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A COMPACT GROUP WHICH IS NOT VALDIVIA COMPACT

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Abstract. A compact space K is *Valdivia compact* if it can be embedded in a Tikhonov cube I^A in such a way that the intersection $K \cap \Sigma$ is dense in K, where Σ is the sigma-product (= the set of points with countably many non-zero coordinates). We show that there exists a compact connected Abelian group of weight ω_1 which is not Valdivia compact, and deduce that Valdivia compact spaces are not preserved by open maps.

1. Introduction

Let $\{X_\alpha : \alpha \in A\}$ be a family of spaces, and let a point x_α^* be given in each X_α. The *sigma-product* Σ of the family $\{X_\alpha\}$ with the base point $\{x_\alpha^*\}$ is the subset of the product $\prod X_\alpha$ consisting of all points $\{x_\alpha\}$ such that the set $\{\alpha \in A : x_\alpha \neq x_\alpha^*\}$ is countable. A compact space K is *Valdivia compact* if it can be embedded in a Tikhonov cube I^A in such a way that $K \cap \Sigma$ is dense in K, where Σ is the sigma-product of intervals with the zero base point. The class of Valdivia compact spaces is a natural extension of the class of Corson compact spaces, which are defined as compact subspaces of sigma-products of intervals. A compact space is Corson compact if and only if it is Valdivia compact and countably tight. While Corson compact spaces are preserved by maps (we use the word "map" to mean a continuous map), this is not true for Valdivia compact spaces (however, if a Valdivia compact space is mapped onto a countably tight space X, then X is Corson compact [9]).

The class of Valdivia compact spaces was introduced by Argyros, Mercourakis and Negrepontis in [1]. They showed among others that these spaces admit 'sufficiently many' retractions, which gives a projectional resolution of the identity on their spaces of continuous functions; see [16]. The name *Valdivia compact* was introduced by Deville and Godefroy in [3]. Valdivia compacta have been extensively studied by Kalenda; we refer to his survey article [11]. Kalenda proved in [10] that an open image of a Valdivia compact space is Valdivia provided it contains a dense set of G_δ points. The general question whether Valdivia compact spaces are preserved by open maps had remained open; see [11]. In the present paper we answer this question in the negative.

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To this end, we construct a compact connected Abelian group G which is not Valdivia compact. Let us see why this yields an example of an open map which does not preserve Valdivia compactness.

Proposition 1.1. *Every compact Abelian group is the image of a Valdivia compact space under an open map.*

Proof. We prove more: every compact Abelian group is a homomorphic image of a product of compact metrizable groups. Indeed, by virtue of the Pontryagin duality this assertion is equivalent to the following: every Abelian group embeds into a direct sum of countable groups. To see that this is true, note that every Abelian group embeds into a divisible group [5, Thm. 24.1], and every Abelian divisible group is a direct sum of countable groups [5, Thm. 23.1].

It remains to observe that:

Fact 1. Every continuous onto homomorphism between compact groups is open.

Indeed, every quotient map $p : G \to G/H$, where G is a topological group and H is a subgroup of G, is open, since for any open set $U \subset G$ its saturation $p^{-1}(p(U)) = UH$ is open, being a union of translates of U. On the other hand, every map between compact Hausdorff spaces is closed, hence quotient if it is onto.

Fact 2. Compact metric spaces are Valdivia.

Indeed, every compact metric space embeds in the product of a countable family of intervals, and for countable families the product is the same as the sigma product.

Fact 3. Valdivia compact spaces are preserved by products.

This is obvious from the definition, and also can be found in [11, Theorem 3.29]. □

Our compact group G is the Pontryagin dual of an uncountable indecomposable torsion-free Abelian group A. The above argument can be made easier in this case: A is a subgroup of the vector space $A \otimes \mathbb{Q}$ over the field $\mathbb{Q}$ of rationals (the tensor product $A \otimes \mathbb{Q}$ is taken over $\mathbb{Z}$, the group of integers), and every vector space over $\mathbb{Q}$ is the direct sum of a family of copies of $\mathbb{Q}$. Hence $G = A^*$ is a homomorphic image of a power of the compact metrizable group $\mathbb{Q}^*$.

The class of Valdivia compact spaces is contained in a wider class that was denoted by $\mathcal{R}$ in [2]. Recall the definition of this class. A map $f : X \to Y$ is *right-invertible* if there exists a map $g : Y \to X$ such that $fg : Y \to Y$ is the identity. A map $f : X \to Y$ is right-invertible if and only if it is homeomorphic to a retraction.

The class $\mathcal{R}$ is defined as the smallest class containing all compact metric spaces which is closed under inverse limits of continuous transfinite sequences whose bonding mappings are right-invertible (in that case the limit projections are right-invertible as well [2, Proposition 4.6]). An inverse sequence $\{X_\alpha; p_\alpha^\beta : \alpha < \beta < \kappa\}$ is *continuous* if for every limit ordinal $\delta < \kappa$ the space X_δ is naturally homeomorphic to $\varprojlim\{X_\alpha; p_\alpha^\beta : \alpha < \beta < \delta\}$. To see that Valdivia compact spaces are in $\mathcal{R}$, note that every Valdivia compact space of uncountable weight is the inverse limit of a continuous transfinite sequence of Valdivia compacta of smaller weights all of whose bonding maps are retractions; see [1] or [11, Thm. 3.6.2].

The compact group that we construct is not in the class $\mathcal{R}$. Thus our example shows that the image of a product of compact metric spaces under an open map need not be in $\mathcal{R}$. Our example has the smallest possible weight, namely ω_1. It has been proved in [12] that a 0-dimensional open image of a Valdivia compact space is Valdivia if its weight does not exceed ω_1.

Let us also mention that every compact group is a Dugundji space [14, 15] and every 0-dimensional Dugundji space is Valdivia [12]. It is unknown whether the class of Valdivia compacta is stable under retractions; in [12] an affirmative answer is given in the case where the retract has weight ω_1.

2. Proof of the main theorem

Theorem 2.1 (Main Theorem). *There exists a compact connected Abelian group of weight ω_1 which is not in the class $\mathcal{R}$ and hence is not Valdivia compact.*

We need some prerequisites on cohomologies. For a compact space X, an Abelian group G and an integer n we denote by $H^n(X, G)$ the cohomology group in the sense of the theory of sheaves [7] or the Čech theory or the Alexander–Spanier theory [13, Chapter 6]. All these theories are naturally isomorphic to each other [7, Theorem 5.10.1], [13, Corollary 6.8.8] and have the following continuity property: If X is the inverse limit of a family of compact spaces (X_α), then the group $H^n(X, G)$ is the direct limit of the family $H^n(X_\alpha, G)$ [13, Theorem 6.6.6 and the following paragraph]. The singular cohomology theory does not have this property, and that is why it is not suitable for our purposes.

Lemma 2.2. *If G is a countable Abelian group, then for every compact metrizable space X and every integer n the cohomology group $H^n(X, G)$ is countable.*

Proof. We have $H^n(X, G) = \varinjlim H^n(\mathcal{U}, G)$, where $\mathcal{U}$ runs over a cofinal collection of open covers of X [13, Section 6.7]. If X is compact metric, there is a cofinal sequence $(\mathcal{U}_i)$ of finite open covers ("cofinal" means that every open cover of X is refined by some $\mathcal{U}_i$). Then $H^n(X, G)$ is the direct limit of the sequence $(H^n(\mathcal{U}_i, G))$ of countable groups and hence is countable.

Alternatively, for each finite simplicial complex K the group $H^n(|K|, G)$ is countable. Since X can be represented as the limit of an inverse sequence of polyhedra $|K|$ (see e.g. [4, Theorem 1.13.2]), we again see that $H^n(X, G)$ is the direct limit of a countable sequence of countable groups. □

Proposition 2.3. *Let a compact space X be in the class $\mathcal{R}$. For every countable Abelian group G and every integer n the cohomology group $H^n(X, G)$ is covered by its countable direct summands.*

Proof. Let $\mathcal{R}'$ be the class of all compact spaces X for which the proposition holds.

The cohomology functor turns inverse limits of compact spaces into direct limits of Abelian groups (that is the continuity property mentioned above) and right-invertible maps into left-invertible homomorphisms, which are injective homomorphisms onto a direct summand. It follows that the class $\mathcal{R}'$ is stable under limits of continuous inverse sequences with right-invertible bonding maps and projections. All compact metrizable spaces are in $\mathcal{R}'$ (Lemma 2.2). Since $\mathcal{R}$ is the smallest class with these properties, we have $\mathcal{R} \subset \mathcal{R}'$. □

Let $\mathbf{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the circle group. For an Abelian group A we denote by A^* its Pontryagin dual (= the group of all characters $\chi : A \to \mathbf{T}$, considered as a compact group; see e.g. [8, Ch. 6]).

Lemma 2.4. *The Pontryagin duality turns direct limits of discrete groups into inverse limits of compact groups.*

Proof. Let $\{A_i : i \in I\}$ be a directed family of discrete groups, $A = \varinjlim A_i$. Each character $\chi \in A^*$ can be identified with a family $\{\chi_i : i \in I\} \in \varprojlim A_i^*$, where χ_i is the restriction of χ to A_i (more precisely, χ_i is the image of χ under the map $A^* \to A_i^*$ dual to the canonical map $A_i \to A$). It is easy to check that the map $A^* \to \varprojlim A_i^*$ arising in this way is a topological isomorphism. □

The following proposition is well known.

Proposition 2.5. *For every torsion-free Abelian group A there exists a natural isomorphism $\phi : A \to H^1(X, \mathbb{Z})$, where $X = A^*$.*

Proof. The homomorphism ϕ can be described as follows. Every $a \in A$ can be identified with a character $\chi_a : X \to \mathbf{T}$. Pick a generator $u \in H^1(\mathbf{T}, \mathbb{Z}) \simeq \mathbb{Z}$, and put $\phi(a) = H^1(\chi_a)(u) \in H^1(X, \mathbb{Z})$.

If $A = \mathbb{Z}^n$ and $X = A^* = \mathbf{T}^n$, it is clear that ϕ is an isomorphism. The general case follows by passing to limits: every torsion-free Abelian group is the direct limit of finitely-generated free groups; the Pontryagin duality turns direct limits of discrete groups into inverse limits of compact groups (Lemma 2.4); and the cohomology functor turns inverse limits of compact spaces back to direct limits. □

Proof of Theorem 2.1. There exists a torsion-free Abelian group A of cardinality ω_1 which is indecomposable [6, Sections 88 and 89]; that is, A has no proper direct summands. Let $X = A^*$. We claim that the compact group X has the required properties.

The duals of torsion-free discrete groups are connected [8, Theorem 24.25], so X is connected. According to Proposition 2.5, the cohomology group $H^1(X, \mathbb{Z})$ is isomorphic to A and therefore indecomposable. Proposition 2.3 implies that X is not in the class $\mathcal{R}$. □

Corollary 2.6. *There exists a compact metric space K and an open onto map $f : K^{\omega_1} \to X$ such that X is not in the class $\mathcal{R}$ and hence not Valdivia.*

Proof. We explained the construction in Section 1: Let A and $X = A^*$ be as in the preceding proof. Embed A into $A \otimes \mathbb{Q} = \mathbb{Q}^{(\omega_1)}$. Passing to the duals, we get a homomorphism of compact groups $K^{\omega_1} \to X$, where $K = \mathbb{Q}^*$ (we consider $\mathbb{Q}$ as a discrete group). □

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