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HOPF ALGEBRAS OF DIMENSION $2p$

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Abstract. Let H be a finite-dimensional Hopf algebra over an algebraically closed field of characteristic 0. If H is not semisimple and $\dim(H) = 2n$ for some odd integer n, then H or H^* is not unimodular. Using this result, we prove that if $\dim(H) = 2p$ for some odd prime p, then H is semisimple. This completes the classification of Hopf algebras of dimension $2p$.

0. Introduction

In recent years, there has been some progress on the classification problems of finite-dimensional Hopf algebras over an algebraically closed field k of characteristic 0 (cf. [Mon98], [And02]). It is shown in [Zhu94] that Hopf algebras of dimension p, where p is a prime, are isomorphic to the group algebra $k[\mathbb{Z}_p]$. In [Ng02b] and [Ng02a], the author completed the classification of Hopf algebras of dimension p^2, which started in [AS98] and [Mas96]. They are group algebras and Taft algebras of dimension p^2 (cf. [Taf71]). However, the classification of Hopf algebras of dimension pq, where p, q are distinct prime numbers, remains open in general.

It is shown in [EG98], [GW00] that semisimple Hopf algebras over k of dimension pq are trivial (i.e. isomorphic to either group algebras or the dual of group algebras). Most recently, Etingof and Gelaki proved that if p, q are odd primes such that $p < q \le 2p + 1$, then any Hopf algebra over k of dimension pq is semisimple [EG]. Meanwhile, the author proved the same result, using a different method, for the case that p, q are twin primes [Ng]. In addition to that, Williams settled the case of dimensions 6 and 10 in [Wil88], and Beattie and Dăscălescu did dimensions 14, 65 in [BD]. Hopf algebras of dimensions 6, 10, 14 and 65 are semisimple, and so they are trivial.

In this paper, we prove that any Hopf algebra of dimension $2p$, where p is an odd prime, over an algebraically closed field k of characteristic 0, is semisimple. By [Mas95], semisimple Hopf algebras of dimension $2p$ are isomorphic to

$$k[\mathbb{Z}_{2p}], \quad k[D_{2p}] \quad \text{or} \quad k[D_{2p}]^*$$

where D_{2p} is the dihedral group of order $2p$. Hence, our main result, Theorem 3.3, completes the classification of Hopf algebras of dimension $2p$.

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1. Notation and preliminaries

Throughout this paper, p is an odd prime, k denotes an algebraically closed field of characteristic 0, and H denotes a finite-dimensional Hopf algebra over k with antipode S. Its comultiplication and counit are respectively denoted by Δ and ϵ. We will use Sweedler's notation [Swe69]:

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \,.$$

A non-zero element $a \in H$ is called group-like if $\Delta(a) = a \otimes a$. The set of all group-like elements $G(H)$ of H is a linearly independent set, and it forms a group under the multiplication of H. For the details of elementary aspects for finite-dimensional Hopf algebras, readers are referred to the references [Swe69] and [Mon93].

Let $\lambda \in H^*$ be a non-zero right integral of H^* and $\Lambda \in H$ a non-zero left integral of H. There exists $\alpha \in \mathrm{Alg}(H,k) = G(H^*)$, independent of the choice of Λ, such that $\Lambda a = \alpha(a)\Lambda$ for $a \in H$. Likewise, there is a group-like element $g \in H$, independent of the choice of λ, such that $\beta\lambda = \beta(g)\lambda$ for $\beta \in H^*$. We call g the distinguished group-like element of H and α the distinguished group-like element of H^*. Then we have Radford's formula [Rad76] for S^4 :

$$S^4(a) = g(\alpha \rightharpoonup a \leftharpoonup \alpha^{-1})g^{-1} \quad \text{for } a \in H\,, \tag{1.1}$$

where $\rightharpoonup$ and $\leftharpoonup$ denote the natural actions of the Hopf algebra H^* on H described by

$$\beta \rightharpoonup a = \sum a_{(1)}\beta(a_{(2)}) \quad \text{and} \quad a \leftharpoonup \beta = \sum \beta(a_{(1)})a_{(2)}$$

for $\beta \in H^*$ and $a \in H$. In particular, we have the following proposition.

Proposition 1.1 ([Rad76])**.** *Let H be a finite-dimensional Hopf algebra with antipode S over the field k. Suppose that g and α are distinguished group-like elements of H and H^* respectively. Then the order of S^4 divides the least common multiple of the order of g and the order of α.* □

For any $a \in H$, the linear operator $r(a) \in \mathrm{End}_k(H)$ is defined by $r(a)(b) = ba$ for $b \in H$. The semisimplicity of a finite-dimensional Hopf algebra can be characterized by the antipode.

Theorem 1.2 ([LR87], [LR88], [Rad94])**.** *Let H be a finite-dimensional Hopf algebra with antipode S over the field k. Then the following statements are equivalent:*

(1) *H is not semisimple;*
(2) *H^* is not semisimple;*
(3) *$S^2 \neq id_H$;*
(4) $\mathrm{Tr}(S^2) = 0$*;*
(5) $\mathrm{Tr}(S^2 \circ r(a)) = 0$ *for all $a \in H$.* □

Proposition 1.3 ([Ng, Corollary 2.2])**.** *Let H be a finite-dimensional Hopf algebra over k with antipode S and g the distinguished group-like element of H. If $\mathrm{lcm}(o(S^4), o(g)) = n$ is an odd integer greater than 1, then the subspace*

$$H_- = \{u \in H \mid S^{2n}(u) = -u\}$$

has even dimension. □

The following lemma is useful in our remaining discussion.

Lemma 1.4. *Let V be a finite-dimensional vector space over the field k. If T is a linear automorphism on V such that $\mathrm{Tr}(T) = 0$ and $o(T) = q^n$ for some prime q and positive integer n, then*

$$q \mid \dim(V)\,.$$

Proof. Let $\omega \in k$ be a primitive q^nth root of unity and

$$V_i = \{u \in V \mid T(u) = \omega^i u\} \quad \text{for} \quad i = 0, \cdots, q^n - 1\,.$$

Consider the integral polynomial

$$f(x) = \sum_{i=0}^{q^n-1} \dim(V_i) x^i\,.$$

Since

$$0 = \mathrm{Tr}(T) = f(\omega)\,,$$

there exists $g(x) \in \mathbb{Z}[x]$ such that

$$f(x) = \Phi_{q^n}(x) g(x)$$

where $\Phi_{q^n}(x)$ is the q^nth cyclotomic polynomial. Hence,

$$\dim(V) = f(1) = \Phi_{q^n}(1) g(1)\,.$$

Since $\Phi_{q^n}(x) = \Phi_q(x^{q^{n-1}})$, $\Phi_{q^n}(1) = \Phi_q(1) = q$. Thus we have

$$q \mid \dim(V)\,.$$

□

2. Unimodularity of Hopf algebras of dimension $2n$

In this section, we prove that if H is a non-semisimple Hopf algebra over k of dimension $2n$, where n is an odd integer, then H or H^* is not unimodular. This result is essential to the proof of our main result in the next section.

Proposition 2.1. *Let H be a finite-dimensional Hopf algebra over the field k with antipode S. If H is unimodular and $o(S^2) = 2$, then*

$$4 \mid \dim(H)\,.$$

Proof. Let λ be a non-zero right integral of H^*. Since H is unimodular, by [Rad94, Theorem 3],

$$\lambda(ab) = \lambda(S^2(b)a)$$

for all $a, b \in H$. Let

$$H_i = \{u \in H \mid S^2(u) = (-1)^i u\} \quad \text{for} \quad i = 0, 1\,.$$

We claim that $(a, b) = \lambda(ab)$ defines a non-degenerate alternating form on H_1. For any $a, b \in H_1$,

$$\lambda(ab) = \lambda(S^2(b)a) = -\lambda(ba)\,.$$

Since $\lambda(u) = \lambda(S^2(u)) = -\lambda(u)$ for all $u \in H_1$, $\lambda(H_1) = \{0\}$. Let $a \in H_1$ such that $\lambda(ab) = 0$ for all $b \in H_1$. Then for all $b \in H_0$, $ab \in H_1$ and so $\lambda(ab) = 0$. By the non-degeneracy of λ on H, $a = 0$. Therefore, $(a, b) = \lambda(ab)$ defines a non-degenerate alternating bilinear form on H_1 and hence $\dim(H_1)$ is even. Since $o(S^2) = 2$, by Theorem 1.2, $\mathrm{Tr}(S^2) = 0$ and so $\dim(H_0) = \dim(H_1)$. Therefore

$$\dim(H) = \dim(H_0) + \dim(H_1) = 2\dim(H_1)$$

is a multiple of 4. □

Corollary 2.2. *Let H be a Hopf algebra over k of dimension $2n$ where n is an odd integer. If H is not semisimple, then H or H^* is not unimodular.*

Proof. If both H and H^* are unimodular, by Proposition 1.1, $S^4 = id_H$. Since H is not semisimple, by Theorem 1.2, $o(S^2) = 2$. It follows from Proposition 2.1 that $\dim(H)$ is then a multiple of 4, which contradicts $\dim(H) = 2n$. □

3. Hopf algebras of dimension $2p$

In this section, we prove, by contradiction, that non-semisimple Hopf algebras over k of dimension $2p$, p an odd prime, do not exist. By [Mas95], semisimple Hopf algebras of dimension $2p$ are

$$k[\mathbb{Z}_{2p}], \quad k[D_{2p}] \quad \text{and} \quad k[D_{2p}]^*$$

where D_{2p} is the dihedral group of order $2p$. Our main result completes the classification of Hopf algebras of dimension $2p$. We begin to prove our main result with the following lemma.

Lemma 3.1. *Let H be a non-semisimple finite-dimensional Hopf algebra over k of dimension $2p$ where p is an odd prime. Suppose that g and α are the distinguished group-like elements of H and H^* respectively. Then*

$$\operatorname{lcm}(o(g), o(\alpha)) = 2 \text{ or } p\,.$$

Proof. Since H is not semisimple, by Theorem 1.2, H^* is also not semisimple. Therefore $|G(H)|$ and $|G(H^*)|$ are strictly less than $2p$. By the Nichols-Zoeller theorem [NZ89],

$$|G(H)|,\ |G(H^*)| \in \{1, 2, p\}\,.$$

It follows from [Ng02b, Lemma 5.1] that

$$\operatorname{lcm}(|G(H)|, |G(H^*)|) = 1, 2 \text{ or } p\,.$$

Since $\operatorname{lcm}(\, o(g), o(\alpha)\,)$ divides $\operatorname{lcm}(\,|G(H)|, |G(H^*)|\,)$, we obtain

$$\operatorname{lcm}(o(g), o(\alpha)) = 1, 2 \text{ or } p\,.$$

By Corollary 2.2, $\operatorname{lcm}(o(g), o(\alpha)) > 1$, and so the result follows. □

Lemma 3.2. *Let H be a finite-dimensional Hopf algebra over k and $a \in G(H)$ of order d. Let $\omega \in k$ be a primitive dth root of unity and*

$$e_i = \frac{1}{d}\sum_{j=0}^{d-1} \omega^{-ij} a^j \quad (i = 0, \dots, d-1).$$

Then $\dim(H)/d$ is an integer, $\dim(He_i) = \dim(H)/d$ and $S^2(He_i) = He_i$ for $i = 0, \dots, d-1$. In addition, if H is not semisimple, then

$$\operatorname{Tr}(S^2|_{He_i}) = 0$$

for $i = 0, \dots, d-1$.

Proof. Let $B = k[a]$. Then B is a Hopf subalgebra of H and $\dim(B) = d$. By the Nichols-Zoeller theorem, H is a free B-module. In particular, $\dim(H)$ is a multiple of d and

$$H \cong B^{\dim(H)/d}$$

as right B-modules. Note that $e_0, \dots, e_{d-1}$ are orthogonal idempotents of B such that

$$1 = e_0 + \cdots + e_{d-1}\,,$$

and $Be_i = ke_i$. Therefore,

$$He_i \cong B^{\dim(H)/d} e_i = (Be_i)^{\dim(H)/d} = (ke_i)^{\dim(H)/d},$$

and so $\dim(He_i) = \dim(H)/d$ for $i = 0, \dots, d-1$. Since $S^2(a) = a$, $S^2(e_i) = e_i$ for $i = 0, \dots, d-1$. Therefore,

$$S^2(He_i) = HS^2(e_i) = He_i\,.$$

If, in addition, H is not semisimple, by Theorem 1.2,

$$\operatorname{Tr}(S^2|_{He_i}) = \operatorname{Tr}(S^2 \circ r(e_i)) = 0 \quad \text{for} \quad i = 0, \dots, d-1. \qquad \square$$

Theorem 3.3. *If p is an odd prime, then any Hopf algebra of dimension $2p$ over the field k is semisimple.*

Proof. Suppose there exists a non-semisimple Hopf algebra H of dimension $2p$. Let g and α be the distinguished group-like elements of H and H^* respectively. By Corollary 2.2, g and α cannot both be trivial. By Theorem 1.2, we may simply assume that g is not trivial and $o(g) = d$. Let $\omega \in k$ be a primitive dth root of unity and

$$e_i = \frac{1}{d} \sum_{j=0}^{d-1} \omega^{-ij} g^j \quad (i = 0, \dots, d-1).$$

By Lemma 3.1,

$$\operatorname{lcm}(o(g), o(\alpha)) = 2 \text{ or } p\,.$$

If $\operatorname{lcm}(o(g), o(\alpha)) = 2$, then $d = 2$ and $S^8 = id_H$ by Proposition 1.1. It follows from Theorem 1.2 that

$$o(S^2) = 2 \quad \text{or} \quad 4\,.$$

It follows from Lemma 3.2 that

$$\dim(He_i) = p \quad \text{and} \quad \operatorname{Tr}(S^2|_{He_i}) = 0 \quad (i = 0, 1)\,.$$

Since $o(S^2) = 2$ or 4, there exists $j \in \{0, 1\}$ such that

$$o(S^2|_{He_j}) = 2 \quad \text{or} \quad 4\,.$$

By Lemma 1.4, $\dim(He_j)$ is even, which contradicts that $\dim(He_j) = p$.

If $\operatorname{lcm}(o(g), o(\alpha)) = p$, then $d = p$ and $S^{4p} = id_H$. By Proposition 1.3, the subspace

$$H_- = \{u \in H \mid S^{2p}(u) = -u\}$$

has even dimension. On the other hand, by Lemma 3.2, we have

$$\dim(He_i) = 2 \quad \text{and} \quad \operatorname{Tr}(S^2|_{He_i}) = 0 \quad \text{for} \quad i = 0, \dots, p-1\,.$$

Thus, for $i \in \{0, \dots, p-1\}$, there is a basis $\{u_i^+, u_i^-\}$ for He_i such that

$$S^2(u_i^+) = \zeta_i u_i^+ \quad \text{and} \quad S^2(u_i^-) = -\zeta_i u_i^-$$

for some pth root of unity ζ_i. Thus, $\{u_0^-, u_1^-, \dots, u_{p-1}^-\}$ forms a basis of H_- and so

$$\dim(H_-) = p\,,$$

a contradiction! $\square$

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