HOPF ALGEBRAS OF DIMENSION $2p$

SIU-HUNG NG

(Communicated by Martin Lorenz)

Abstract. Let $H$ be a finite-dimensional Hopf algebra over an algebraically closed field of characteristic 0. If $H$ is not semisimple and $\dim(H) = 2n$ for some odd integer $n$, then $H$ or $H^*$ is not unimodular. Using this result, we prove that if $\dim(H) = 2p$ for some odd prime $p$, then $H$ is semisimple. This completes the classification of Hopf algebras of dimension $2p$.

0. Introduction

In recent years, there has been some progress on the classification problems of finite-dimensional Hopf algebras over an algebraically closed field $k$ of characteristic 0 (cf. [Mon98], [And02]). It is shown in [Zhu94] that Hopf algebras of dimension $p$, where $p$ is a prime, are isomorphic to the group algebra $k[Z_p]$. In [Ng02b] and [Ng02a], the author completed the classification of Hopf algebras of dimension $p^2$, which started in [AS98] and [Mas96]. They are group algebras and Taft algebras of dimension $p^2$ (cf. [Taf71]). However, the classification of Hopf algebras of dimension $pq$, where $p, q$ are distinct prime numbers, remains open in general.

It is shown in [EG98], [GW00] that semisimple Hopf algebras over $k$ of dimension $pq$ are trivial (i.e. isomorphic to either group algebras or the dual of group algebras). Most recently, Etingof and Gelaki proved that if $p, q$ are odd primes such that $p < q \leq 2p + 1$, then any Hopf algebra over $k$ of dimension $pq$ is semisimple [EG]. Meanwhile, the author proved the same result, using a different method, for the case that $p, q$ are twin primes [Ng]. In addition to that, Williams settled the case of dimensions 6 and 10 in [Wil88], and Beattie and Dăscălescu did dimensions 14, 65 in [BD]. Hopf algebras of dimensions 6, 10, 14 and 65 are semisimple, and so they are trivial.

In this paper, we prove that any Hopf algebra of dimension $2p$, where $p$ is an odd prime, over an algebraically closed field $k$ of characteristic 0, is semisimple. By [Mas95], semisimple Hopf algebras of dimension $2p$ are isomorphic to

$$k[Z_{2p}], \quad k[D_{2p}] \quad \text{or} \quad k[D_{2p}]^*$$

where $D_{2p}$ is the dihedral group of order $2p$. Hence, our main result, Theorem 3.3, completes the classification of Hopf algebras of dimension $2p$. 

©2005 American Mathematical Society
Reverts to public domain 28 years from publication

2237
1. Notation and preliminaries

Throughout this paper, \( p \) is an odd prime, \( k \) denotes an algebraically closed field of characteristic 0, and \( H \) denotes a finite-dimensional Hopf algebra over \( k \) with antipode \( S \). Its comultiplication and counit are respectively denoted by \( \Delta \) and \( \epsilon \). We will use Sweedler’s notation \[\text{Swe69}\]:

\[
\Delta(x) = \sum x_{(1)} \otimes x_{(2)}.\]

A non-zero element \( a \in H \) is called group-like if \( \Delta(a) = a \otimes a \). The set of all group-like elements \( G(H) \) of \( H \) is a linearly independent set, and it forms a group under the multiplication of \( H \). For the details of elementary aspects for finite-dimensional Hopf algebras, readers are referred to the references \[\text{Swe69} \] and \[\text{Mon93}\].

Let \( \lambda \in H^* \) be a non-zero right integral of \( H^* \) and \( \Lambda \in H \) a non-zero left integral of \( H \). There exists \( \alpha \in \text{Alg}(H, k) = G(H^*) \), independent of the choice of \( \Lambda \), such that \( \Lambda \alpha = \alpha \Lambda \) for \( a \in H \). Likewise, there is a group-like element \( g \in H \), independent of the choice of \( \lambda \), such that \( g \beta = \beta g \) for \( \beta \in H^* \). We call \( g \) the distinguished group-like element of \( H \) and \( \alpha \) the distinguished group-like element of \( H^* \). Then we have Radford’s formula \[\text{Rad76}\] for \( S^4 \):

\[
S^4(a) = g(\alpha \to a \leftarrow \alpha^{-1})g^{-1} \quad \text{for} \quad a \in H,
\]

where \( \to \) and \( \leftarrow \) denote the natural actions of the Hopf algebra \( H^* \) on \( H \) described by

\[
\beta \rightarrow a = \sum a_{(1)} \beta(a_{(2)}) \quad \text{and} \quad a \leftarrow \beta = \sum \beta(a_{(1)})a_{(2)}
\]

for \( \beta \in H^* \) and \( a \in H \). In particular, we have the following proposition.

**Proposition 1.1** (\[\text{Rad76}\]). Let \( H \) be a finite-dimensional Hopf algebra with antipode \( S \) over the field \( k \). Suppose that \( g \) and \( \alpha \) are distinguished group-like elements of \( H \) and \( H^* \) respectively. Then the order of \( S^4 \) divides the least common multiple of the order of \( g \) and the order of \( \alpha \). \[\square\]

For any \( a \in H \), the linear operator \( r(a) \in \text{End}_k(H) \) is defined by \( r(a)(b) = ba \) for \( b \in H \). The semisimplicity of a finite-dimensional Hopf algebra can be characterized by the antipode.

**Theorem 1.2** (\[\text{LR87}, \text{LR88}, \text{Rad94}\]). Let \( H \) be a finite-dimensional Hopf algebra with antipode \( S \) over the field \( k \). Then the following statements are equivalent:

1. \( H \) is not semisimple;
2. \( H^* \) is not semisimple;
3. \( S^2 \neq id_H \);
4. \( \text{Tr}(S^2) = 0 \);
5. \( \text{Tr}(S^2 \circ r(a)) = 0 \) for all \( a \in H \). \[\square\]

**Proposition 1.3** (\[\text{Ng}, \text{Corollary 2.2}\]). Let \( H \) be a finite-dimensional Hopf algebra over \( k \) with antipode \( S \) and \( g \) the distinguished group-like element of \( H \). If \( \text{lcm}(o(S^4), o(g)) = n \) is an odd integer greater than 1, then the subspace

\[
H_- = \{ u \in H \mid S^{2n}(u) = -u \}
\]

has even dimension. \[\square\]
The following lemma is useful in our remaining discussion.

**Lemma 1.4.** Let $V$ be a finite-dimensional vector space over the field $k$. If $T$ is a linear automorphism on $V$ such that $\text{Tr}(T) = 0$ and $o(T) = q^n$ for some prime $q$ and positive integer $n$, then $q \mid \dim(V)$.

**Proof.** Let $\omega \in k$ be a primitive $q^n$th root of unity and $V_i = \{u \in V \mid T(u) = \omega^i u\}$ for $i = 0, \cdots, q^n - 1$.

Consider the integral polynomial

$$f(x) = \sum_{i=0}^{q^n - 1} \dim(V_i)x^i.$$ 

Since $0 = \text{Tr}(T) = f(\omega)$, there exists $g(x) \in \mathbb{Z}[x]$ such that $f(x) = \Phi_{q^n}(x)g(x)$ where $\Phi_{q^n}(x)$ is the $q^n$th cyclotomic polynomial. Hence,

$$\dim(V) = f(1) = \Phi_{q^n}(1)g(1).$$

Since $\Phi_{q^n}(x) = \Phi_q(x^{q^{n-1}})$, $\Phi_{q^n}(1) = q$. Thus we have $q \mid \dim(V)$. \hfill $\square$

## 2. Unimodularity of Hopf Algebras of Dimension $2n$

In this section, we prove that if $H$ is a non-semisimple Hopf algebra over $k$ of dimension $2n$, where $n$ is an odd integer, then $H$ or $H^*$ is not unimodular. This result is essential to the proof of our main result in the next section.

**Proposition 2.1.** Let $H$ be a finite-dimensional Hopf algebra over the field $k$ with antipode $S$. If $H$ is unimodular and $o(S^2) = 2$, then $4 \mid \dim(H)$.

**Proof.** Let $\lambda$ be a non-zero right integral of $H^*$. Since $H$ is unimodular, by [Rad94, Theorem 3],

$$\lambda(ab) = \lambda(S^2(b)a)$$

for all $a, b \in H$. Let $H_i = \{u \in H \mid S^2(u) = (-1)^iu\}$ for $i = 0, 1$.

We claim that $(a, b) = \lambda(ab)$ defines a non-degenerate alternating form on $H_1$. For any $a, b \in H_1$,

$$\lambda(ab) = \lambda(S^2(b)a) = -\lambda(ba).$$

Since $\lambda(u) = \lambda(S^2(u)) = -\lambda(u)$ for all $u \in H_1$, $\lambda(H_1) = \{0\}$. Let $a \in H_1$ such that $\lambda(ab) = 0$ for all $b \in H_1$. Then for all $b \in H_0$, $ab \in H_1$ and so $\lambda(ab) = 0$. By the non-degeneracy of $\lambda$ on $H$, $a = 0$. Therefore, $(a, b) = \lambda(ab)$ defines a non-degenerate alternating bilinear form on $H_1$ and hence $\dim(H_1)$ is even. Since $o(S^2) = 2$, by Theorem 1.2, $\text{Tr}(S^2) = 0$ and so $\dim(H_0) = \dim(H_1)$. Therefore

$$\dim(H) = \dim(H_0) + \dim(H_1) = 2\dim(H_1)$$

is a multiple of 4. \hfill $\square$
Corollary 2.2. Let $H$ be a Hopf algebra over $k$ of dimension $2n$ where $n$ is an odd integer. If $H$ is not semisimple, then $H$ or $H^*$ is not unimodular.

Proof. If both $H$ and $H^*$ are unimodular, by Proposition 1.1 $S^4 = id_H$. Since $H$ is not semisimple, by Theorem 1.2 $o(S^2) = 2$. It follows from Proposition 2.1 that $\dim(H)$ is then a multiple of 4, which contradicts $\dim(H) = 2n$. □

3. Hopf algebras of dimension $2p$

In this section, we prove, by contradiction, that non-semisimple Hopf algebras over $k$ of dimension $2p$, $p$ an odd prime, do not exist. By [Mas95], semisimple Hopf algebras of dimension $2p$ are

$$k[\mathbb{Z}_{2p}], \ k[D_{2p}] \quad \text{and} \quad k[D_{2p}]^*$$

where $D_{2p}$ is the dihedral group of order $2p$. Our main result completes the classification of Hopf algebras of dimension $2p$. We begin to prove our main result with the following lemma.

Lemma 3.1. Let $H$ be a non-semisimple finite-dimensional Hopf algebra over $k$ of dimension $2p$ where $p$ is an odd prime. Suppose that $g$ and $\alpha$ are the distinguished group-like elements of $H$ and $H^*$ respectively. Then

$$\lcm(o(g), o(\alpha)) = 2 \text{ or } p.$$ 

Proof. Since $H$ is not semisimple, by Theorem 1.2 $H^*$ is also not semisimple. Therefore $|G(H)|$ and $|G(H^*)|$ are strictly less than $2p$. By the Nichols-Zoeller theorem [NZ89],

$$|G(H)|, \ |G(H^*)| \in \{1, 2, p\}.$$ 

It follows from [Ng02b, Lemma 5.1] that

$$\lcm(|G(H)|, |G(H^*)|) = 1, 2 \text{ or } p.$$ 

Since $\lcm(o(g), o(\alpha))$ divides $\lcm(|G(H)|, |G(H^*)|)$, we obtain

$$\lcm(o(g), o(\alpha)) = 1, 2 \text{ or } p.$$ 

By Corollary 2.2 $\lcm(o(g), o(\alpha)) > 1$, and so the result follows. □

Lemma 3.2. Let $H$ be a finite-dimensional Hopf algebra over $k$ and $a \in G(H)$ of order $d$. Let $\omega \in k$ be a primitive $d$th root of unity and

$$e_i = \frac{1}{d} \sum_{j=0}^{d-1} \omega^{-ij}a^j \quad (i = 0, \ldots, d-1).$$

Then $\dim(H)/d$ is an integer, $\dim(He_i) = \dim(H)/d$ and $S^2(He_i) = He_i$ for $i = 0, \ldots, d-1$. In addition, if $H$ is not semisimple, then

$$\Tr(S^2|_{He_i}) = 0$$

for $i = 0, \ldots, d-1$.

Proof. Let $B = k[a]$. Then $B$ is a Hopf subalgebra of $H$ and $\dim(B) = d$. By the Nichols-Zoeller theorem, $H$ is a free $B$-module. In particular, $\dim(H)$ is a multiple of $d$ and

$$H \cong B^{\dim(H)/d}$$
as right $B$-modules. Note that $e_0, \ldots, e_{2^d - 1}$ are orthogonal idempotents of $B$ such that
\[ 1 = e_0 + \cdots + e_{2^d - 1}, \]
and $Be_i = ke_i$. Therefore,
\[ He_i \cong B^{\dim(H)/d} e_i = (Be_i)^{\dim(H)/d} = (ke_i)^{\dim(H)/d}, \]
and so $\dim(He_i) = \dim(H)/d$ for $i = 0, \ldots, 2^d - 1$. Since $S^2(u) = a$, $S^2(e_i) = e_i$ for $i = 0, \ldots, 2^d - 1$. Therefore,
\[ S^2(He_i) = HS^2(e_i) = He_i. \]
If, in addition, $H$ is not semisimple, by Theorem 1.2
\[ \text{Tr}(S^2|_{He_i}) = \text{Tr}(S^2 \circ r(e_i)) = 0 \quad \text{for} \quad i = 0, \ldots, 2^d - 1. \]

**Theorem 3.3.** If $p$ is an odd prime, then any Hopf algebra of dimension $2p$ over the field $k$ is semisimple.

**Proof.** Suppose there exists a non-semisimple Hopf algebra $H$ of dimension $2p$. Let $g$ and $\alpha$ be the distinguished group-like elements of $H$ and $H^*$ respectively. By Corollary 2.2, $g$ and $\alpha$ cannot both be trivial. By Theorem 1.2 we may simply assume that $g$ is not trivial and $o(g) = d$. Let $\omega \in k$ be a primitive $d$th root of unity and
\[ e_i = \frac{1}{d} \sum_{j=0}^{d-1} \omega^{-ij} g^j \quad (i = 0, \ldots, d - 1). \]

By Lemma 3.1
\[ \text{lcm}(o(g), o(\alpha)) = 2 \text{ or } p. \]

If $\text{lcm}(o(g), o(\alpha)) = 2$, then $d = 2$ and $S^8 = id_H$ by Proposition 1.1. It follows from Theorem 1.2 that
\[ o(S^2) = 2 \text{ or } 4. \]
It follows from Lemma 3.2 that
\[ \dim(He_i) = p \quad \text{and} \quad \text{Tr}(S^2|_{He_i}) = 0 \quad (i = 0, 1). \]
Since $o(S^2) = 2$ or $4$, there exists $j \in \{0, 1\}$ such that
\[ o(S^2|_{He_j}) = 2 \text{ or } 4. \]
By Lemma 1.3, $\dim(He_j)$ is even, which contradicts that $\dim(He_j) = p$.

If $\text{lcm}(o(g), o(\alpha)) = p$, then $d = p$ and $S^{2p} = id_H$. By Proposition 1.3 the subspace
\[ H_- = \{ u \in H \mid S^{2p}(u) = -u \} \]
has even dimension. On the other hand, by Lemma 3.2 we have
\[ \dim(He_i) = 2 \quad \text{and} \quad \text{Tr}(S^2|_{He_i}) = 0 \quad \text{for} \quad i = 0, \ldots, p - 1. \]
Thus, for $i \in \{0, \ldots, p - 1\}$, there is a basis $\{u_i^+, u_i^-\}$ for $He_i$ such that
\[ S^2(u_i^+) = \zeta_i u_i^+ \quad \text{and} \quad S^2(u_i^-) = -\zeta_i u_i^- \]
for some $p$th root of unity $\zeta_i$. Thus, $\{u_0^-, u_1^-, \ldots, u_{p-1}^-\}$ forms a basis of $H_-$ and so
\[ \dim(H_-) = p, \]
a contradiction! \[ \square \]
ACKNOWLEDGEMENT

The author would like to thank L. Long for her useful suggestions to this paper.

REFERENCES


DEPARTMENT OF MATHEMATICS, IOWA STATE UNIVERSITY, AMES, IOWA 50011

E-mail address: rng@math.iastate.edu

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use