$|T(r)(-A)^{\beta}| \leq M_1 e^{-\epsilon r} r^{-\beta}, \quad r > 0. \tag{3.9}$$

Next, we observe that the function $s \mapsto (-A)^{-\beta}P(s)$ is a uniformly bounded function $\mathbb{R} \to B(\mathbb{X}, D((-A)^{\alpha-\beta})$. Indeed, it is the composition of $P(\cdot) : \mathbb{R} \to B(\mathbb{X}, D((-A)^{\alpha}))$, which is bounded by $[P]$, with $(-A)^{-\beta}$, an isometry from $D((-A)^{\alpha})$ onto $D((-A)^{\alpha-\beta})$. Thus

$$\sup_{t\in\mathbb{R}} \|P(t)\|_{B(\mathbb{X},D((-A)^{\alpha-\beta}))} \leq [P].$$

Now combining the estimates in (3.8) and (3.9), we deduce

$$|(G\varphi)(t)| \leq \int_{-\infty}^{t} M_1 e^{-\epsilon(t-s)}(t-s)^{-\beta}[P]\mathcal{M}(\|\varphi\|_{\infty})ds.$$

Letting $r = t - s$ in the integral gives

$$|(G\varphi(t)| \leq \int_{0}^{\infty} M_1 e^{-r} r^{-\beta}[P]\mathcal{M}(\|\varphi\|)dr;$$

that is,

$$|(G\varphi)(t)| \leq C_1(\beta)\mathcal{M}(\|\varphi\|_{\infty}), \tag{3.10}$$

where $C_1(\beta)$ depends on β, M_1, ϵ and $[P]$. Next, for $t_2 > t_1$, we have

$$\begin{aligned} &|(G\varphi)(t_2) - (G\varphi)(t_1)| \\ &\quad \leq |(\int_{-\infty}^{t_2} - \int_{-\infty}^{t_1})T(t_2-s)(-A)^{\beta}(-A)^{-\beta}P(s)Q(\varphi(s))ds| \\ &\qquad + |\int_{-\infty}^{t_1}(T(t_2-s) - T(t_1-s))(-A)^{\beta}(-A)^{-\beta}P(s)Q(\varphi(s))ds| \\ &\quad \leq \int_{t_1}^{t_2} |T(t_2-s)(-A)^{\beta}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\qquad + \int_{-\infty}^{t_1} |(T(t_2-t_1) - I)T(t_1-s)(-A)^{\beta}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\quad = J_1 + J_2. \end{aligned}$$

By the same argument leading to (3.10) we get

$$\begin{aligned} J_1 &\leq \int_0^{t_2-t_1} M_1 e^{-\epsilon r} r^{-\beta}[P]\mathcal{M}(\|\varphi\|_\infty)dr \\ &\leq C_2(\beta)\mathcal{M}(\|\varphi\|_\infty)(t_2-t_1)^{1-\beta}. \end{aligned}$$

Also, we have

$$\begin{aligned} J_2 &\leq \int_{-\infty}^{t_1} |(T(t_2-t_1)-I)(-A)^{-\gamma}(T(t_1-s)(-A)^{(\beta-\gamma)}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\leq \int_{-\infty}^{t_1} |(T(t_2-t_1)-I)(-A)^{-\gamma}| \\ &\qquad \cdot|(T(t_1-s)(-A)^{(\beta-\gamma)}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\leq |(T(t_2-t_1)-I)(-A)^{-\gamma}| \\ &\qquad \cdot\int_{-\infty}^{t_1} |T(t_1-s)(-A)^{(\beta-\gamma)}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\leq |(T(t_2-t_1)-I)(-A)^{-\gamma}|C_3(\beta,\gamma)\mathcal{M}(\|\varphi\|_\infty) \end{aligned}$$

provided $0<\gamma<\beta$. Next recall that $(T(r)-I)g=\int_0^r T(s)Agds$ for $g\in D(A)$, by the fundamental theorem of calculus. Thus, for $f\in\mathbb{Y}$,

$$\begin{aligned} |(T(r)-I)(-A)^{-\gamma}f| &= \|\int_0^r T(s)(-A)^{1-\gamma-\alpha}(-A)^{\alpha}fds\| \\ &\leq \|(-A)^{\alpha}f\|\int_0^r M_1e^{-\epsilon s}s^{1-\gamma-\alpha}ds \\ &= C_4(\gamma,\epsilon,M_1)r^{2-\gamma-\alpha}|f|, \end{aligned}$$

since $1-\gamma-\alpha>-1$, because $0<\gamma<\beta<\alpha<1$.

In other words, $|(T(r)-I)(-A)^{-\gamma}|\leq C_4 r^{2-\gamma-\alpha}$; consequently,

$$J_2\leq C_4(t_2-t_1)^{2-\gamma-\alpha}C_3\mathcal{M}(\|\varphi\|_\infty).$$

For $\delta=\min(2-\gamma-\alpha,1-\beta)>0$, it follows that

$$|(G\varphi)(t_2)-(G\varphi(t_1)|\leq C_5|t_2-t_1|^{\delta}\mathcal{M}(\|\varphi\|_\infty), \tag{3.11}$$

where C_5 depends on $\epsilon, M_1, [P], \alpha, \beta, \gamma$ and $\mathbb{Y}$, that is, on parameters of the problem.

It follows that, for $\varphi\in BC(\mathbb{R},\mathbb{X})$ with $\|\varphi(t)\|\leq R$ for all $t\in\mathbb{R}$, then $G\varphi\in BC^{\delta}(\mathbb{R},\mathbb{Y})$ with $\|G\varphi(t)\|\leq R_1$ for all $t\in\mathbb{R}$ and some R_1 that depends on R. This completes the proof. □

Proposition 3.2. *The function G maps bounded sets of $AA(\mathbb{X})$ into bounded sets of $BC^{\delta}(\mathbb{R},\mathbb{Y})\cap AA(\mathbb{X})$ for $0<\delta<\alpha$.*

Proof. We just need to check that

$$G(AA(\mathbb{X}))\subset AA(\mathbb{X}).$$

To this end, let $\varphi\in AA(\mathbb{X})$. Then given a sequence $(s_n')\subset\mathbb{R}$, there exists a subsequence $(s_n)\subset(s_n')$ such that

$$\psi(t)=\lim_{n\to\infty}\varphi(t+s_n)$$

is well defined for each $t\in\mathbb{R}$ and

$$\lim_{n\to\infty}\psi(t-s_n)=\varphi(t)$$

for each $t \in \mathbb{R}$. Since $\psi \in BC(\mathbb{R}, \mathbb{X})$, then

$$(G\varphi)(t + s_n) = \int_{-\infty}^{t+s_n} T(t + s_n - s)P(s)Q(\varphi(s))ds.$$

Let $\sigma = s - s_n$. Then

$$\begin{aligned}(G\varphi)(t + s_n) &= \int_{-\infty}^{t} T(t - \sigma)P(\sigma + s_n)Q(\varphi(\sigma + s_n))d\sigma \\ &= \int_{-\infty}^{t} T(t - \sigma)P_n(\sigma)Q_n(\sigma)d\sigma,\end{aligned}$$

where $P_n(\sigma) = P(\sigma + s_n), Q_n(\sigma) = Q(\varphi(\sigma + s_n))$, $n = 1, 2, \cdots$, $\sigma \in \mathbb{R}$.

Since $P \in AA(\mathbb{Z})$, there exists a subsequence of (s_n), which we still denote by (s_n), such that

$$\hat{P}(\sigma) = \lim_{n\to\infty} P_n(\sigma)$$

exists for each $\sigma \in \mathbb{R}$ and

$$\lim_{n\to\infty} \hat{P}(\sigma - s_n) = P(\sigma)$$

for each $\sigma \in \mathbb{R}$. Clearly we also have, by passing to a subsequence if necessary,

$$\lim_{n\to\infty} \varphi(t + s_n) = \psi(t)$$

and

$$\lim_{n\to\infty} \psi(t - s_n) = \varphi(t),$$

for each $t \in \mathbb{R}$. By the Bochner integral version of Lebesgue's dominated convergence theorem, we get

$$(G\varphi)(t + s_n) = \int_{-\infty}^{t} T(t - \sigma)P_n(\sigma)Q_n(\sigma)d\sigma$$

$$\longrightarrow \int_{-\infty}^{t} T(t - \sigma)\hat{P}(\sigma)Q(\varphi(\sigma))d\sigma = \chi(t)$$

for each $t \in \mathbb{R}$, and

$$\begin{aligned}\chi(t - s_n) &= \int_{-\infty}^{t-s_n} T(t - s_n - \sigma)\hat{P}(\sigma)Q(\psi(\sigma))d\sigma \\ &= \int_{-\infty}^{t} T(t - r)\hat{P}(r - s_n)Q(\psi(r - s_n))dr\end{aligned}$$

by letting $r = \sigma + s_n$. Thus we obtain

$$\chi(t - s_n) \longrightarrow \int_{-\infty}^{t} T(t - r)P(r)Q(\varphi(r))dr = (G\varphi)(t),$$

again by Lebesgue's dominated convergence theorem. This shows that $G(AA(\mathbb{X})) \subset AA(\mathbb{X})$, and the proof is now complete. □

Proposition 3.3. *$BC^{\delta}(\mathbb{R}, \mathbb{Y})$ is compactly contained in $BC(\mathbb{R}, \mathbb{X})$; in other words, the canonical injection $id : BC^{\delta}(\mathbb{R}, \mathbb{Y}) \to BC(\mathbb{R}, \mathbb{X})$ is compact, which implies that*

$$id : BC^{\delta}(\mathbb{R}, \mathbb{Y}) \cap AA(\mathbb{X}) \to AA(\mathbb{X})$$

is compact too.

Proof. We show that id maps bounded sets of $BC^\delta(\mathbb{R},\mathbb{Y})$ into relatively compact sets of $BC(\mathbb{R},\mathbb{X})$. To this end, let (φ_ν) be a bounded sequence in $BC^\delta(\mathbb{R},\mathbb{Y})$. Let $\mathbb{Q}=\{r_n\}$ be the set of all rational numbers. Then $(\varphi_\nu(r_n))$ is a bounded sequence in $\mathbb{Y}$, for each n. By the well-known Cantor diagonalization process, there exists a subsequence (φ_{ν_k}) such that

$$\varphi_{\nu_k}(r_n)\to\varphi(r_n),$$

as $k\to\infty$ in $\mathbb{X}$, for each n, and some $\varphi:\mathbb{Q}\to\mathbb{X}$. But the sequence (φ_n) is an equicontinuous family in $BUC(\mathbb{R},\mathbb{Y})\subset BUC(\mathbb{R},\mathbb{X})$, because of the uniform Hölder condition. Thus, as in the proof of the Arzela-Ascoli theorem, there is a further subsequence (which we still denote by (φ_{ν_k})) satisfying

$$\varphi_{\nu_k}(t)\to\varphi(t),\ \text{as } k\to\infty \tag{3.12}$$

in $\mathbb{X}$, for all $t\in\mathbb{R}$. In addition the convergence is uniform in $t\in\mathbb{R}$. Note that $BUC(\mathbb{R},\mathbb{X})$ can be identified with $C(K,\mathbb{X})$ for a suitable Hausdorff compactification K of $\mathbb{R}$ (see for instance [3]). Thus the convergence $\varphi_{\nu_k}\to\varphi$ holds in $BUC(\mathbb{R},\mathbb{X})\subset BC(\mathbb{R},\mathbb{X})$. This completes the proof. □

Proposition 3.4. *The function G has a fixed point in $AA(\mathbb{X})$.*

Proof. Let us recall that the estimates (3.10)-(3.11), $|G\varphi|_\infty\le C_1(\beta)\mathcal{M}(\|\varphi\|_\infty)$ and $|(G\varphi)(t_2)-(G\varphi(t_1)|\le C_5|t_2-t_1|\delta\mathcal{M}(\|\varphi\|_\infty)$, hold for all $\varphi\in BC(\mathbb{R},\mathbb{Y})$ and all $t_1,t_2\in\mathbb{R}$ with t_2 not equal to t_1. It follows that there exists a constant $C_6=C_6(\epsilon,M,M_1,\alpha,\beta,\gamma)$ such that

$$\begin{aligned}\varphi\in BC(\mathbb{R},\mathbb{X})\quad&\text{and}\quad\|\varphi\|_\infty<R\text{ imply}\\ G\varphi\in BC^\delta(\mathbb{R},\mathbb{Y})\quad&\text{and}\quad|G\varphi|<R_1,\end{aligned}$$

where $R_1=C_6\mathcal{M}(R)$.

Since $\mathcal{M}(R)/R\to0$ as $R\to\infty$, and since $\|y\|\le C_7|y|$ holds for some constant C_7 and all $y\in\mathbb{Y}$, it follows that there exists $\rho>0$ such that for all $R\ge\rho$, we have

$$G(B_{AA(\mathbb{X})}(0,R))\subset B_{BC^\delta(\mathbb{R},\mathbb{Y})}(0,R)\cap B_{AA(\mathbb{X})}(0,R). \tag{3.13}$$

Since G leaves $AA(\mathbb{X})\subset BC(\mathbb{R},\mathbb{X})$ invariant, the estimate (3.13) along with the continuity properties of G imply that G is a continuous, compact mapping $S\to S$, where S is the ball of radius R in $AA(\mathbb{X})$ and $R\ge\rho$. By the Schauder fixed point theorem, G has a fixed point in S, φ_0. Obviously, φ_0 is a mild solution of (1.1). □

Finally, the above results can be summarized as follows.

Theorem 3.5. *Let A generate an exponentially stable C_0-semigroup $\mathcal{T}$ in $\mathcal{B}(\mathbb{X})$. Assume assumptions* (1.1) *and* (3.2)-(3.5). *Then* (1.1) *has a mild solution in $AA(\mathbb{X})$.*

Now we end this paper with the following

Example of nonuniqueness. Let $\mathbb{X}=\mathbb{R}$, $A=-1$ and

$$u(t)=\begin{cases}t^{3/2}e^{1-t}, & \text{for } t\in[0,\frac32],\\ 0, & \text{for } t\in[-\frac32,0].\end{cases}$$

Then for $t\in[0,\frac32]$ we have

$$u'(t)=-u(t)+\frac32t^{1/2}e^{(1-t)}=-u(t)+\frac32u(t)^{1/3}e^{\frac23(1-t)}=-u(t)+f(t,u(t))$$

where

$$f(t,\varphi)=\begin{cases}\frac{3}{2}\varphi^{1/3}e^{\frac{2}{3}(1-t)}, & \text{for } t\in[0,\frac{3}{2}]\times\mathbb{R},\\ \frac{3}{2}\varphi^{1/3}e^{2/3}, & \text{for } t\in[-\frac{3}{2},0]\times\mathbb{R}.\end{cases}$$

Note that $u'(\frac{3}{2})=0$ and $u(\frac{3}{2})=(\frac{3}{2})^{\frac{3}{2}}e^{-\frac{3}{2}}$.

Now let $f(t,\varphi)=f(\frac{3}{2},\varphi)$ on $[\frac{3}{2},3]\times\mathbb{R}$ and $f(t,\varphi)=f(\frac{9}{2}-t,\varphi)$ on $[3,\frac{9}{2}]\times\mathbb{R}$; let $u(t)=u(\frac{3}{2})$ on $[\frac{3}{2},3]$, and $u(t)=u(\frac{9}{2}-t)$ on $[3,\frac{9}{2}]$. Then $u'=-u+f(t,u)$ on $[-\frac{3}{2},\frac{9}{2}]$, together with $u(0)=0$.

Extend u to be a periodic function of period 6 (hence an almost automorphic function). Then u and $v\equiv 0$ both satisfy

$$\frac{dx}{dt}=-x+f(t,x),\quad x(0)=0.$$

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