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MULTIPLE SOLUTIONS FOR STRONGLY RESONANT NONLINEAR ELLIPTIC PROBLEMS WITH DISCONTINUITIES

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Abstract. We examine a nonlinear strongly resonant elliptic problem driven by the p-Laplacian and with a discontinuous nonlinearity. We assume that the discontinuity points are countable and at them the nonlinearity has an upward jump discontinuity. We show that the problem has at least two nontrivial solutions without using a multivalued interpretation of the problem as it is often the case in the literature. Our approach is variational based on the nonsmooth critical point theory for locally Lipschitz functions.

1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary ∂Z. We consider the following nonlinear Dirichlet Problem:

$$(1) \qquad \left\{ \begin{array}{l} -\mathrm{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) - \lambda_1|x(z)|^{p-2}x(z) = f\big(z, x(z)\big) \quad \text{a.e. on } Z \\ x|_{\partial Z} = 0. \end{array} \right\}$$

The special feature that this problem has, is that the nonlinearity $f\colon Z \times \mathbb{R} \to \mathbb{R}$ is discontinuous in the second variable. More precisely, we assume that $f(z,\cdot)$ is discontinuous on a sequence $\{r_n\}_{n\geq 1}$ with no finite accumulation point. Also, we assume that we have "strong resonance" at infinity, namely if $F(z,x) = \int_0^x f(z,r)dr$ is the corresponding potential function, we have $F(z,x) \to F_\pm(z)$ as $x \to \pm\infty$ and $\lim_{|x|\to\infty} f(z,x) = 0$. Strongly resonant problems, are more difficult to deal with, since they exhibit a lack of compactness. They were first studied by Thews [14], Bartolo-Benci-Fortunato [3] (who coined the term "strong resonance") and Ward [15]. The problems studied in these works are semilinear (i.e. $p = 2$), with a continuous nonlinearity independent of z. "Discontinuous" strongly resonant semilinear problems, were investigated by Arcoya-Canada [1] using the dual variational principle. Their equation is semilinear and the discontinuous right-hand side nonlinearity, which is independent of z, satisfies a certain type of perturbed monotonicity. In contrast here the equation under consideration is driven by the p-Laplacian and we do not impose any kind of monotonicity condition on $f(z,\cdot)$. We only require that at the

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discontinuity points $f(z,\cdot)$ has an upward jump discontinuity. Moreover, here we establish the existence of at least two nontrivial solutions. For additional works dealing with discontinuous elliptic problems, we refer to Stuart [12], Chang [4], Costa-Goncalves [6], Badiale [2], Kourogenis-Papageorgiou [10] and the references therein. With the exception of Stuart [12], in all the other works the authors pass to a multivalued problem (by filling in the gaps at the discontinuity points) in order to obtain existence results. In addition, none of the above works deal with strongly resonant problems.

2. Mathematical background

Our approach is variational based on the nonsmooth critical point for locally Lipschitz functions (see Chang [4] and Kourogenis-Papageorgiou [10]). If X is a Banach space, a function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ there exists a neighborhood U of x and a constant $k_U > 0$ such that $|\varphi(y) - \varphi(v)| \leq k_U \|y - v\|$ for all $y, v \in U$. For such a function we define the generalized directional derivative at $x \in X$ in the direction $h \in X$, by

$$\varphi^0(x;h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

The function $h \to \varphi^0(x;h)$ is sublinear, continuous. So we can define the set

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x;h) \text{ for all } h \in X\}.$$

Here by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). Evidently, $\partial\varphi(x)$ is nonempty, convex and w^*-compact and it is called the generalized (or Clarke) subdifferential of φ at $x \in X$. If φ is also convex, then it coincides with the subdifferential in the sense of convex analysis given by $\partial_c\varphi(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq \varphi(y) - \varphi(x) \text{ for all } y \in X\}$. If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$. If $\varphi, \psi : X \to \mathbb{R}$ are both locally Lipschitz and $\lambda \in \mathbb{R}$, then $\partial(\varphi + \psi) \subseteq \partial\varphi + \partial\psi$ and $\partial(\lambda\varphi) = \lambda\partial\varphi$. For details see Clarke [5] and Denkowski-Migórski-Papageorgiou [7].

Given a locally Lipschitz function $\varphi\colon X \to \mathbb{R}$, $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. It is easy to see that a local extremum (i.e. a local minimum or a local maximum), is a critical point. We say that φ satisfies the nonsmooth C_c-condition if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\varphi(x_n) \to c$ and $(1 + \|x_n\|)m(x_n) \to 0$ ($m(x_n) = \inf\{\|x^*\| : x^* \in \partial\varphi(x)\}$) has a strongly convergent subsequence.

It is well known that $\left(-\triangle_p, W_0^{1,p}(Z)\right)$ (i.e. the negative p-Laplacian with Dirichlet boundary conditions), has a first eigenvalue $\lambda_1 > 0$, which is also simple, isolated, and has a variational characterization

$$\lambda_1 = \inf\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} \colon x \in W_0^{1,p}(Z),\ x \neq 0\right]. \tag{2}$$

The infimum is realized at the normalized principal eigenvalue u_1, $\quad u_1 \in C_0^{1,\beta}(\bar{Z})$ with $0 < \beta < 1$ and $u_1(z) > 0$ for all $z \in Z$. Also since $\lambda_1 > 0$ is isolated and if $V = \{x \in W_0^{1,p}(Z)\colon \int_Z x(z)u_1(z)^{p-1}dz = 0\}$ (a topological complement of the eigenspace $\mathbb{R}u_1$), then

$$\widehat{\lambda}_2 = \inf\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} \colon x \in V,\ x \neq 0\right] > \lambda_1. \tag{3}$$

If $p = 2$, then $\widehat{\lambda}_2 > 0$ is the second eigenvalue of $\big(-\triangle, H_0^1(Z)\big)$. If $p \neq 2$, we do not know if this is the case. For further details see Lindqvist [11] and Denkowski-Migórski-Papageorgiou [8].

3. Multiple solutions

In what follows $F(z,x) = \int_0^x f(z,r)dr$ and $C = \{r_n\}_{n\geq 1} \subseteq \mathbb{R}$ with no finite accumulation point. Our hypotheses on the discontinuous nonlinearity f are the following:

H(f): $f\colon Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) f is sup-measurable (i.e. for all $x\colon Z \to \mathbb{R}$ measurable, the function $z \to f(z, x(z))$ is measurable);

(ii) for almost all $z \in Z$, $x \to f(z,x)$ is continuous in $\mathbb{R}\backslash C$ and has upward jump discontinuity at all $x \in C$;

(iii) for every $M > 0$, there exists $\alpha_M \in L^{s'}(Z)$ $\left(1 \leq s < p^* - 1,\ \frac{1}{s} + \frac{1}{s'} = 1,\ p^* = \begin{cases} \frac{Np}{N-p} & \text{if } N > p \\ +\infty & \text{if } N \leq p \end{cases}\right)$, such that for almost all $z \in Z$ and all $|x| \leq M$ we have $|f(z,x)| \leq \alpha_M(z)$;

(iv) there exist $F_\pm \in L^1(Z)$ such that $F(z,x) \to F_\pm(z)$ uniformly for almost all $z \in Z$ and $f(z,x) \to 0$ a.e. on Z as $|x| \to \infty$;

(v) for almost all $z \in Z$ and all $x \in \mathbb{R}$, $F(z,x) \leq (\widehat{\lambda}_2 - \lambda_1)|x|^p$;

(vi) there exist $\beta_- < 0 < \beta_+$ such that

$$0 < \int_Z F_\pm(z)dz < \int_Z F\big(z, \beta_\pm u_1(z)\big)dz.$$

We consider the energy functional $\varphi\colon W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{p}\|Dx\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p - \int_Z F\big(z, x(z)\big)dz,\ x \in W_0^{1,p}(Z).$$

We know that $x \to \int_Z F\big(z, x(z)\big)dz$ and $x \to \varphi(x)$ are locally Lipschitz on $W_0^{1,p}(Z)$ (see Denkowski-Migórski-Papageorgiou [7]).

Proposition 1. **If** *hypotheses* H(f) *hold,* **then** φ *satisfies the nonsmooth* C_c*-condition for* $c < -\int_Z F_\pm(z)dz$.

Proof. Consider a sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ such that $(1 + \|x_n\|)m(x_n) \to 0$ and $\varphi(x_n) \to c$, with $c < -\int_Z F_\pm(z)dz$. Since $\partial\varphi(x_n) \subseteq W^{-1,q}(Z)$ $\big(\frac{1}{p} + \frac{1}{q} = 1\big)$ is weakly compact and the norm functional is weakly lower semicontinuous, we can find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \geq 1$. We have

$$x_n^* = A(x_n) - \lambda_1|x_n|^{p-2}x_n - \widehat{u}_n,\ n \geq 1.$$

Here $A\colon W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N}dz,\ x, y \in W_0^{1,p}(Z)$$

$\big($hereafter by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $\big(W_0^{1,p}(Z), W^{-1,q}(Z)\big)\big)$ and $\widehat{u}_n \in L^{s'}(Z)$, with $\widehat{u}_n(z) \in \partial F\big(z, x_n(z)\big)$ a.e. on Z. (see Denkowski-Migórski-Papageorgiou [7], p. 617). It is easy to check that A is demicontinuous, monotone, hence maximal monotone (see Denkowski-Migórski-Papageorgiou [8], p. 37).

We claim that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. Suppose that it is not true. We may assume that $\|x_n\| \to \infty$. Set $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. We can say that $y_n \xrightarrow{w} y$ in $W_0^{1,p}(Z)$, $y_n \to y$ in $L^p(Z)$ and in $L^{s'}(Z)$, $y_n(z) \to y(z)$ a.e. on Z, $|y_n(z)| \leq k(z)$ a.e. on Z with $k \in L^m(Z)$ and $m = \max\{p, s\}$. From the choice of $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$, we have

$$\begin{aligned} \frac{|\varphi(x_n)|}{\|x_n\|^p} &= \left|\frac{1}{p}\|Dy_n\|_p^p - \frac{\lambda_1}{p}\|y_n\|_p^p - \int_Z \frac{F\big(z, x_n(z)\big)}{\|x_n\|^p}dz\right| \leq \frac{M_1}{\|x_n\|^p}, \\ &\qquad \text{for some } M_1 > 0, \text{ all } n \geq 1. \end{aligned} \tag{4}$$

Using hypotheses H(f)(iv), (iii) and the mean value theorem for locally Lipschitz functions (see Denkowski-Migórski-Papageorgiou [7], p. 609), we can see that for almost all $z \in Z$ and all $x \in \mathbb{R}$, $|F(z,x)| \leq \alpha(z)$ with $\alpha \in L^1(Z)$. So $\big|\int_Z \frac{F(z,x_n(z))}{\|x_n\|^p}dz\big| \leq \frac{\|\alpha\|_1}{\|x_n\|^p} \longrightarrow 0$ as $n \to \infty$. Therefore, if we pass to the limit as $n \to \infty$ in (4) we obtain

$$\begin{aligned} & \|Dy\|_p^p \leq \lambda_1 \|y\|_p^p \\ \Rightarrow \quad & \|Dy\|_p^p = \lambda_1 \|y\|_p^p \qquad \text{(see (2)), hence } y = 0 \quad \text{or} \quad y = \pm u_1. \end{aligned}$$

If $y = 0$, then $\|Dy_n\|_p \to 0$ and so $y_n \to 0$ in $W_0^{1,p}(Z)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \geq 1$. So $y = \pm u_1$. Assume $y = u_1$ (the reasoning is similar if $y = -u_1$). Since $\varphi(x_n) \to c$, given $\varepsilon > 0$ we can find $n_0 = n_0(\varepsilon) \geq 1$ such that for all $n \geq n_0$ we have

$$\begin{aligned} & c - \varepsilon \leq \varphi(x_n) = \frac{1}{p}\|Dx_n\|_p^p - \frac{\lambda_1}{p}\|x_n\|_p^p - \int_Z F\big(z, x_n(z)\big)dz \leq c + \varepsilon, \\ \Rightarrow \quad & -\int_Z F\big(z, x_n(z)\big)dz \leq c + \varepsilon \qquad \text{(see (2))}. \end{aligned} \tag{5}$$

Because we have assumed that $y = u_1$ and $u_1(z) > 0$ for all $z \in Z$, it follows that $x_n(z) \to +\infty$ a.e. on Z and so $F\big(z, x_n(z)\big) \to F_+(z)$ a.e. on Z. Then by virtue of the dominated convergence theorem, we have that $\int_Z F\big(z, x_n(z)\big)dz \longrightarrow \int_Z F_+(z)dz$, and so from (5) it follows that

$$-\int_Z F_+(z)dz \leq c + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we let $\varepsilon \downarrow 0$ and have $c \geq -\int_Z F_+(z)dz$, a contradiction to the choice of c. This means that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded and so we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and $x_n \to x$ in $L^p(Z)$ and in $L^s(Z)$ (recall $s < p^*$). From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$, we have

$$\big|\langle A(x_n), x_n - x\rangle - \lambda_1 \int_Z |x_n|^{p-2}x_n(x_n - x)dz - \int_Z \widehat{u}_n(x_n - x)dz\big| \leq \varepsilon_n,$$

with $\varepsilon_n \downarrow 0$.

Evidently, $\int_Z |x_n|^{p-2}x_n(x_n - x)dz$, $\int_Z \widehat{u}_n(x_n - x)dz \longrightarrow 0$ as $n \to \infty$. So it follows that

$$\lim_{n\to\infty} \langle A(x_n), x_n - x\rangle = 0.$$

Since A is maximal monotone, it is generalized pseudomonotone (see Denkowski-Migórski-Papageorgiou [8], p. 58), and so $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle$, hence $\|Dx_n\|_p \to \|Dx\|_p$. Since $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$ and $L^p(Z, \mathbb{R}^N)$ is uniformly convex, from the Kadec-Klee property it follows that $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$, and so $x_n \to x$ in $W_0^{1,p}(Z)$. □

Let $V = \{x \in W_0^{1,p}(Z)\colon \int_Z xu_1^{p-1}dz = 0\}$. We know that $W_0^{1,p}(Z) = \mathbb{R}u_1 \oplus V$. Let $G_+ = \{\beta u_1 + v\colon v \in V,\ \beta \geq 0\}$ and $G_- = \{\beta u_1 + v\colon v \in V,\ \beta \leq 0\}$. These are closed, half spaces and from hypothesis H(f)(vi), we have

$$\inf_{G_\pm} \varphi \leq -\int F\big(\beta_\pm u_1(z)\big)dz < 0. \tag{6}$$

Proposition 2. **If** *hypotheses* H(f) *hold,* **then** $\inf_V \varphi = 0$.

Proof. By virtue of hypothesis H(f)(v) and (3), for every $v \in V$, we have

$$\begin{aligned}\varphi(v) &= \frac{1}{p}\|Dv\|_p^p - \frac{\lambda_1}{p}\|v\|_p^p - \int_Z F\big(z, v(z)\big)dz \\ &\geq \left(\frac{\widehat{\lambda}_2 - \lambda_1}{p}\right)\|v\|_p^p - \left(\frac{\widehat{\lambda}_2 - \lambda_1}{p}\right)\|v\|_p^p = 0 \\ \Rightarrow \inf_V \varphi &= 0.\end{aligned}$$

□

Now we are ready for the multiplicity result concerning problem (1).

Theorem 3. **If** *hypotheses* H(f) *hold,* **then** *problem* (1) *has at least two nontrivial solutions.*

Proof. Let $\varphi_+\colon W_0^{1,p}(Z) \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be defined by

$$\varphi_+(x) = \begin{cases} \varphi(x) & \text{if } x \in G_+ \\ +\infty & \text{otherwise.}\end{cases}$$

Evidently, φ_+ is lower semicontinuous and bounded below (see (2) and recall that for almost all $z \in Z$ and all $x \in \mathbb{R}$, $|F(z,x)| \leq \alpha(z)$ with $\alpha \in L^1(Z)_+$; see the proof of Proposition 1). Invoking the generalized Ekeland variational principle (see Denkowski-Migórski-Papageorgiou [8], p. 97), we can find $\{y_n\}_{n\geq 1} \subseteq \mathrm{int}G_+$ such that

$$\begin{aligned}&\varphi(y_n) = \varphi_+(y_n) \downarrow \inf_{W_0^{1,p}(Z)} \varphi_+ = \inf_{G_+} \varphi \quad \text{and} \\ &\varphi(y_n) = \varphi_+(y_n) \leq \varphi_+(v) + \frac{1}{n(1 + \|y_n\|)}\|v - y_n\| \quad \text{for all } v \in W_0^{1,p}(Z).\end{aligned} \tag{7}$$

Since $y_n \in \text{int}G_+$, for every $h \in W_0^{1,p}(Z)$, we can find $t_h^* > 0$ such that for all $t \in (0, t_h^*]$, we have $y_n + th \in G_+$. In (7) let $v = y_n + th,\ t \in (0, t_h^*]$. We have

$$\begin{aligned} & -\frac{\|h\|}{n(1+\|y_n\|)} \leq \frac{\varphi(y_n+th)-\varphi(y_n)}{t} \quad (\text{since} \quad \varphi_+\big|_{G_+} = \varphi) \\ \Rightarrow \quad & -\frac{\|h\|}{n(1+\|y_n\|)} \leq \varphi^0(y_n;h). \end{aligned}$$

Invoking Lemma 3.1 of Szulkin [13], we can find $u_n^* \in W^{-1,q}(Z)$ with $\|u_n^*\| \leq 1$, such that

$$\langle u_n^*, h\rangle \leq n(1+\|y_n\|)\varphi^0(y_n;h) \quad \text{for all } h \in W_0^{1,p}(Z).$$

It follows that $v_n^* = \frac{u_n^*}{n(1+\|y_n\|)} \in \partial\varphi(x_n),\ n \geq 1$ and so $(1+\|x_n\|)m(x_n) \leq (1+\|x_n\|)\|v_n^*\| \leq \frac{1}{n} \to 0$ as $n \to \infty$. Since $\inf\limits_{W_0^{1,p}(Z)} \varphi_+ = \inf\limits_{G_+} \varphi \leq \varphi(\beta_+ u_1) = -\int_Z F\big(z, \beta_+ u_1(z)\big)dz < -\int_Z F_+(z)dz$ (see hypothesis H(f)(iv) and recall that $\|Du_1\|_p^p = \lambda_1 \|u_1\|_p^p$), we can apply Proposition 1 and assume that $y_n \to y$ in $W_0^{1,p}(Z)$. Then we have $\varphi_+(y) \leq \inf\limits_{n\to\infty} \varphi_+(y_n)$ (since φ_+ is lower semicontinuous) and $y \in G_+$. Therefore $\varphi_+(y) = \varphi(y) = \inf\limits_{G_+} \varphi$. If $y \in \partial G_+ = V$, then by Proposition 2 we have $\varphi(y) \geq 0$, a contradiction to (6). So $y \in \text{int}G_+$ and it follows that y is a local minimizer of φ, which implies that $0 \in \partial\varphi(y)$. From this inclusion it follows that

$$A(y) = \widehat{u} \tag{8}$$

with $\widehat{u} \in L^{s'}(Z)$ and

$$\widehat{u}(z) \in \begin{cases} \big\{f\big(z,(y(z)\big)\big\} & \text{a.e. on } \{y \notin C\}, \\ \big[f\big(z,y(z)^-\big), f\big(z,y(z)^+\big)\big] & \text{a.e. on } \{y \in C\} \end{cases} \tag{9}$$

(see Hu-Papageorgiou [9], p. 317). Recall that $C = \{r_n\}_{n\geq 1}$ is the set of discontinuity points at which $f(z,\cdot)$ has an upward jump discontinuity (see hypothesis H(f)(ii)). Since $y \in \text{int}G_+$ is a local minimizer of φ, for $\varepsilon > 0$ small we have

$$\begin{aligned} 0 \leq \frac{\varphi(y+\varepsilon u_1)-\varphi(y)}{\varepsilon} \quad &= \quad \frac{1}{p}\left[\frac{\|D(y+\varepsilon u_1)\|_p^p - \|Dy\|_p^p}{\varepsilon}\right] \\ & \quad -\frac{1}{\varepsilon}\int_Z \big(F(z,y+\varepsilon u_1) - F(z,y)\big)dz. \end{aligned}$$

Passing to the limit as $\varepsilon \downarrow 0$, we obtain

$$\begin{aligned} 0 \quad &\leq \quad \langle A(y), u_1\rangle - \int_Z f\big(z,y(z)^+\big)u_1(z)dz \\ &= \quad \int_Z \big(\widehat{u}(z) - f(z,y(z)^+)\big)u_1(z)dz \qquad (\text{see} \quad (8)) \\ &= \quad \int\limits_{\{y\in C\}} \big(\widehat{u}(z) - f(z,y(z)^+)\big)u_1(z)dz \qquad (\text{see} \quad (9)). \end{aligned}$$

Because $\widehat{u}(z) \leq f(z,y(z)^+)$ a.e. on Z (see (9)), it follows that

$$\widehat{u}(z) = f(z,y(z)^+) \quad \text{a.e. on } Z. \tag{10}$$

Also, since $y \in \mathrm{int}G_+$, for $\varepsilon > 0$ small $x - \varepsilon u_1 \in G_+$, and so because $\inf_{G_+} \varphi = \varphi(y)$, we have

$$\begin{aligned} 0 &\leq \frac{\varphi(y-\varepsilon u_1)-\varphi(y)}{\varepsilon} = \frac{1}{p}\left[\frac{\|D(y-\varepsilon u_1)\|_p^p - \|Dy\|_p^p}{\varepsilon}\right] \\ &\quad -\frac{1}{\varepsilon}\int_Z \big(F(z,y-\varepsilon u_1)-F(z,y)\big)dz, \\ 0 &\leq \langle A(y), -u_1\rangle + \int_Z f\big(z,y(z)^-\big)u_1(z)dz \\ &= \int\limits_{\{y\in C\}} \big(f(z,y(z)^-)-\widehat{u}(z)\big)u_1(z)dz \qquad \text{(see (8) and (9))}. \end{aligned}$$

Since $f(z,y(z)^-) \leq \widehat{u}(z)$ a.e. on Z (see (9)), it follows that

$$\widehat{u}(z) = f(z,y(z)^-) \qquad \text{a.e. on } Z. \tag{11}$$

From (10) and (11) and because $f(z,y(z)^-) < f(z,y(z)^+)$ a.e. on $\{y \in C\}$, we infer that $\big|\{y \in C\}\big|_N = 0$ (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$). Therefore, $y(z) \notin C$ a.e. on Z and this combined with the inclusion $0 \in \partial\varphi(y)$, implies that y is a nontrivial solution of problem (1).

Similarly, working on G_- and defining φ_- in an analogous way we obtain $v \in \mathrm{int}G_-$ such that $\varphi(v) = \inf_{G_-}\varphi$. Since $v \in \mathrm{int}G_-$, $\inf_{G_-}\varphi = \varphi(v)$ and $0 \in \partial\varphi(v)$, working as above we show that $v(z) \notin C$ a.e. on Z, and so it is a nontrivial solution of problem (1). □

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