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SPECTRAL RADII OF REFINEMENT AND SUBDIVISION OPERATORS

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ABSTRACT. The spectral radii of refinement and subdivision operators considered on the space L_2 can be estimated by using norms of their symbols. In several cases, including those arising in wavelet analysis, the exact value of the spectral radius is found. For example, if $\mathbb{T}$ is the unit circle and if the symbol a of a refinement operator satisfies the conditions $|a(z)|^2+|a(-z)|^2=4, z\in\mathbb{T}$, and $a(1)=2$, then the spectral radius of this operator is equal to $\sqrt{2}$.

INTRODUCTION

Let q be a positive integer not equal to one, and let $a(e^{ix})\sim\sum_{k\in\mathbb{Z}}a_ke^{ikx}, x\in\mathbb{R}$, be an essentially bounded measurable function on the unit circle $\mathbb{T}$, where a_k denotes the k-th Fourier coefficient of a. The function a generates two operators widely used in wavelet analysis. The refinement operator $D_a^{(q)}$ is an operator on $L_2(\mathbb{R})$ defined by

$$D_a^{(q)}f(x):=\sum_{k\in\mathbb{Z}}a_kf(qx-k).$$

The subdivision operator $A_a^{(q)}$ associated with the function a is an operator on $L_2(\mathbb{T})$ with the matrix representation with respect to the standard basis $\{z^k: z\in\mathbb{T}, k\in\mathbb{Z}\}$

$$A_a^{(q)}\sim\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & a_q & a_0 & a_{-q} & a_{-2q} & \cdots\\ \cdots & a_{1+q} & a_1 & a_{1-q} & a_{1-2q} & \cdots\\ \cdots & a_{2+q} & a_2 & a_{2-q} & a_{2-2q} & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

It is well known that the operators $D_a^{(q)}$ and $A_a^{(q)}$ play outstanding roles in wavelet analysis [1, 6, 7] as well as in curve and surface modelling [2]. In particular, the spectral radius $\rho(A_a^{(q)})$ of $A_a^{(q)}$ is, in a sense, responsible for the regularity of wavelets and refinable functions [6, 8, 15]. This observation led to intensive studies of the spectral radius of $A_a^{(q)}$. The most popular approach in these investigations is based on using different characteristics of finite matrices, for example, eigenvalues, norms,

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joint spectral radii [10, 11, 17]. Note that in these papers the function a is assumed to be a polynomial. In [13] for a continuous function a the spectral radius of $A_a^{(2)}$ is represented as the limit of a sequence of spectral radii of some finite Toeplitz matrices. On the other hand, in some cases the spectral radii of the operators $A_a^{(q)}$ and $D_a^{(q)}$ can be estimated without using any characteristic of an auxiliary matrix but only by careful study of the symbol a. Thus the inequality

$$\rho(A_a^{(2)}) \geq \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \ln|a(e^{ix})|dx\right) \tag{0.1}$$

for the spectral radius of the operator $A_a^{(2)}$ has been proved in [12], using ergodic theory.

Let $\mathbf{PC}(\mathbb{T})$ stand for the set of all functions which are continuous everywhere on $\mathbb{T}$, except for a finite number of points where they have finite one-sided limits and are continuous from the left. In the present paper, lower bounds for the spectral radii of the operators $A_a^{(q)}$ and $D_a^{(q)}$ for $a \in \mathbf{PC}(\mathbb{T})$ are established. In particular, we present an estimate for $\rho(D_a^{(q)})$ analogous to (0.1) and prove it without employing ergodic theorems. Moreover, it is shown that for some classes of symbols the spectral radius of $D_a^{(q)}$ satisfies the equality

$$\rho(D_a^{(q)}) = \frac{1}{\sqrt{q}} \max_{z\in\mathbb{T}} |a(z)|. \tag{0.2}$$

For example, for dyadic wavelets $q = 2$, and the symbol a of the corresponding subdivision and refinement operators often satisfies the conditions (cf. [1])

$$\begin{aligned} &|a(z)|^2 + |a(-z)|^2 = 4, \quad z \in \mathbb{T}, \\ &a(1) = 2. \end{aligned}$$

For such symbols we show that

$$\sqrt{2} \leq \rho(A_a^{(2)}) \leq 2, \qquad \rho(D_a^{(2)}) = \sqrt{2}.$$

In addition, some sufficient conditions for equality (0.2) to be true are given.

1. q-cyclic m-tuples and lower bounds for spectral radii

Let $a \in L_\infty(\mathbb{T})$. For convenience, from now on the unit circle $\mathbb{T}$ is assumed to have a 1-periodic parametrization, so the function a is also 1-periodic.

Recall that the operator $A_a^{(q)} : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ can be identified with the weighted composition operator

$$A_a^{(q)} f(z) = a(z) f(z^q), \quad z = e^{i2\pi x}, \quad x \in \mathbb{R}. \tag{1.1}$$

Adopting the notations $a(x)$ for $a(e^{i2\pi x})$ and $f(x)$ for $f(e^{i2\pi x})$, equation (1.1) can be rewritten as

$$A_a^{(q)} f(x) = a(x) f(qx), \quad x \in \mathbb{R}.$$

Hence for any $n \in \mathbb{N}$ we have

$$(A_a^{(q)})^n f(x) = \prod_{j=0}^{n-1} a(q^j x) f(q^n x).$$

Lemma 1.1. *Let $a \in \mathbf{PC}(\mathbb{T})$ and let $\{x_n\}$ be a sequence of the real numbers. Then the spectral radius $\rho(A_a^{(q)})$ of the operator $A_a^{(q)}$ satisfies the inequality*

$$\frac{1}{\sqrt{q}} \limsup_{n\to\infty} \left(\prod_{j=0}^{n-1} |a(q^{-j}x_n)| \right)^{1/n} \le \rho(A_a^{(q)}) \le \lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left(\prod_{j=0}^{n-1} |a(q^{-j}x)| \right)^{1/n} . \tag{1.2}$$

Proof. Let $\mathcal{A}_n^{(q)} = \mathcal{A}_n^{(q)}(x), x \in \mathbb{R}$, denote the product $\prod_{j=0}^{n-1} a(q^j x)$. Then taking into account the periodicity of the function f one obtains

$$\begin{aligned} ||(A_a^{(q)})^n f||_2^2 &= \int_0^1 |\mathcal{A}_n^{(q)}(x)|^2 |f(q^n x)|^2 dx \\ &= \frac{1}{q^n} \int_0^{q^n} |\mathcal{A}_n^{(q)}(q^{-n}x)|^2 |f(x)|^2 dx \\ &= \int_0^1 \left(\frac{1}{q^n} \sum_{j=0}^{q^n-1} |\mathcal{A}_n^{(q)}(q^{-n}(x+j))|^2 \right) |f(x)|^2 dx. \end{aligned}$$

Hence the norm of the operator $(A_a^{(q)})^n$ is equal to the norm of the operator of multiplication by the function $\mathcal{B}_n^{(q)}(x) = ((1/q^n) \sum_{j=0}^{q^n-1} |\mathcal{A}_n^{(q)}(q^{-n}(x+j))|^2)^{1/2}$, viz,

$$||(A_a^{(q)})^n|| = \operatorname*{ess\,sup}_{x\in[0,1)} \left(\frac{1}{q^n} \sum_{j=0}^{q^n-1} |\mathcal{A}_n^{(q)}(q^{-n}(x+j))|^2 \right)^{1/2} . \tag{1.3}$$

It is easily seen that $\mathcal{B}_n^{(q)}$ is a 1-periodic function, and since $a \in \mathbf{PC}(\mathbb{T})$ we can rewrite (1.3) as

$$||(A_a^{(q)})^n|| = \sup_{x\in\mathbb{R}} \left(\frac{1}{q^n} \sum_{j=0}^{q^n-1} |\mathcal{A}_n^{(q)}(q^{-n}(x+j))|^2 \right)^{1/2} .$$

Therefore for any $n \in \mathbb{N}$ and for any $x_n \in \mathbb{R}$

$$\left(\frac{1}{q}\right)^{n/2} |\mathcal{A}_n^{(q)}(q^{-n+1}x_n)| \le ||(A_a^{(q)})^n|| \le \sup_{x\in\mathbb{R}} \left(\prod_{j=0}^{n-1} |a(q^{-j}x)| \right),$$

which implies (1.2). □

As far as the operator $D_a^{(q)}, a \in L_\infty(\mathbb{T})$, is concerned, it was shown in [9] that

$$\rho(D_a^{(q)}) = \frac{1}{\sqrt{q}} \lim_{n\to\infty} \operatorname*{ess\,sup}_{x\in\mathbb{R}} \left(\prod_{j=0}^{n-1} |a(q^{-j}x)| \right)^{1/n} . \tag{1.4}$$

Thus any estimates for the product $\prod_{j=0}^{n-1} |a(q^{-j}x)|$ provide estimates for the spectral radii of the operators $A_a^{(q)}$ and $D_a^{(q)}$.

Definition 1.2. Let $t = 0.\overline{p_1 p_2 \dots p_m}$, $0 \le p_j < q$, $p_j \in \mathbb{N}, j = 1, 2, \dots, m$, be a repeating fraction with the base q, i.e. t can be represented in the form

$$t = \left(\frac{p_1}{q} + \frac{p_2}{q^2} + \ldots + \frac{p_m}{q^m} \right) + \left(\frac{p_1}{q^{m+1}} + \frac{p_2}{q^{m+2}} + \ldots + \frac{p_m}{q^{2m}} \right) + \ldots .$$

The set

$$\begin{aligned} t_1 &= 0.\overline{p_1 p_2 \dots p_m}, \\ t_2 &= 0.\overline{p_m p_1 \dots p_{m-1}}, \\ &\dots\dots\dots\dots\dots\dots \\ t_m &= 0.\overline{p_2 p_3 \dots p_1} \end{aligned}$$

of m points from $[0, 1)$ is called a q-cyclic m-tuple corresponding to the point t and is denoted by $[t]$.

By $\mathcal{C}_q^{(m)}$ we denote the set of all q-cyclic m-tuples. Some useful properties of q-cyclic m-tuples are listed below.

Lemma 1.3. 1. *Let $[t^{(1)}], [t^{(2)}] \in \mathcal{C}_q^{(m)}$. If there exist $t_j^{(1)} \in [t^{(1)}]$ and $t_k^{(2)} \in [t^{(2)}]$ such that $t_j^{(1)} = t_k^{(2)}$, then $[t^{(1)}] = [t^{(2)}]$.*

2. *Let $[t] \in \mathcal{C}_q^{(m)}$ and let $t_j, t_k \in [t], 1 \le j, k \le m$. If m is a prime number and if $j \ne k$, then the equality $t_j = t_k$ holds if and only if $t_j = 0.\overline{pp \dots p}$, where p is an integer such that $0 \le p \le q - 1$.*
3. *Let $m, q \in \mathbb{N}, q \ge 2$. If we identify the set $\mathcal{C}_q^{(m)}$ with the decimal representations of the real numbers comprising the subsets $[t]$ of $\mathcal{C}_q^{(m)}$, then*

$$\mathcal{C}_q^{(m)} = \left\{ 0, \frac{1}{q^m - 1}, \frac{2}{q^m - 1}, \dots, \frac{q^m - 2}{q^m - 1} \right\}.$$

4. *If $m, d, q \in \mathbb{N}$ and $q \ge 2$, then $\mathcal{C}_q^{(m)} \subset \mathcal{C}_q^{(dm)}$.*

Proof. The first and second properties are obvious. The third property follows from the formula for the sum of an infinite geometric progression. With respect to the fourth property, if $t_k = 0.\overline{p_1 p_2 \dots p_m} \in \mathcal{C}_q^{(m)}$, then it can be written in the form

$$t_k = 0.\underbrace{\overline{(p_1 p_2 \dots p_m)(p_1 p_2 \dots p_m) \dots (p_1 p_2 \dots p_m)}}_{d\text{-times}},$$

i.e. $t_k \in \mathcal{C}_q^{(dm)}$. □

Proposition 1.4. *Let $a = a(x)$ be a piece-wise continuous 1-periodic function, and let $[t] \in \mathcal{C}_q^{(m)}, [t] = \{t_1, t_2, \dots, t_m\}$. Then the inequality*

$$\left(\prod_{j=1}^{m} |a(t_j)| \right)^{1/m} \le \lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left(\prod_{j=0}^{n-1} |a(q^{-j} x)| \right)^{1/n} \tag{1.5}$$

holds.

Proof. Let $t_1 = 0.\overline{p_1 p_2 \dots p_m}$ be a repeating fraction from $\mathcal{C}_q^{(m)}$. Any natural number n may be represented in the form $n = dm + r, 0 \le r < m$, and we consider the real number $x_n = x_n(t_1)$ defined by

$$x_n = p_{m-r+1} \dots p_m \underbrace{(p_1 p_2 \dots p_m)(p_1 p_2 \dots p_m) \dots (p_1 p_2 \dots p_m)}_{d\text{-times}}.\overline{p_1 p_2 \dots p_m},$$

with the representation

$$\begin{aligned} x_n &= p_{m-r+1}q^{n-1}+\ldots+p_m q^{n-r} \\ &+ \underbrace{(p_1 q^{n-r-1}+\ldots+p_m q^{n-m-r})+\ldots+(p_1 q^{m-1}+\ldots+p_m)}_{d\text{-times}} \\ &+ \left(\frac{p_1}{q}+\frac{p_2}{q^2}+\ldots+\frac{p_m}{q^m}\right)+\left(\frac{p_1}{q^{m+1}}+\frac{p_2}{q^{m+2}}+\ldots+\frac{p_m}{q^{2m}}\right)+\ldots. \end{aligned}$$

Let $R_n(a)$ denote the expression

$$R_n(a) = \sup_{x\in\mathbb{R}} \left(\prod_{j=0}^{n-1} |a(q^{-j}x)|\right)^{1/n}. \tag{1.6}$$

Obviously,

$$R_n(a) \geq \left(\prod_{j=0}^{n-1} |a(q^{-j}x_n)|\right)^{1/n}. \tag{1.7}$$

However, using the periodicity of the function a we obtain

$$a(q^{-1}x_n) = a(p_{m-r+1}\ldots p_{m-2}p_{m-1}.\overline{p_m p_1\ldots p_{m-1}}) = a(0.\overline{p_m p_1\ldots p_{m-1}}) = a(t_2).$$

Analogously,

$$a(q^{-2}x_n) = a(t_3), \quad a(q^{-3}x_n) = a(t_4), \quad \ldots, \quad a(q^{-(n-1)}x_n) = a(t_r).$$

Therefore,

$$\left(\prod_{j=0}^{n-1} |a(q^{-j}x_n)|\right)^{1/n} = \left(\prod_{j=1}^{r} |a(t_j)| \left(\prod_{j=1}^{m} |a(t_j)|\right)^{d}\right)^{1/(md+r)}.$$

(Note: the product $\prod_{j=1}^{r}|a(t_j)|$ in the last equality is replaced by 1 in the case when $r=0$.) Taking into account inequality (1.7) and letting n tend to infinity we obtain (1.5). □

Corollary 1.5. *Let $a \in \mathbf{PC}(\mathbb{T})$. Then:*

1. *The spectral radius $\rho(D_a^{(q)})$ of the operator $D_a^{(q)}$ satisfies the inequality*

$$\rho(D_a^{(q)}) \geq \frac{1}{\sqrt{q}} \sup_{m\in\mathbb{N}} \max_{[t]\in\mathcal{C}_q^{(m)}} \left(\prod_{j=1}^{m} |a(t_j)|\right)^{1/m}.$$

2. *The spectral radius $\rho(A_a^{(q)})$ of the operator $A_a^{(q)}$ satisfies the inequality*

$$\rho(A_a^{(q)}) \geq \frac{1}{\sqrt{q}} \sup_{m\in\mathbb{N}} \max_{[t]\in\mathcal{C}_q^{(m)}} \left(\prod_{j=1}^{m} |a(t_j)|\right)^{1/m}.$$

Corollary 1.6. *Fix $m \in \mathbb{N}, m \geq 2$, and consider the sequence $\{c_k\}$ defined by*

$$c_k := \frac{1}{\sqrt{q}} \max_{[t]\in \mathcal{C}_q^{(md^k)}} \left(\prod_{j=1}^{md^k} |a(t_j)| \right)^{1/md^k}.$$

If $a \in \mathbf{PC}(\mathbb{T})$, then the limit $\lim_{k\to\infty} c_k$ exists and

$$\rho(A_a^{(q)}) \geq \lim_{k\to\infty} c_k.$$

Proof. It follows from Lemma 1.3, item 4, that the sequence $\{c_k\}$ is monotonically increasing. Moreover, it is bounded by $\max_{x\in[0,1)} |a(x)|$. To finish the proof one has to apply (1.2). □

It has been shown in [12] that if the function $\ln |a|$ is Lebesgue integrable, then $\rho(A_a^{(2)}) > 0$. The above estimates allow us to get the following more general result.

Corollary 1.7. *If $a \in \mathbf{PC}(\mathbb{T})$ and if there exists a natural number m such that $a(t_j) \neq 0, j = 1, 2, \ldots, m$, at least for one q-cyclic m-tuple $\{t_1, t_2, \ldots, t_m\}$, then $\rho(A_a^{(q)}) > 0$ and $\rho(D_a^{(q)}) > 0$.*

Let us note the difference in the behavior of the spectral radii of semi-continuous $D_a^{(q)}$ and continuous $W_c^{(q)}$ convolution dilation operators. Recall that the latter is defined by

$$W_c^{(q)} f(x) := \int_{\mathbb{R}} c(qx - y) f(y) dy.$$

It was shown in [9] that for $c \in L_1(\mathbb{R})$ the spectral radius of $W_c^{(q)}$ is

$$\rho(W_c^{(q)}) = \frac{|a(0)|}{\sqrt{q}},$$

where

$$a(x) = \int_{\mathbb{R}} e^{ixy} c(y) dy.$$

Therefore, $\rho(W_c^{(q)}) = 0$ if and only if $a(0) = 0$.

The following theorem represents an integral estimate for $\rho(D_a^{(q)})$. In contrast to [12] where an analogous result for the operator $A_a^{(2)}$ is established, the proof given here does not use ergodic theorems.

Theorem 1.8. *Let $a \in \mathbf{PC}(\mathbb{T})$ and let the function $\ln |a(x)|$ be Lebesgue integrable on $[0, 1]$. Then the spectral radius $\rho(D_a^{(q)})$ of the operator $D_a^{(q)}$ satisfies the inequality*

$$\rho(D_a^{(q)}) \geq \frac{1}{\sqrt{q}} \exp\left(\int_0^1 \ln |a(x)| dx \right).$$

Proof. We start with the case where the function $\ln |a(x)|$ is Riemann integrable on $[0, 1]$. Choose a prime number m. By Lemma 1.3 all q-cyclic m-tuples differ from each other. Moreover, in the interval $[0, 1)$ there are exactly $q^m - 1$ different points from $\mathcal{C}_q^{(m)}$. Let $\widetilde{\mathcal{C}}_q^{(m)} := \{[t] \in \mathcal{C}_q^{(m)} : [t] \neq [0.\overline{r}], r = 0, 1, \ldots, q-2\}$. Then for any $[t] \in \widetilde{\mathcal{C}}_q^{(m)}$ all points $t_1, t_2, \ldots, t_m$ differ from each other. It is also worth mentioning that if s denotes the number of elements in $\widetilde{\mathcal{C}}_q^{(m)}$, then $ms+q-1 = q^m-1$. Moreover, for any $[t^{(k)}] \in \widetilde{\mathcal{C}}_q^{(m)}, k = 1, 2, \ldots, s,$ we have

$$\left(\prod_{j=1}^{m} |a(t_j^{(k)})|\right)^{1/m} \leq \sqrt{q}\rho(D_a^{(q)}). \tag{1.8}$$

Taking the logarithm of both sides of (1.8) for $k = 1, 2, \ldots, s$ and adding all resulting inequalities yields

$$\frac{1}{m}\sum_{k=1}^{s}\sum_{j=1}^{m} \ln|a(t_j^{(k)})| \leq s\ln(\sqrt{q}\rho(D_a^{(q)})). \tag{1.9}$$

Dividing by s and using Lemma 1.3 once again we can rewrite (1.9) as

$$\frac{1}{sm}\sum_{i=0}^{q^m-2} \ln\left|a\left(\frac{i}{q^m-1}\right)\right| - \frac{1}{sm}\sum_{r=0}^{q-2}\ln|a(0.\overline{r})| \leq \ln(\sqrt{q}\rho(D_a^{(q)}))$$

or as

$$\frac{q^m-1}{sm}\left(\frac{1}{q^m-1}\sum_{i=0}^{q^m-2} \ln\left|a\left(\frac{i}{q^m-1}\right)\right|\right) - \frac{1}{sm}\sum_{r=0}^{q-2}\ln|a(0.\overline{r})| \leq \ln(\sqrt{q}\rho(D_a^{(q)})). \tag{1.10}$$

The sum in round brackets is a Riemann sum for the function $\ln|a(x)|$. Since the function $\ln|a(x)|$ is assumed to be Riemann integrable on $[0,1]$, relation (1.10) implies

$$\int_0^1 \ln|a(x)|dx \leq \ln(\sqrt{q}\rho(D_a^{(q)})).$$

This completes the proof in the case where the function $\ln|a(x)|$ is Riemann integrable.

Now suppose the function a vanishes at some points in $[0,1)$, but $\ln|a(x)|$ is Lebesgue integrable on $[0,1]$. Then $\ln|a(x)|$ is unbounded, and the previous considerations are not applicable. However, standard arguments can be used to extend the result in this case. For the convenience of the reader let us sketch the proof. Introducing the functions $a_r(x) := |a(x)| + 1/r, r = 1, 2, \ldots$, and the corresponding operators $D_{a_r}^{(q)}$ from formula (1.4) for the spectral radii of the operators $D_a^{(q)}$ and $D_{a_r}^{(q)}$ we obtain

$$\rho(D_a^{(q)}) = \rho(D_{|a|}^{(q)}) \leq \rho(D_{a_r}^{(q)}), \quad r = 1, 2, \ldots,$$

and hence,

$$\rho(D_a^{(q)}) \leq \liminf_{r\to\infty} \rho(D_{a_r}^{(q)}). \tag{1.11}$$

On the other hand, since $||D_{a_r}^{(q)} - D_{|a|}^{(q)}|| \to 0$ as $r \to \infty$, the upper semi-continuity of the spectral radius [3] implies

$$\limsup_{r\to\infty} \rho(D_{a_r}^{(q)}) \leq \rho(D_a^{(q)}). \tag{1.12}$$

Comparing inequalities (1.11) and (1.12) we obtain

$$\rho(D_a^{(q)}) = \lim_{r\to\infty} \rho(D_{a_r}^{(q)}) \geq \frac{1}{\sqrt{q}} \lim_{r\to\infty} \exp\left(\int_0^1 \ln a_r(x)dx\right).$$

The sequence $\{a_r(x)\}$ is monotonically decreasing, so the application of Fatou's theorem completes the proof. $\square$

Using the same method and known inequalities for means (cf. [14], pp. 14-15) one can establish another estimate for the spectral radius.

Theorem 1.9. *Let $a \in \mathbf{PC}(\mathbb{T})$ and for an $r > 0$ let the function $|a(x)|^{-r}$ be Lebesgue integrable on $[0,1]$. Then the spectral radius $\rho(D_a^{(q)})$ of the operator $D_a^{(q)}$ satisfies the inequality*

$$\rho(D_a^{(q)}) \geq \frac{1}{\sqrt{q}} \lim_{r \to 0+} \left(\int_0^1 |a(x)|^{-r} dx \right)^{-1/r} . \tag{1.13}$$

Remark 1.10. In most of the above statements, the requirement $a \in \mathbf{PC}(\mathbb{T})$ can be replaced by a weaker one. For example, the following result is true.

Corollary 1.11. *Let a be an essentially bounded measurable function. If there exists a q-cyclic m-tuple $[t] = \{t_1, t_2, \dots, t_m\}$ such that a is continuous at least from one side at all the points $t_j, j = 1, 2, \dots, m$, then*

$$\rho(A_a^{(q)}) \geq \frac{1}{\sqrt{q}} \left(\prod_{j=1}^m |a(t_j)| \right)^{1/m} .$$

2. Exact value for spectral radius of the operator $D_a^{(q)}$

In this section we consider some cases when the spectral radius of the refinement operator can be calculated exactly. One of the immediate consequences of Proposition 1.4 is the following result.

Theorem 2.1. *Let the function $|a| \in \mathbf{PC}(\mathbb{T})$ achieve its maximum at m points $\widetilde{t}_1, \widetilde{t}_2, \dots, \widetilde{t}_m$. If there exists a q-cyclic m-tuple $[t] = \{t_1, t_2, \dots, t_m\}$ such that $t_j = \widetilde{t}_j$ for all $j = 1, 2, \dots m$, then*

$$\rho(D_a^{(q)}) = \frac{1}{\sqrt{q}} \max_{x \in [0,1)} |a(x)|$$

and

$$\frac{1}{\sqrt{q}} \max_{x \in [0,1)} |a(x)| \leq \rho(A_a^{(q)}) \leq \max_{x \in [0,1)} |a(x)|. \tag{2.1}$$

Remark 2.2. Estimates (2.1) are sharp and cannot be improved. Thus, let c be a real number and let $a(x) = c$ for any $x \in \mathbb{R}$. Then $\rho(A_a^{(q)}) = |c| = \max |a(x)|$. Consider now the function $\widetilde{a}(x) = 1 + \exp(i2\pi x), x \in \mathbb{R}$. Then $\max |\widetilde{a}(x)| = 2 = a(0)$ and (2.1) implies $\sqrt{2} \leq \rho(A_{\widetilde{a}}^{(2)})$. But it is known [11], p. 325, that $\rho(A_{\widetilde{a}}^{(2)}) = \sqrt{2}$, which gives us the lower bound in (2.1).

Theorem 2.1 can provide us with the exact values of the spectral radii of known refinement operators.

Example 2.3. Consider the refinement operator $D_{2P}^{(2)}$, which generates the Daubechies scaling function φ_3^D. According to [1], pp. 129–130, the symbol of $D_{2P}^{(2)}$ is

$$2P(z) \quad = \quad \frac{1+\sqrt{3}}{4} + \frac{3+\sqrt{3}}{4} z + \frac{3-\sqrt{3}}{4} z^2 + \frac{1-\sqrt{3}}{4} z^3, \quad z \in \mathbb{T}.$$

Elementary calculations show that

$$|2P(x)|^2 = (1 + \cos 2\pi x)^2(2 - \cos 2\pi x), \quad z = e^{i2\pi x},\, x \in \mathbb{R}.$$

It is easily seen that $\max_{x\in\mathbb{R}} |2P(x)| = 2$ and that the polynomial $2P$ achieves its maximum at the point $x = 0$. Since $0.\overline{0}$ is a q-cyclic 1-tuple for any $q \in \mathbb{N}$, the spectral radius $\rho(D^{(2)}_{2P})$ of the corresponding refinement operator is

$$\rho(D^{(2)}_{2P}) = \sqrt{2}.$$

Example 2.4. The polynomial P above is a particular case of two-scale symbols used for the construction of orthogonal dyadic wavelets. More precisely, the corresponding polynomial $P(z) = (1/2)\sum_{k=0}^{N} p_k z^k$ has to satisfy ([1], p. 166)

$$|P(z)|^2 + |P(-z)|^2 = 1, \quad z \in \mathbb{T},$$
$$P(1) = 1,$$

which implies that the symbol $a(z) = 2P(z), z \in \mathbb{T}$, of the corresponding subdivision operator $D^{(2)}_a$ satisfies the inequality

$$\max_{x\in[0,1)} |a(e^{i2\pi x})| = 2 = |a(1)| \geq |a(z)| \quad \text{for all} \quad z \in \mathbb{T},$$

hence $\rho(D^{(2)}_a) = \sqrt{2}$ by Theorem 2.1.

Example 2.5. In [16] the author studies the spectral radius of subdivision operator $A^{(2)}_p$ with a polynomial symbol $p(z) = \sum_{k=-N}^{N} a_k z^k$. He introduces the notion of ergodic sign property and shows that if the coefficients $p_k, k \in \mathbb{Z}$, of the polynomial $|p|^2$ satisfy this property, then

$$\rho^2(A^{(2)}_p) \leq p_0 + \left|\sum_{k\neq 0} p_k\right|. \tag{2.2}$$

Note that inequality (2.2) was obtained by estimating a product which is similar to product (1.6). Therefore, under the additional assumption $\sum_{k\neq 0} p_k \geq 0$, Corollary 1.5 and Theorem 2.1 give the exact value for $\rho(D^{(2)}_p)$, viz,

$$\rho^2(D^{(2)}_p) = \frac{1}{2}\left(p_0 + \sum_{k\neq 0} p_k\right).$$

Remark 2.6. In [5] (see also [4]) the authors considered symbols of the form $a(x) = ((1 + e^{i2\pi x})/2)^N p_N(e^{i2\pi x}), x \in \mathbb{R}$, where N is a natural number and p_N is a polynomial. They used the 2-cyclic 2-tuple $[0.\overline{01}]$ to estimate the regularity of scaling functions and wavelets. Note that the polynomial p_N in [5] has to satisfy some additional requirements.

One of the consequences of Theorem 2.1 can be formulated in terms of Fourier coefficients of the symbol of the corresponding operator.

Corollary 2.7. *Let $a \in \mathbf{PC}(\mathbb{R})$. If the Fourier coefficients of the function a or the function $|a|^2$ are non-negative, then $\rho(D^{(q)}_a) = (1/\sqrt{q}) \max_{x\in[0,1)} |a(x)|$.*

For the proof one has to mention that under the above conditions the function $|a|$ achieves its maximum at the q-cyclic 1-tuple $0.\overline{0}$.

Now let us consider a general situation. Fix $\varepsilon > 0$ and introduce the set

$$E_a^\varepsilon := \left\{ x \in [0,1) : |a(x)| \geq \max_{x\in[0,1)} |a(x)| - \sqrt{q}\,\varepsilon \right\}.$$

For every $m \in \mathbb{N}$ we also consider the set

$$\mathcal{N}_m^q(E_a^\varepsilon) := \{ j \in \mathbb{Z} : j/(q^m - 1) \in E_a^\varepsilon \}.$$

Let $t = 0.\overline{p_1 p_2 \ldots p_m} \in [0,1)$ be a repeating fraction with the base q. By Lemma 1.3, the number t can be represented in the form $t = r_t/(q^m - 1)$, where $0 \leq r_t \leq q^m - 2, r_t \in \mathbb{Z}$.

For each $m \in \mathbb{N}$ and for each q-cyclic m-tuple $[t] = \{t_1, t_2, \ldots, t_m\}$ we consider a system of congruences

$$\begin{cases} pr_{t_1} \equiv k_1 & \mod \quad (q^m - 1), \\ pr_{t_2} \equiv k_2 & \mod \quad (q^m - 1), \\ \quad \vdots & \quad\quad \vdots \\ pr_{t_m} \equiv k_m & \mod \quad (q^m - 1). \end{cases} \tag{2.3}$$

Theorem 2.8. *Let $a \in \mathbf{PC}(\mathbb{T})$ and let ε, q be as above. If there exists an $m \in \mathbb{N}$ such that the problem* (2.3) *is solvable for at least one q-cyclic m-tuple $[t]$ and for at least one right-hand side $(k_1, k_2, \ldots, k_m)^T$ with $k_j \in \mathcal{N}_m^q(E_a^\varepsilon), j = 1, 2, \ldots, m$, then the spectral radii of the operators $D_a^{(q)}$ and $A_a^{(q)}$ satisfy the inequalities*

$$\rho(A_a^{(q)}) \geq \frac{1}{\sqrt{q}} \max_{x\in[0,1)} |a(x)| - \varepsilon, \quad \rho(D_a^{(q)}) \geq \frac{1}{\sqrt{q}} \max_{x\in[0,1)} |a(x)| - \varepsilon.$$

Proof. We only consider the operator $D_a^{(q)}$. Let $\{t_1^*, t_2^*, \ldots, t_m^*\} \in \mathcal{C}_q^{(m)}$ and $(k_1^*, k_2^*, \ldots, k_m^*)$, $k_j^* \in \mathcal{N}_m^q(E_a^\varepsilon), j = 1, 2, \ldots, m$, denote, respectively, the corresponding q-cyclic m-tuple and the right-hand side for which the system (2.3) is solvable. If p^* denotes the corresponding solution of (2.3), consider a function $\widetilde{a}_{p^*}$ defined by

$$\widetilde{a}_{p^*}(x) = a(p^* x), \quad x \in \mathbb{R}.$$

Since a is a 1-periodic, the function $\widetilde{a}_{p^*}$ is also 1-periodic, and it is easily seen that $R_n(a) = R_n(\widetilde{a}_{p^*})$, where $R_n(a)$ is defined in (1.6). Therefore $\rho(D_a^{(q)}) = \rho(D_{\widetilde{a}_{p^*}}^{(q)})$. On the other hand, applying Corollary 1.5 to the operator $D_{\widetilde{a}_{p^*}}^{(q)}$ one obtains

$$\rho(D_{\widetilde{a}_{p^*}}^{(q)}) \geq \frac{1}{\sqrt{q}} \left(\prod_{j=1}^m |\widetilde{a}_{p^*}(t_j^*)| \right)^{1/m} = \frac{1}{\sqrt{q}} \left(\prod_{j=1}^m \left| a\left(\frac{p^* r_{t_j^*}}{q^m - 1} \right) \right| \right)^{1/m}$$

$$= \frac{1}{\sqrt{q}} \left(\prod_{j=1}^m \left| a\left(\frac{k_j^*}{q^m - 1} \right) \right| \right)^{1/m} \geq \frac{1}{\sqrt{q}} \max_{x\in[0,1)} |a(x)| - \varepsilon.$$

□

The next result follows immediately from Theorem 2.8.

Corollary 2.9. *Let* $a \in \mathbf{PC}(\mathbb{T})$ *and let for any* $\varepsilon > 0$ *the system* (2.3) *is solvable in the sense of Theorem* 2.8. *Then*

$$\rho(D_a^{(q)}) = \frac{1}{\sqrt{q}} \max_{x\in[0,1)} |a(x)|.$$

Remark 2.10. Let $s \geq 2$ be an integer, and assume that $a : \mathbb{T}^s \to \mathbb{C}$ is an essentially bounded function. Let M be a dilation matrix, i.e. a matrix with integer entries and with all eigenvalues of modulus greater than one. The above method can be used to study the spectral radii of the corresponding multivariate refinement $D_a^{(M)}$ and subdivision $A_a^{(M)}$ operators. However, additional assumptions on the dilation matrix are necessary. In particular, if the matrix M has s linearly independent integer eigenvectors and if all eigenvalues of M are natural numbers, then one can derive estimates for the spectral radii of $D_a^{(M)}$ and $A_a^{(M)}$, which are similar to those obtained in the one-dimensional case. Note that these conditions are satisfied for a number of known multivariate subdivision and refinement operators.

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