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# ALMOST AUTOMORPHIC SOLUTIONS OF SEMILINEAR EVOLUTION EQUATIONS

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ABSTRACT. We are concerned with the semilinear differential equation in a Banach space $\mathbb{X}$,

$$x'(t) = Ax(t) + F(t, x(t)), \quad t \in \mathbb{R},$$

where $A$ generates an exponentially stable $C_0$-semigroup and $F(t,x) : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a function of the form $F(t,x) = P(t)Q(x)$. Under appropriate conditions on $P$ and $Q$, and using the Schauder fixed point theorem, we prove the existence of an almost automorphic mild solution to the above equation.

## 1. Introduction

Consider in a Banach space $(\mathbb{X}, \|\cdot\|)$ the semilinear differential equation

$$x'(t) = Ax(t) + F(t, x(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where the linear operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ generates an exponentially stable $C_0$-semigroup $\mathcal{T} = (T(t))_{t \geq 0}$; that is, $\mathcal{T}$ satisfies the estimate

$$\|T(t)\| \leq Me^{-\epsilon t}, \tag{1.2}$$

for some constants $M > 0, \epsilon > 0$ and all $t \geq 0$. Let $F : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be jointly continuous. A *mild solution* to (1.1) is a function $x \in C(\mathbb{R}, \mathbb{X})$ satisfying the integral equation

$$x(t) = T(t-a)x(a) + \int_a^t T(t-s)F(s, x(s))ds \tag{1.3}$$

for every $a \in \mathbb{R}$ and every $t \geq a$.

A fundamental problem is the existence of almost automorphic mild solutions to (1.1). Recently, G. M. N'Guérékata [5] showed, using the Banach fixed point theorem, that if

i) $F$ is Lipschitzian in $x \in \mathbb{X}$, uniformly in $t \in \mathbb{R}$, that is,

$$\|F(t,x) - F(t,y)\| \leq L\|x - y\| \tag{1.4}$$

for all $x, y \in \mathbb{X}$, and $t \geq 0$, and $L$ is sufficiently small, namely $L < \dfrac{\epsilon}{M}$, where $\epsilon$ and $M$ are as in (1.2), and

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ii) $F(t,x)$ is almost automorphic in $t \in \mathbb{R}$ for each $x \in \mathbb{X}$,

then problem (1.1) has a unique almost automorphic mild solution.

In this paper, we are going to prove the existence of almost automorphic mild solutions to (1.1), $F$ being not necessarily Lipschitzian. But first, let us recall some definitions.

**Definition 1.1.** A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be *almost automorphic* if for every sequence of real numbers $(s'_n)$, there exists a subsequence $(s_n)$ such that

$$g(t) = \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$.

It is well known that the range $\mathcal{R}_f = \{f(t)|t \in \mathbb{R}\}$ of an almost automorphic function $f$ is relatively compact in $\mathbb{X}$, thus bounded in norm (see [6], Theorem 2.13). The function $g$ in the definition is also bounded and strongly measurable. Also, the set $AA(\mathbb{X})$ of all almost automorphic functions $f : \mathbb{R} \to \mathbb{X}$ equipped with the sup-norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|,$$

is a Banach space (see [6], page 20).

Also, given two Banach spaces $(\mathbb{X}_1, \|\cdot\|_1)$ and $(\mathbb{X}_2, \|\cdot\|_2)$, $B(\mathbb{X}_1, \mathbb{X}_2)$ will denote the Banach space of bounded linear operators $L : \mathbb{X}_1 \to \mathbb{X}_2$, $BC(\mathbb{R}, \mathbb{X}_1)$ is the Banach space of all continuous and bounded functions $f : \mathbb{R} \to \mathbb{X}_1$, and $BUC(\mathbb{R}, \mathbb{X}_1)$ is the Banach space of all bounded and uniformly continuous functions $f : \mathbb{R} \to \mathbb{X}_1$.

## 2. Preliminaries

In this paper $(\mathbb{Y}, |\cdot|)$ will denote a Banach space algebraically contained in $\mathbb{X}$ such that the canonical injection $\mathbb{Y} \to \mathbb{X}$ is compact. An example of such a space $\mathbb{Y}$ is an abstract Sobolev space that we construct as follows:

Let $A$ be as in (1.1), (1.2). By (1.2), $0 \in \rho(A)$, so that the fractional powers $(-A)^\alpha$, $0 < \alpha < 1$, are well defined. Also, since $0 \in \rho(A)$, the norm

$$|f| = \|(-A)^\alpha f\| \tag{2.1}$$

is equivalent to the graph norm

$$\|f\|_\alpha = \|(-A)^\alpha f\| + \|f\|.$$

Now we take $\mathbb{X} = L^p(\Omega)$, where $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain in $\mathbb{R}^n$. Let $A$ be a linear uniformly elliptic operator (with suitable boundary conditions), of order $2m$. Then let $\mathbb{Y}$ be the domain of $(-A)^\alpha$ with norm (2.1); we have

$$W_0^{2m\alpha,p}(\Omega) \subset \mathbb{Y} \subset W^{2m\alpha,p}(\Omega)$$

and the norm $|\cdot|$ in $\mathbb{Y}$ is equivalent to the usual norm in $W^{2m\alpha,p}(\Omega)$. Also, the injection $\mathbb{Y} \to \mathbb{X}$ is compact in this case, by Sobolev embedding.

## 3. Main results

Now let $\mathbb{Y} = D((-A)^\alpha)$, the domain of $(-A)^\alpha$, with norm

$$|y| = \|(-A)^\alpha y\|, \quad y \in D((-A)^\alpha),$$

where $0 < \alpha < 1$ is fixed. We get

$$|T(t)y| = \|T(t)(-A)^\alpha y\| \leq Me^{-\epsilon t}\|(-A)^\alpha y\| = Me^{-\epsilon t}|y| \tag{3.1}$$

for each $y \in \mathbb{Y}$ and every $t \geq 0$, by (1.2).

We also make the following assumptions:

$$F(t,x) = P(t)Q(x), \quad \text{for all} \quad t \in \mathbb{R}, x \in \mathbb{X}, \tag{3.2}$$

where $P(t) \in AA(\mathbb{Z})$ for each $t \in \mathbb{R}$ with $\mathbb{Z} = B(\mathbb{X}, \mathbb{Y})$; $P$ is continuous from $\mathbb{R}$ to $AA(\mathbb{Z})$, and $Q : BC(\mathbb{R}, \mathbb{X}) \to BC(\mathbb{R}, \mathbb{X})$ is continuous and satisfies the estimate

$$\|Q\varphi\|_\infty \leq \mathcal{M}(\|\varphi\|_\infty), \tag{3.3}$$

where $\|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\|$ and $\mathcal{M} \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies

$$\lim_{r\to\infty} \frac{\mathcal{M}(r)}{r} = 0. \tag{3.4}$$

Note that $\mathcal{M}$ can be unbounded but must grow slower than a linear function. Let

$$[P] := \sup_{t\in\mathbb{R}} \|P(t)\|_{\mathbb{Z}} < \infty. \tag{3.5}$$

Define $G : BC(\mathbb{R}, \mathbb{X}) \to BC(\mathbb{R}, \mathbb{Y})$ by

$$(G\varphi)(t) = \int_{-\infty}^{t} T(t-s)F(s, \varphi(s))ds. \tag{3.6}$$

For $\varphi \in BC(\mathbb{R}, \mathbb{X})$, this integral exists. Indeed, we have

$$\begin{aligned} |(G\varphi)(t)| &\leq \int_{-\infty}^{t} |T(t-s)||P(t)Q(\varphi(s))|ds \\ &\leq \int_{-\infty}^{t} Me^{-\epsilon(t-s)}[P]\mathcal{M}(\|\varphi\|_\infty)ds \end{aligned}$$

using (3.1), (3.3) and (3.5). Consequently

$$\begin{aligned} |G\varphi|_\infty &= \sup_{t\in\mathbb{R}} |(G\varphi)(t)| \\ &\leq M\epsilon^{-1}[P]\mathcal{M}(\|\varphi\|_\infty). \end{aligned} \tag{3.7}$$

Continuity of $G$ is straightforward by virtue of continuity of both $P$ and $Q$. Thus we have

$$G(BC(\mathbb{R}, \mathbb{X})) \subset BC(\mathbb{R}, \mathbb{Y}).$$

Finally, for $0 < \delta \leq 1$, let

$$BC^\delta(\mathbb{R}, \mathbb{Y}) \equiv \{f \in BC(\mathbb{R}, \mathbb{Y}) : |f|_{\delta,\mathbb{Y}} < \infty\},$$

where

$$|f|_{\delta,\mathbb{Y}} \equiv \sup_{t\in\mathbb{R}} |f(t)| + \delta \sup_{t,s\in\mathbb{R}, t\neq s} \frac{|f(t) - f(s)|}{|t-s|^\delta}.$$

With the norm $|\cdot|_{\delta,\mathbb{Y}}$, $BC^\delta(\mathbb{R}, \mathbb{Y})$ turns out to be a Banach space of all bounded Hölder continuous $\mathbb{Y}$-valued functions on $\mathbb{R}$ of Hölder exponent $\delta$.

**Proposition 3.1.** *The function $G$ defined above maps bounded sets of $BC(\mathbb{R},\mathbb{X})$ into bounded sets of $BC^{\delta}(\mathbb{R},\mathbb{Y})$ for any $\delta > 0$ satisfying $\delta < \alpha$, where $0 < \alpha < 1$ is the exponent defining $\mathbb{Y} = D(-A)^{-\alpha}$.*

*Proof.* The proof is basically a modification of the above remarks. Let $0 < \beta < \alpha$. Then

$$\begin{aligned} |(G\varphi)(t)| &= |\int_{-\infty}^{t} T(t-s)(-A)^{\beta}(-A)^{-\beta}F(s,\varphi(s))ds| \\ &\leq \int_{-\infty}^{t} |T(t-s)(-A)^{\beta}||(-A)^{-\beta}P(s)||Q(\varphi(s)|ds. \end{aligned} \tag{3.8}$$

Now, by semigroup theory (see for instance [4]), there exists a constant $M_1$ such that

$$\|T(r)(-A)^{\beta}\| \leq \frac{M_1 e^{-\epsilon r}}{r^{\beta}}$$

for all $r > 0$. Thus we obtain, as previously,

$$|T(r)(-A)^{\beta}| \leq M_1 e^{-\epsilon r} r^{-\beta}, \quad r > 0. \tag{3.9}$$

Next, we observe that the function $s \mapsto (-A)^{-\beta}P(s)$ is a uniformly bounded function $\mathbb{R} \to B(\mathbb{X}, D((-A)^{\alpha-\beta})$. Indeed, it is the composition of $P(\cdot) : \mathbb{R} \to B(\mathbb{X}, D((-A)^{\alpha}))$, which is bounded by $[P]$, with $(-A)^{-\beta}$, an isometry from $D((-A)^{\alpha})$ onto $D((-A)^{\alpha-\beta})$. Thus

$$\sup_{t\in\mathbb{R}} \|P(t)\|_{B(\mathbb{X},D((-A)^{\alpha-\beta}))} \leq [P].$$

Now combining the estimates in (3.8) and (3.9), we deduce

$$|(G\varphi)(t)| \leq \int_{-\infty}^{t} M_1 e^{-\epsilon(t-s)}(t-s)^{-\beta}[P]\mathcal{M}(\|\varphi\|_{\infty})ds.$$

Letting $r = t - s$ in the integral gives

$$|(G\varphi(t)| \leq \int_{0}^{\infty} M_1 e^{-r} r^{-\beta}[P]\mathcal{M}(\|\varphi\|)dr;$$

that is,

$$|(G\varphi)(t)| \leq C_1(\beta)\mathcal{M}(\|\varphi\|_{\infty}), \tag{3.10}$$

where $C_1(\beta)$ depends on $\beta, M_1, \epsilon$ and $[P]$. Next, for $t_2 > t_1$, we have

$$\begin{aligned} &|(G\varphi)(t_2) - (G\varphi)(t_1)| \\ &\quad \leq |(\int_{-\infty}^{t_2} - \int_{-\infty}^{t_1})T(t_2-s)(-A)^{\beta}(-A)^{-\beta}P(s)Q(\varphi(s))ds| \\ &\qquad + |\int_{-\infty}^{t_1}(T(t_2-s) - T(t_1-s))(-A)^{\beta}(-A)^{-\beta}P(s)Q(\varphi(s))ds| \\ &\quad \leq \int_{t_1}^{t_2} |T(t_2-s)(-A)^{\beta}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\qquad + \int_{-\infty}^{t_1} |(T(t_2-t_1) - I)T(t_1-s)(-A)^{\beta}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\quad = J_1 + J_2. \end{aligned}$$

By the same argument leading to (3.10) we get

$$\begin{aligned} J_1 &\leq \int_0^{t_2-t_1} M_1 e^{-\epsilon r} r^{-\beta} [P] \mathcal{M}(\|\varphi\|_\infty) dr \\ &\leq C_2(\beta) \mathcal{M}(\|\varphi\|_\infty)(t_2-t_1)^{1-\beta}. \end{aligned}$$

Also, we have

$$\begin{aligned} J_2 &\leq \int_{-\infty}^{t_1} |(T(t_2-t_1)-I)(-A)^{-\gamma}(T(t_1-s)(-A)^{(\beta-\gamma)}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\leq \int_{-\infty}^{t_1} |(T(t_2-t_1)-I)(-A)^{-\gamma}| \\ &\qquad \cdot |(T(t_1-s)(-A)^{(\beta-\gamma)}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\leq |(T(t_2-t_1)-I)(-A)^{-\gamma}| \\ &\qquad \cdot \int_{-\infty}^{t_1} |T(t_1-s)(-A)^{(\beta-\gamma)}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\leq |(T(t_2-t_1)-I)(-A)^{-\gamma}| C_3(\beta,\gamma)\mathcal{M}(\|\varphi\|_\infty) \end{aligned}$$

provided $0<\gamma<\beta$. Next recall that $(T(r)-I)g = \int_0^r T(s)Agds$ for $g \in D(A)$, by the fundamental theorem of calculus. Thus, for $f \in \mathbb{Y}$,

$$\begin{aligned} |(T(r)-I)(-A)^{-\gamma}f| &= \|\int_0^r T(s)(-A)^{1-\gamma-\alpha}(-A)^{\alpha} f ds\| \\ &\leq \|(-A)^{\alpha} f\| \int_0^r M_1 e^{-\epsilon s} s^{1-\gamma-\alpha} ds \\ &= C_4(\gamma,\epsilon,M_1) r^{2-\gamma-\alpha}|f|, \end{aligned}$$

since $1-\gamma-\alpha > -1$, because $0<\gamma<\beta<\alpha<1$.

In other words, $|(T(r)-I)(-A)^{-\gamma}| \leq C_4 r^{2-\gamma-\alpha}$; consequently,

$$J_2 \leq C_4(t_2-t_1)^{2-\gamma-\alpha} C_3 \mathcal{M}(\|\varphi\|_\infty).$$

For $\delta = \min(2-\gamma-\alpha, 1-\beta) > 0$, it follows that

$$|(G\varphi)(t_2) - (G\varphi(t_1)| \leq C_5 |t_2-t_1|^{\delta} \mathcal{M}(\|\varphi\|_\infty), \tag{3.11}$$

where $C_5$ depends on $\epsilon, M_1, [P], \alpha, \beta, \gamma$ and $\mathbb{Y}$, that is, on parameters of the problem.

It follows that, for $\varphi \in BC(\mathbb{R},\mathbb{X})$ with $\|\varphi(t)\| \leq R$ for all $t \in \mathbb{R}$, then $G\varphi \in BC^{\delta}(\mathbb{R},\mathbb{Y})$ with $\|G\varphi(t)\| \leq R_1$ for all $t \in \mathbb{R}$ and some $R_1$ that depends on $R$. This completes the proof. □

**Proposition 3.2.** *The function $G$ maps bounded sets of $AA(\mathbb{X})$ into bounded sets of $BC^{\delta}(\mathbb{R},\mathbb{Y}) \cap AA(\mathbb{X})$ for $0<\delta<\alpha$.*

*Proof.* We just need to check that

$$G(AA(\mathbb{X})) \subset AA(\mathbb{X}).$$

To this end, let $\varphi \in AA(\mathbb{X})$. Then given a sequence $(s'_n) \subset \mathbb{R}$, there exists a subsequence $(s_n) \subset (s'_n)$ such that

$$\psi(t) = \lim_{n\to\infty} \varphi(t+s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} \psi(t-s_n) = \varphi(t)$$

for each $t \in \mathbb{R}$. Since $\psi \in BC(\mathbb{R}, \mathbb{X})$, then

$$(G\varphi)(t+s_n) = \int_{-\infty}^{t+s_n} T(t+s_n-s)P(s)Q(\varphi(s))ds.$$

Let $\sigma = s - s_n$. Then

$$\begin{aligned}(G\varphi)(t+s_n) &= \int_{-\infty}^{t} T(t-\sigma)P(\sigma+s_n)Q(\varphi(\sigma+s_n))d\sigma \\ &= \int_{-\infty}^{t} T(t-\sigma)P_n(\sigma)Q_n(\sigma)d\sigma,\end{aligned}$$

where $P_n(\sigma) = P(\sigma+s_n), Q_n(\sigma) = Q(\varphi(\sigma+s_n))$, $n = 1, 2, \cdots$, $\sigma \in \mathbb{R}$.

Since $P \in AA(\mathbb{Z})$, there exists a subsequence of $(s_n)$, which we still denote by $(s_n)$, such that

$$\hat{P}(\sigma) = \lim_{n\to\infty} P_n(\sigma)$$

exists for each $\sigma \in \mathbb{R}$ and

$$\lim_{n\to\infty} \hat{P}(\sigma - s_n) = P(\sigma)$$

for each $\sigma \in \mathbb{R}$. Clearly we also have, by passing to a subsequence if necessary,

$$\lim_{n\to\infty} \varphi(t+s_n) = \psi(t)$$

and

$$\lim_{n\to\infty} \psi(t-s_n) = \varphi(t),$$

for each $t \in \mathbb{R}$. By the Bochner integral version of Lebesgue's dominated convergence theorem, we get

$$(G\varphi)(t+s_n) = \int_{-\infty}^{t} T(t-\sigma)P_n(\sigma)Q_n(\sigma)d\sigma$$

$$\longrightarrow \int_{-\infty}^{t} T(t-\sigma)\hat{P}(\sigma)Q(\varphi(\sigma))d\sigma = \chi(t)$$

for each $t \in \mathbb{R}$, and

$$\begin{aligned}\chi(t-s_n) &= \int_{-\infty}^{t-s_n} T(t-s_n-\sigma)\hat{P}(\sigma)Q(\psi(\sigma))d\sigma \\ &= \int_{-\infty}^{t} T(t-r)\hat{P}(r-s_n)Q(\psi(r-s_n))dr\end{aligned}$$

by letting $r = \sigma + s_n$. Thus we obtain

$$\chi(t-s_n) \longrightarrow \int_{-\infty}^{t} T(t-r)P(r)Q(\varphi(r))dr = (G\varphi)(t),$$

again by Lebesgue's dominated convergence theorem. This shows that $G(AA(\mathbb{X})) \subset AA(\mathbb{X})$, and the proof is now complete. □

**Proposition 3.3.** *$BC^\delta(\mathbb{R}, \mathbb{Y})$ is compactly contained in $BC(\mathbb{R}, \mathbb{X})$; in other words, the canonical injection $id : BC^\delta(\mathbb{R}, \mathbb{Y}) \to BC(\mathbb{R}, \mathbb{X})$ is compact, which implies that*

$$id : BC^\delta(\mathbb{R}, \mathbb{Y}) \cap AA(\mathbb{X}) \to AA(\mathbb{X})$$

*is compact too.*

*Proof.* We show that $id$ maps bounded sets of $BC^\delta(\mathbb{R},\mathbb{Y})$ into relatively compact sets of $BC(\mathbb{R},\mathbb{X})$. To this end, let $(\varphi_\nu)$ be a bounded sequence in $BC^\delta(\mathbb{R},\mathbb{Y})$. Let $\mathbb{Q} = \{r_n\}$ be the set of all rational numbers. Then $(\varphi_\nu(r_n))$ is a bounded sequence in $\mathbb{Y}$, for each $n$. By the well-known Cantor diagonalization process, there exists a subsequence $(\varphi_{\nu_k})$ such that

$$\varphi_{\nu_k}(r_n) \to \varphi(r_n),$$

as $k \to \infty$ in $\mathbb{X}$, for each $n$, and some $\varphi : \mathbb{Q} \to \mathbb{X}$. But the sequence $(\varphi_n)$ is an equicontinuous family in $BUC(\mathbb{R},\mathbb{Y}) \subset BUC(\mathbb{R},\mathbb{X})$, because of the uniform Hölder condition. Thus, as in the proof of the Arzela-Ascoli theorem, there is a further subsequence (which we still denote by $(\varphi_{\nu_k})$) satisfying

$$\varphi_{\nu_k}(t) \to \varphi(t), \text{ as } k \to \infty \tag{3.12}$$

in $\mathbb{X}$, for all $t \in \mathbb{R}$. In addition the convergence is uniform in $t \in \mathbb{R}$. Note that $BUC(\mathbb{R},\mathbb{X})$ can be identified with $C(K,\mathbb{X})$ for a suitable Hausdorff compactification $K$ of $\mathbb{R}$ (see for instance [3]). Thus the convergence $\varphi_{\nu_k} \to \varphi$ holds in $BUC(\mathbb{R},\mathbb{X}) \subset BC(\mathbb{R},\mathbb{X})$. This completes the proof. □

**Proposition 3.4.** *The function $G$ has a fixed point in $AA(\mathbb{X})$.*

*Proof.* Let us recall that the estimates (3.10)-(3.11), $|G\varphi|_\infty \leq C_1(\beta)\mathcal{M}(\|\varphi\|_\infty)$ and $|(G\varphi)(t_2) - (G\varphi(t_1)| \leq C_5|t_2 - t_1|\delta\mathcal{M}(\|\varphi\|_\infty)$, hold for all $\varphi \in BC(\mathbb{R},\mathbb{Y})$ and all $t_1, t_2 \in \mathbb{R}$ with $t_2$ not equal to $t_1$. It follows that there exists a constant $C_6 = C_6(\epsilon, M, M_1, \alpha, \beta, \gamma)$ such that

$$\begin{aligned} \varphi \in BC(\mathbb{R},\mathbb{X}) \quad &\text{and} \quad \|\varphi\|_\infty < R \text{ imply} \\ G\varphi \in BC^\delta(\mathbb{R},\mathbb{Y}) \quad &\text{and} \quad |G\varphi| < R_1, \end{aligned}$$

where $R_1 = C_6\mathcal{M}(R)$.

Since $\mathcal{M}(R)/R \to 0$ as $R \to \infty$, and since $\|y\| \leq C_7|y|$ holds for some constant $C_7$ and all $y \in \mathbb{Y}$, it follows that there exists $\rho > 0$ such that for all $R \geq \rho$, we have

$$G(B_{AA(\mathbb{X})}(0,R)) \subset B_{BC^\delta(\mathbb{R},\mathbb{Y})}(0,R) \cap B_{AA(\mathbb{X})}(0,R). \tag{3.13}$$

Since $G$ leaves $AA(\mathbb{X}) \subset BC(\mathbb{R},\mathbb{X})$ invariant, the estimate (3.13) along with the continuity properties of $G$ imply that $G$ is a continuous, compact mapping $S \to S$, where $S$ is the ball of radius $R$ in $AA(\mathbb{X})$ and $R \geq \rho$. By the Schauder fixed point theorem, $G$ has a fixed point in $S$, $\varphi_0$. Obviously, $\varphi_0$ is a mild solution of (1.1). □

Finally, the above results can be summarized as follows.

**Theorem 3.5.** *Let $A$ generate an exponentially stable $C_0$-semigroup $\mathcal{T}$ in $\mathcal{B}(\mathbb{X})$. Assume assumptions* (1.1) *and* (3.2)-(3.5)*. Then* (1.1) *has a mild solution in* $AA(\mathbb{X})$.

Now we end this paper with the following

**Example of nonuniqueness.** Let $\mathbb{X} = \mathbb{R}$, $A = -1$ and

$$u(t) = \begin{cases} t^{3/2}e^{1-t}, & \text{for } t \in [0, \frac{3}{2}], \\ 0, & \text{for } t \in [-\frac{3}{2}, 0]. \end{cases}$$

Then for $t \in [0, \frac{3}{2}]$ we have

$$u'(t) = -u(t) + \frac{3}{2}t^{1/2}e^{(1-t)} = -u(t) + \frac{3}{2}u(t)^{1/3}e^{\frac{2}{3}(1-t)} = -u(t) + f(t, u(t))$$

where

$$f(t,\varphi) = \begin{cases} \frac{3}{2}\varphi^{1/3}e^{\frac{2}{3}(1-t)}, & \text{for } t \in [0,\frac{3}{2}] \times \mathbb{R}, \\ \frac{3}{2}\varphi^{1/3}e^{2/3}, & \text{for } t \in [-\frac{3}{2},0] \times \mathbb{R}. \end{cases}$$

Note that $u'(\frac{3}{2}) = 0$ and $u(\frac{3}{2}) = (\frac{3}{2})^{\frac{3}{2}}e^{-\frac{3}{2}}$.

Now let $f(t,\varphi) = f(\frac{3}{2},\varphi)$ on $[\frac{3}{2},3] \times \mathbb{R}$ and $f(t,\varphi) = f(\frac{9}{2}-t,\varphi)$ on $[3,\frac{9}{2}] \times \mathbb{R}$; let $u(t) = u(\frac{3}{2})$ on $[\frac{3}{2},3]$, and $u(t) = u(\frac{9}{2}-t)$ on $[3,\frac{9}{2}]$. Then $u' = -u + f(t,u)$ on $[-\frac{3}{2},\frac{9}{2}]$, together with $u(0) = 0$.

Extend $u$ to be a periodic function of period 6 (hence an almost automorphic function). Then $u$ and $v \equiv 0$ both satisfy

$$\frac{dx}{dt} = -x + f(t,x), \quad x(0) = 0.$$

## References


[1] S. Bochner, Continuous mappings of almost automorphic and almost periodic functions, *Proc. Nat. Acad. Sci. USA* **52**(1964), 407-410. MR0168997 (29:6252)

[2] T. Diagana, G. M. N'Guérékata and N. V. Minh, Almost automorphic solutions of evolution equations, *Proc. Amer. Math. Soc.* **132** (2004), 3289-3298. MR2073304

[3] N. Dunford and J. T. Schwartz, *Linear Operators*. Vol. I, Interscience, New York, 1964. MR0117523 (22:8302)

[4] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford University Press, Oxford, 1985. MR0790497 (87c:47056)

[5] G. M. N'Guérékata, Existence and uniqueness of almost automorphic mild solutions to some semilinear abstract differential equations, *Semigroup Forum* **69** (2004), no. 1, 80-86. MR2063980

[6] G. M. N'Guérékata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic/ Plenum Publishers, New York, 2001. MR1880351 (2003d:43001)



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