

HYDRODYNAMIC APPROACH TO CONSTRUCTING SOLUTIONS OF THE NONLINEAR SCHRÖDINGER EQUATION IN THE CRITICAL CASE

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ABSTRACT. Proceeding from the hydrodynamic approach, we construct exact solutions to the nonlinear Schrödinger equation with special properties. The solutions describe collapse, in finite time, and scattering, over infinite time, of wave packets. They generalize known blow-up solutions based on the “ground state”.

1. PRELIMINARIES

Consider the initial value problem for the nonlinear Schrödinger equation (NLS) in \mathbb{R}^n :

$$(1) \quad i\Psi'_t + \Delta\Psi + |\Psi|^\sigma\Psi = 0, \quad \Psi(\mathbf{x}, t) : \mathbb{R}_x^n \times \mathbb{R}_t^+ \rightarrow \mathbb{C},$$

$$(2) \quad \Psi(0, \mathbf{x}) = \Psi_0(\mathbf{x}) \in H^1(\mathbb{R}^n).$$

It is well known that the Cauchy problem (1),(2) has, locally in time, a solution of class $C([0, T]; H^1(\mathbb{R}^n))$, $T \leq \infty$ [1], [2], [3]. The Cauchy problem is also locally well posed in $L^2(\mathbb{R}^n)$ [4], and this space is optimal [5].

Moreover, $T = \infty$ for $\sigma < \frac{4}{n}$, where dispersion dominates, and for $\sigma \geq \frac{4}{n}$ the solution may “blow up” in finite time under certain initial conditions; e.g. [6], [7]. More exactly, there exist initial data $\Psi_0(\mathbf{x})$ and a positive constant $T(\Psi_0(\mathbf{x})) < \infty$, such that $\lim_{t \rightarrow T(\Psi_0(\mathbf{x}))} \int_{\mathbb{R}^n} |\nabla\Psi(\mathbf{x}, t)|^2 d\mathbf{x} = \infty$.

The blow-up corresponds to self-trapping of beams in the laser propagation.

A very good review with references can be found in [8]

For solutions to (1),(2) the following quantities are conserved:

$$(I) \quad N[\Psi] = \|\Psi(\mathbf{x}, t)\|_{L^2(\mathbb{R}^n)} = \|\Psi_0(\mathbf{x})\|_{L^2(\mathbb{R}^n)} = N \quad (\text{the probability}),$$

$$(II) \quad P[\Psi] = \text{Im} \int_{\mathbb{R}^n} \Psi(\mathbf{x}, t) \nabla \bar{\Psi}(\mathbf{x}, t) d\mathbf{x} = P \quad (\text{the linear momentum}),$$

and

$$(III) \quad H[\Psi] = \|\nabla\Psi(\mathbf{x}, t)\|_{L^2(\mathbb{R}^n)}^2 - \frac{2}{\sigma+2} \|\Psi(\mathbf{x}, t)\|_{L^{\sigma+2}(\mathbb{R}^n)}^{\sigma+2} = H \quad (\text{the energy}).$$

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Let $\sigma = \frac{4}{n}$. Assume additionally a stronger decay of initial data at infinity, namely, $|\mathbf{x}|\Psi_0(\mathbf{x}) \in L_2(\mathbb{R}^n)$. Then the following important identity holds:

$$(3) \quad M''(t) = 8H,$$

where $M(t) = \int_{\mathbb{R}^n} |\Psi(\mathbf{x}, t)|^2 |\mathbf{x}|^2 d\mathbf{x}$. This identity appears in many papers, but [9] seems to be the earliest one. It implies a very simple sufficient condition for the blow up. It is easy to see that $M(t) > 0$ for the solution of class $C([0, T]; H^1(\mathbb{R}^n))$, $T \leq \infty$. At the same time (3) implies that

$$(4) \quad M(t) = 4Ht^2 + M'(0)t + M(0).$$

The blow up corresponds to vanishing of $M(t)$. It signifies the concentration of the solution support in a set of zero measure. From elementary algebra arguments we get that $M(t) \rightarrow 0$ at a finite moment of time T if $K \leq 0$, where

$$(5) \quad K = 16HM(0) - (M'(0))^2.$$

This time T is positive if $H < 0$ or $H \geq 0$, $M'(0) < 0$.

In [10] it was proved that for radial initial conditions $\Psi_0(\mathbf{x})$ with $H < 0$ the solution blows up in finite time in $H^1(\mathbb{R}^n)$ with no integrability conditions on $|\Psi|^2|\mathbf{x}|^2$ (see also [11] in the context).

2. HYDRODYNAMIC INTERPRETATION

Below we use the hydrodynamic approach due to Madelung; e.g. [12]. Namely, we represent the solution in trigonometric form, that is,

$$\Psi(t, \mathbf{x}) = A(t, \mathbf{x}) \exp(i\phi(t, \mathbf{x})),$$

where $A(t, \mathbf{x}) \geq 0$ and $\phi(t, \mathbf{x})$ are real functions, the amplitude and the phase of the wave, respectively.

Note that if $A(t, \mathbf{x})$ is compactly supported, then $\Psi(t, \mathbf{x})$ is compactly supported, too. The support does not depend on the phase function ϕ .

Substituting this representation in (1) and taking the real and imaginary parts of the resulting equation, we obtain the following system:

$$(6) \quad A'_t + 2(\nabla A, \nabla \phi) + A\Delta\phi = 0,$$

$$(7) \quad A\phi'_t + A|\nabla\phi|^2 = \Delta A + A^{\sigma+1}.$$

Further, we multiply (6) by $2A$ and apply the gradient operator to (7). Denote by ρ the probability density, $A^2 = |\Psi|^2$, and by V double the gradient of the phase function, $2\nabla\phi$. Thus, the final hydrodynamic form of (1) consists of two equations

$$(8) \quad \rho'_t + \rho \operatorname{div} V + (V, \nabla \rho) = 0,$$

$$(9) \quad \rho(V'_t + (V, \nabla)V) = 2(A\nabla(\Delta A + A^{\sigma+1}) - \nabla A(\Delta A + A^{\sigma+1})), \quad A = \rho^{1/2}.$$

The only difference from the traditional gas dynamics is the “exotic” pressure on the right-hand side of (9). This type of pressure changes the character of singularity completely, but allows us to use the same methods as in the gas dynamics.

The data

$$\rho(0, \mathbf{x}) = \rho_0(\mathbf{x}), \quad V(0, \mathbf{x}) = V_0(\mathbf{x})$$

complete the statement of the Cauchy problem for (8),(9).

It suffices to demand that

$$\rho_0^{1/2}(\mathbf{x}) \in H^1(\mathbb{R}^n), \quad \rho_0^{1/2}(\mathbf{x})V_0(\mathbf{x}) \in L_2(\mathbb{R}^n),$$

to ensure that the corresponding initial function $\Psi_0(\mathbf{x})$ belongs to the class $H^1(\mathbb{R}^n)$.

Note also that ρ vanishes as $|\mathbf{x}| \rightarrow \infty$, however, the same is not necessary for V .

The conservation laws (I)–(III) in the new terms are as follows:

$$(I') \quad \int_{\mathbb{R}^n} \rho d\mathbf{x} = N \quad (\text{the mass}),$$

$$(II') \quad \int_{\mathbb{R}^n} \rho V d\mathbf{x} = \tilde{\mathbf{P}} \quad (\text{the linear momentum})$$

and

$$(III') \quad \int_{\mathbb{R}^n} \left(\frac{\rho|V|^2}{4} + \frac{|\nabla\rho|^2}{4\rho} - \frac{2}{\sigma+2} \rho^{\sigma/2+1} \right) d\mathbf{x} = H \quad (\text{the energy}).$$

One can see that the “kinetic energy” component in H is the same as in gas dynamics (up to the multiplier).

Note that in gas dynamics terms we have

$$M(t) = \int_{\mathbb{R}^n} \rho(\mathbf{x}, t) |\mathbf{x}|^2 d\mathbf{x}.$$

By virtue of (8),

$$M'(t) = 2 \int_{\mathbb{R}^n} (\mathbf{x}, V(\mathbf{x}, t)) \rho(\mathbf{x}, t) d\mathbf{x};$$

here \mathbf{x} is the radius-vector of a point in the space.

Let us introduce one more functional:

$$Q_{\Lambda}(t) = \int_{\mathbb{R}^n} (\mathbf{x}, \Lambda) \rho(\mathbf{x}, t) d\mathbf{x},$$

where Λ is a constant vector from \mathbb{R}^n . From (8) and the conservation of linear momentum (II') we have $Q'_{\Lambda}(t) = (\tilde{\mathbf{P}}, \Lambda) := \tilde{\mathbf{P}}_{\Lambda} = \text{const}$, and

$$(10) \quad Q_{\Lambda}(t) = \tilde{\mathbf{P}}_{\Lambda} t + Q_{\Lambda}(0).$$

From the Hölder inequality we also have

$$Q_{\Lambda}^2(t) \leq |\Lambda|^2 N M(t).$$

In the domains where the amplitude $A > 0$, instead of (9) we consider the equivalent equation

$$V'_t + (V, \nabla)V = 2\nabla\left(\frac{\Delta A}{A} + A^{\sigma}\right).$$

Now we use the following idea, recently applied to construct solutions to the gas dynamics equations (see [13], [14], [15], [16]). Namely, let the “velocity field” have the form

$$(11) \quad V = a(t)\mathbf{x},$$

where $a(t)$ is a time-dependent function. Thus, the phase function can be restored as

$$\phi(t, \mathbf{x}) = a(t) \frac{|\mathbf{x}|^2}{4} + \gamma(t),$$

with an unknown function $\gamma(t)$. Then, from the (linear in ρ) equation (8) we find the density as

$$\rho(t, \mathbf{x}) = \exp\left(-n \int_0^t a(\tau) d\tau\right) \rho_0 \left(\mathbf{x} \exp\left(-\int_0^t a(\tau) d\tau\right) \right)$$

or

$$A(t, \mathbf{x}) = \exp\left(-\frac{n}{2} \int_0^t a(\tau) d\tau\right) A_0 \left(\mathbf{x} \exp\left(-\int_0^t a(\tau) d\tau\right) \right),$$

with $A_0(\mathbf{x}) = |\Psi_0(\mathbf{x})|$.

Note that in the critical case, $\sigma = \frac{4}{n}$,

$$(12) \quad \frac{\Delta A(t, \mathbf{x})}{A(t, \mathbf{x})} + A^\sigma(t, \mathbf{x}) = \exp\left(-2 \int_0^t a(\tau) d\tau\right) \left(\frac{\Delta A_0(\xi)}{A_0(\xi)} + A_0^\sigma(\xi) \right),$$

where $\xi = \mathbf{x} \exp\left(-\int_0^t a(\tau) d\tau\right)$.

Further, we have from (7) and (12)

$$(a'(t) + a^2(t)) \frac{|\mathbf{x}|^2}{4} + \gamma'(t) = \exp\left(-2 \int_0^t a(\tau) d\tau\right) \left(\frac{\Delta A_0(\xi)}{A_0(\xi)} + A_0^\sigma(\xi) \right)$$

or

$$(a'(t) + a^2(t)) \frac{|\xi|^2}{4} + \gamma'(t) \exp\left(-2 \int_0^t a(\tau) d\tau\right) = \exp\left(-4 \int_0^t a(\tau) d\tau\right) \left(\frac{\Delta A_0(\xi)}{A_0(\xi)} + A_0^\sigma(\xi) \right).$$

The variables t and ξ can be separated if

$$(13) \quad \gamma'(t) = \gamma_0 \exp\left(-2 \int_0^t a(\tau) d\tau\right), \quad \gamma_0 \in \mathbb{R}^1.$$

In this case

$$(14) \quad (a'(t) + a^2(t)) |\xi|^2 = 4 \exp\left(-4 \int_0^t a(\tau) d\tau\right) \left(\frac{\Delta A_0(\xi)}{A_0(\xi)} + A_0^\sigma(\xi) - \gamma_0 \right).$$

It follows from (14) that

$$(15) \quad a'(t) + a^2(t) = 4k \exp\left(-4 \int_0^t a(\tau) d\tau\right),$$

$$(16) \quad \Delta A_0(\mathbf{x}) + A_0^{\sigma+1}(\mathbf{x}) = (k|\mathbf{x}|^2 + \gamma_0) A_0(\mathbf{x}),$$

where k is a constant.

Thus, we seek a special solution to (1) in the form

$$(17) \quad \Psi(t, \mathbf{x}) = \exp\left(-\frac{n}{2} \int_0^t a(\tau) d\tau\right) A_0 \left[\mathbf{x} \exp\left(-\int_0^t a(\tau) d\tau\right) \right] \\ \times \exp\left(ia(t) \frac{|\mathbf{x}|^2}{4}\right) \exp\left(i\gamma_0 \int_0^t \exp\left(-2 \int_0^\tau a(\tau_1) d\tau_1\right) d\tau\right) \exp(i\theta),$$

with $\theta \in \mathbb{R}^1$.

Because the origin is not a particular point, without loss of generality we may consider the velocity field $V = a(t)(\mathbf{x} - \mathbf{x}_0)$, where \mathbf{x}_0 is an arbitrary fixed point. Then the solution takes the form

$$(17') \quad \Psi(t, \mathbf{x}) = \exp\left(-\frac{n}{2} \int_0^t a(\tau) d\tau\right) A_0 \left[(\mathbf{x} - \mathbf{x}_0) \exp\left(-\int_0^t a(\tau) d\tau\right) \right] \\ \times \exp\left(ia(t) \frac{|\mathbf{x} - \mathbf{x}_0|^2}{4}\right) \exp\left(i\gamma_0 \int_0^t \exp\left(-2 \int_0^\tau a(\tau_1) d\tau_1\right) d\tau\right) \exp(i\theta).$$

The decay properties of the solution as $|\mathbf{x}| \rightarrow \infty$ depend on $A_0(\mathbf{x})$. The physical sense requires that the solution should be of the class $L^2(\mathbb{R}^n)$. If we wish to consider solutions from the space $C([0, T]; H^1(\mathbb{R}^n))$, $T \leq \infty$, natural for the existence and uniqueness to the Cauchy problem (1), (2), we have to choose the initial data

$$\Psi_0(\mathbf{x}) = A_0(\mathbf{x} - \mathbf{x}_0) \exp\left(ia_0 \frac{|\mathbf{x} - \mathbf{x}_0|^2}{4}\right) \exp(i\theta),$$

where $A_0(\mathbf{x})$ is a nonnegative solution to (14) belonging to $H^1(\mathbb{R}^n)$ and $|\mathbf{x}|A_0(\mathbf{x}) \in L_2(\mathbb{R}^n)$. Note that with the function A_0 of this class we also have conservation laws (I)–(III) for solutions (17) and (17').

3. TIME EVOLUTION

Let us investigate the qualitative behavior of $a(t)$ governed by (15). It is easy to see that in the case $k < 0$ for any initial datum $a(0) = a_0$ there exists a moment T_* such that $a(t) \rightarrow -\infty, t \rightarrow T_*$. Really, as $a'(t) < \epsilon < 0$, then under any a_0 there exists a moment $T_1 \geq 0$ when $a(T_1) < 0$. Further, the comparison theorem shows that $a(t) \leq \tilde{a}(t)$, where $\tilde{a}(t)$ is a solution to the Cauchy problem $\tilde{a}(t)' = -\tilde{a}^2(t), \tilde{a}(T_1) = a(T_1), t \geq T_1$. This means that if $k < 0$, the solution Ψ to (1) of the form (17), with the amplitude A_0 satisfying (16), localizes at the origin within a finite interval of time, provided $N < \infty$.

We can also analyze the differential corollary of (15)

$$a''(t) + 6a(t)a'(t) + 4a^3(t) = 0,$$

with the initial data $a(0) = a_0, a'(0) = -a_0^2 + 4k$. This gives, in particular, that in the case $k > 0$ the solution vanishes at infinity as $O(t^{-1})$.

If $k = 0$, any nontrivial solution to (15) blows up at a finite time $T = -a_0^{-1}$. The time is positive if $a_0 < 0$. Moreover, if $|\mathbf{x}|\Psi_0(\mathbf{x}) \in L^2(\mathbb{R}^n)$, we can find $a(t)$ explicitly for any k . Observe that, if $V = a(t)\mathbf{x}$, then

$$(18) \quad M'(t) = 2a(t)M(t),$$

therefore $a(t) = \frac{M'(t)}{2M(t)}$. The explicit form of $M(t)$ is known; see (4). Thus,

$$(19) \quad a(t) = \frac{8Ht + M'(0)}{2(4Ht^2 + M'(0)t + M(0))}.$$

If $K = 16HM(0) - (M'(0))^2 \leq 0$ (see (5)), then $M(t)$ tends to zero ($a(t)$ goes to infinity, respectively) within a finite interval of time. Moreover, this interval can be readily calculated. If $K > 0$, then $a(t) \sim t^{-1}$, $t \rightarrow \infty$.

Now taking into account (15), (18), (19) we can compute

$$(20) \quad k = \frac{K}{16M^2(0)} = \frac{16HM(0) - (M'(0))^2}{16M^2(0)}.$$

There are situations where we can express $a(t)$ through $Q_{\Lambda}(t)$. Namely, for $V = a(t)\mathbf{x}$ we get $\tilde{\mathbf{P}}_{\Lambda} = a(t)Q_{\Lambda}(t)$. Therefore, if there is $\Lambda \in \mathbb{R}^n$ such that $Q_{\Lambda}(t) \neq 0$ ($\tilde{\mathbf{P}}_{\Lambda} \neq 0$), then taking into account (10) we have

$$a(t) = \frac{1}{t + Q_{\Lambda}(0)\tilde{\mathbf{P}}_{\Lambda}^{-1}} = \frac{1}{t + a_0^{-1}}.$$

From (18) we now obtain $M(t) = M(0)a_0^2(t + a_0^{-1})^2$. Comparing the result with (4) we can see that $4H = M(0)a_0^2$ and $K = k = 0$.

Note that if A_0 is radial, then $Q_{\Lambda}(t) = 0$ for any Λ .

4. EVOLUTION OF WAVE PACKETS

Summarizing the above results, we can formulate the following theorem:

Theorem 1. *Suppose that (16) has at least one nonnegative solution $A_0(\mathbf{x})$. Then equation (1) has a special solution given by the explicit formula (17)((17')), with the function $a(t)$ governed by equation (15).*

If $\mathbf{x}A_0(\mathbf{x}) \in L^2(\mathbb{R}^n)$, then the formula (17') can be written as

$$(21) \quad \begin{aligned} \Psi(t, \mathbf{x}) &= \left(\frac{M(0)}{M(t)}\right)^{\frac{n}{4}} A_0 \left[\left(\frac{M(0)}{M(t)}\right)^{\frac{1}{2}} (\mathbf{x} - \mathbf{x}_0) \right] \\ &\times \exp\left(i\frac{M'(t)}{8M(t)}|\mathbf{x} - \mathbf{x}_0|^2\right) \exp\left(i\gamma_0 M(0) \int_0^t M^{-1}(\tau) d\tau\right) \exp(i\theta), \end{aligned}$$

with the quadratic function $M(t)$ having the explicit form (4).

The behavior of the solution depends on the sign of the constant k (see (20)).

If $k > 0$, then the solution decays. Namely,

$$\max_{\mathbf{x} \in \mathbb{R}^n} |\Psi(t, \mathbf{x})| = A^+ \left(\frac{M(0)}{4H}\right)^{\frac{n}{4}} t^{-n/2}, \quad A^+ = \max_{\mathbb{R}^n} A_0(\mathbf{x}).$$

If $k \leq 0$, then the solution blows up at the point \mathbf{x}_0 at a finite moment of time T . This time is positive in the following cases:

- (i) *if $H = 0$, $M'(0) < 0$, then $T = \frac{-M(0)}{M'(0)}$;*
- (ii) *if $H > 0$, $M'(0) < 0$, then $T = \frac{-M'(0) - \sqrt{|K|}}{2H}$;*
- (iii) *if $H < 0$, then $T = \frac{-M'(0) + \sqrt{|K|}}{2H}$.*

Moreover, for $k = 0$,

$$\max_{\mathbf{x} \in \mathbb{R}^n} |\Psi(t, \mathbf{x})| = A^+ \left(\frac{M(0)}{4H}\right)^{\frac{n}{4}} (T - t)^{-n/2},$$

and for $k < 0$,

$$\max_{\mathbf{x} \in \mathbb{R}^n} |\Psi(t, \mathbf{x})| = A^+ \left(\frac{M(0)}{\sqrt{|K|}} \right)^{\frac{n}{4}} (T - t)^{-n/4}.$$

Remark 1. If there exists $\Lambda \in \mathbb{R}^n$ such that $Q_\Lambda(t) \neq 0$ ($\tilde{\mathbf{P}}_\Lambda \neq 0$), then only the situation with $k = 0$ may be realized.

Remark 2. For the solution of the form (19) we have

$$\begin{aligned} \|\nabla \Psi(\mathbf{x}, t)\|_{L^2(\mathbb{R}^n)}^2 &= \exp(-2 \int_0^t a(\tau) d\tau) \|\nabla A_0(\mathbf{x})\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad + \frac{1}{4} a^2(t) \exp(2 \int_0^t a(\tau) d\tau) \|\mathbf{x} A_0(\mathbf{x})\|_{L^2(\mathbb{R}^n)}^2 \\ &= \frac{M(0)}{M(t)} \|\nabla A_0(\mathbf{x})\|_{L^2(\mathbb{R}^n)}^2 + \frac{(M'(t))^2}{16M(t)} = O(T - t)^{-\lambda}, \end{aligned}$$

where $\lambda = 1$ (the lower estimate for the blow-up order; see [17]) for $k < 0$, and $\lambda = 2$ for $k = 0$.

5. COMPARISON WITH PREVIOUS RESULTS

In the theory of NLS for the critical case $\sigma = \frac{4}{n}$ the crucial role is played by the so-called ground state, i.e. the positive radially-decreasing solution to the elliptic problem

$$(22) \quad \Delta u + |u|^\sigma u - u = 0, \quad u \in H^1(\mathbb{R}^n).$$

It is known [18] that the solution with such properties is unique and exists at least for $n = 1, 2, 3$. The solution belongs to $C^2(\mathbb{R}^n)$, and $|D^\alpha u| \leq C \exp(-\delta|x|)$, where C, δ are positive constants. We denote it by $R(\mathbf{x})$.

It is known that $H[R(\mathbf{x})] = 0$ [19].

It was proved [19] that if $\|\Psi_0(\mathbf{x})\|_{L^2(\mathbb{R}^n)} < \|R(\mathbf{x})\|_{L^2(\mathbb{R}^n)}$, then the solution to the problem (1), (2) is global in time. If the solution blows up, then $\|\Psi_0(\mathbf{x})\|_{L^2(\mathbb{R}^n)} > \|R(\mathbf{x})\|_{L^2(\mathbb{R}^n)}$. In the case $\|\Psi_0(\mathbf{x})\|_{L^2(\mathbb{R}^n)} = \|R(\mathbf{x})\|_{L^2(\mathbb{R}^n)}$ the solution either blows up or not.

If it does blow up, it necessarily has the following special form based on the ground state [20, 21]:

$$(23) \quad \Psi(\mathbf{x}, t) = \exp(i\theta) \exp\left(i\left(-\frac{\omega^2}{t-T} + \frac{|\mathbf{x} - \mathbf{x}_0|^2}{4(t-T)}\right)\right) \left(\frac{\omega}{t-T}\right)^{\frac{n}{2}} R\left(\frac{\omega(\mathbf{x} - \mathbf{x}_0)}{t-T} - \mathbf{x}_1\right),$$

with certain $\mathbf{x}_0 \in \mathbb{R}^n, \mathbf{x}_1 \in \mathbb{R}^n, \theta \in \mathbb{R}, \omega \in \mathbb{R}_+, T \in \mathbb{R}$. However, (23) coincides with (21) for $K = k = 0$ and $\gamma_0 = 1$. Indeed, in this case $R(x)$ is a solution to (14), $M(t) = H(t + \frac{M'(0)}{2H})^2 = H(t - (-\frac{M(0)}{H})^{\frac{1}{2}})^2, \omega^2 = \frac{M(0)}{H}, T = \omega = -\frac{M'(0)}{2H}$. Here we choose $M'(0) < 0$ to guarantee the positivity of T . Note that $H > 0$ for solutions of the form (23).

It can be readily demonstrated that

$$\begin{aligned} H[\Psi] &= \frac{M(0)}{M(t)} (\|\nabla A_0(\mathbf{x})\|_{L^2(\mathbb{R}^n)}^2 - \frac{2}{\sigma+2} \|\nabla A_0(\mathbf{x})\|_{L^{\sigma+2}(\mathbb{R}^n)}^{\sigma+2}) + \frac{(M'(t))^2}{16M(t)} \\ &= \frac{M(0)}{M(t)} H[A_0] + \frac{(M'(t))^2}{16M(t)}. \end{aligned}$$

Thus, for $A_0 = R$, we get $H = \frac{(M'(t))^2}{16M(t)} > 0$.

It follows from the above results that if $A_0(\mathbf{x})$ is a solution to (16) with $k < 0$ from $H^1(\mathbb{R}^n)$, then

$$\|A_0(\mathbf{x})\|_{L^2(\mathbb{R}^n)} > \|R(\mathbf{x})\|_{L^2(\mathbb{R}^n)},$$

because in this case the solution (17) blows up and it is not of the form (23).

The profile of our solution in the case $k < 0$ is different from that of the solution (23) (corresponding to $k = 0$), and so is the rate of blow up (see Remark 2 of Section 4).

Note that in the case $k < 0$ the solution to equation (16), considered in the space \mathbb{R}^n , oscillates as $|x| \rightarrow \infty$. For the linearized equation ($n = 1$) we can even get the explicit solution:

$$u = |x|^{-1/2} (C_1 W_1(\frac{i\gamma_0}{2\sqrt{2k}}, \frac{1}{4}, \frac{1}{2}i\sqrt{2k}x^2) + C_2 W_2(\frac{i\gamma_0}{2\sqrt{2k}}, \frac{1}{4}, \frac{1}{2}i\sqrt{2k}x^2)),$$

where W_1, W_2 are the Whittaker functions, C_1, C_2 are constants. This highly oscillating function does not belong to $L^2(\mathbb{R}^n)$. So we cannot hope that the solution to nonlinear perturbed problem (16) is positive and belongs to $L^2(\mathbb{R}^n)(H^1(\mathbb{R}^n))$.

However, we can consider (in higher dimensions, too) the solution to (16) given in $\Omega \subset \mathbb{R}^n$. To be exact, now we deal with the Dirichlet problem for (16) with zero boundary conditions. Regarding this Dirichlet problem, for example, there is the following result due to [22].

Theorem 2 ([22]). *Let Ω be a bounded, smooth domain in $\mathbb{R}^n, n \geq 2$, and let $g : \mathbb{R}^+ \times \bar{\Omega} \rightarrow \mathbb{R}^+$ be a locally Lipschitzian map. Consider the elliptical boundary problem*

$$(*) \quad \Delta u + u^p + \epsilon g(x, u) = 0, u|_{\partial\Omega} = 0.$$

If $1 < p < \frac{n+1}{n-2}$ ($p > 1$ for $n = 1, 2$), there exists $\epsilon_0 > 0$ such that, for $0 \leq \epsilon < \epsilon_0$, () has a solution $u = u_\epsilon$, which is positive on Ω .*

In our situation $g(x, u) = \frac{-k|x|^2 - \gamma_0}{\epsilon} u$ satisfies the theorem condition for $k < 0$ in a ball from \mathbb{R}^n ; for $\gamma_0 < 0$ this condition holds for all \mathbb{R}^n .

The solution to the Dirichlet problem is classical, that is, it belongs to $C^2(\Omega) \cap C(\bar{\Omega})$, therefore it can be extended to the whole space \mathbb{R}^n , at least as a solution from $L^2(\mathbb{R}^n)$.

6. FURTHER GENERALIZATION

Let us consider the following velocity field:

$$(24) \quad \mathbf{V} = a(t)\mathbf{x} + b(t)\mathbf{\Lambda},$$

with a constant vector $\mathbf{\Lambda}$. Then the phase function takes the form

$$\phi(t, \mathbf{x}) = a(t) \frac{|\mathbf{x}|^2}{4} + \frac{1}{2} b(t)(\mathbf{\Lambda}, x) + \gamma(t).$$

Proceeding in the spirit of Section 2, we obtain

$$A(t, \mathbf{x}) = \exp\left(-\frac{n}{2} \int_0^t a(\tau) d\tau\right) A_0 \left(\mathbf{x} \exp\left(-\int_0^t a(\tau) d\tau\right) - \mathbf{\Lambda} \int_0^t b(\tau) \exp\left(-\int_0^\tau a(\tau_1) d\tau_1\right) d\tau \right).$$

Denote $\xi = \mathbf{x} \exp\left(-\int_0^t a(\tau) d\tau\right) - \mathbf{\Lambda} \int_0^t b(\tau) \exp\left(-\int_0^\tau a(\tau_1) d\tau_1\right) d\tau$. From (7) and (12) we get

$$\begin{aligned} & [a'(t) + a^2(t)] \frac{|\xi|^2}{4} \\ & + \left[(a'(t) + a^2(t)) \int_0^t b(\tau) \exp\left(-\int_0^\tau a(\tau_1) d\tau_1\right) d\tau \right. \\ & \quad \left. + (b'(t) + a(t)b(t)) \exp\left(-\int_0^t a(\tau) d\tau\right) \right] \frac{(\mathbf{\Lambda}, \mathbf{x})}{2} \\ & + \left[(a'(t) + a^2(t)) \left(\int_0^t b(\tau) \exp\left(-\int_0^\tau a(\tau_1) d\tau_1\right) d\tau \right)^2 \right. \\ & \quad \left. + 2(b'(t) + a(t)b(t)) \exp\left(-\int_0^t a(\tau) d\tau\right) \int_0^t b(\tau) \exp\left(-\int_0^\tau a(\tau_1) d\tau_1\right) d\tau \right. \\ & \quad \left. + b^2(t) \exp\left(-2 \int_0^t a(\tau) d\tau\right) \right] \frac{|\mathbf{\Lambda}|^2}{4} \\ & - \gamma'(t) \exp\left(-2 \int_0^t a(\tau) d\tau\right) = \exp\left(-4 \int_0^t a(\tau) d\tau\right) \left(\frac{\Delta A_0(\xi)}{A_0(\xi)} + A_0^\sigma(\xi) \right). \end{aligned}$$

There are two possibilities for the separation of variables.

I. The functions $a(t), \gamma(t)$ and $A_0(\mathbf{x}_1), \mathbf{x}_1 = \mathbf{x} + \mathbf{\Lambda}$ satisfy equations (15), (13) and (16), respectively, $b(t) = a(t)$. So we return to the formula (17') considered above.

II. The functions $a(t), b(t), \gamma(t)$ and $A_0(\mathbf{x})$ satisfy the following equations:

(25)
$$a'(t) + a^2(t) = 0,$$

(26)
$$b'(t) + a(t)b(t) = 2k_1 \exp\left(-3 \int_0^t a(\tau) d\tau\right),$$

$$\gamma'(t) = \gamma_0 \exp\left(-2 \int_0^t a(\tau) d\tau\right)$$

(27)
$$- \left[4k_1 \exp\left(-2 \int_0^t a(\tau) d\tau\right) + \int_0^t b(\tau) \exp\left(-\int_0^\tau a(\tau_1) d\tau_1\right) d\tau + b^2(t) \right] \frac{|\mathbf{\Lambda}|^2}{4},$$

(28)
$$\Delta A_0(\mathbf{x}) + A_0^{\sigma+1}(\mathbf{x}) = (k_1(\mathbf{\Lambda}, \mathbf{x}) + \gamma_0) A_0(\mathbf{x}),$$

where γ_0 and k_1 are constants.

The functions $a(t), b(t), \gamma(t)$ can be found explicitly.

Note that if $k_1 = 0, \gamma_0 = 1, b(t) = 0$, we again obtain a solution that can be represented by formula (23).

The simplest situation is $a(t) = 0$, $b(t) = 2k_1t + b_0$, $\gamma(t) = -\frac{|\mathbf{\Lambda}|^2}{12k_1}(2k_1t + b_0)^3 + \gamma_0t + \gamma_1$, $\gamma_0, \gamma_1 \in \mathbb{R}^n$. The corresponding solution has the form

$$\Psi(t, \mathbf{x}) = A_0[\mathbf{x} - \mathbf{\Lambda}t(k_1t + b_0)] \exp\left(i(k_1t + \frac{b_0}{2})(\mathbf{\Lambda}, \mathbf{x}) + \gamma(t)\right),$$

with the function $A_0(\mathbf{x})$ satisfying (28). For $k_1 = 0$, $\gamma_0 = 1$, $b_0 = 0$ we get the solitary wave solution $R(\mathbf{x}) \exp(i(t + \gamma_1))$.

In the general case

$$\begin{aligned} a(t) &= \frac{1}{t + a_0^{-1}}, \\ b(t) &= \frac{b_0a_0^{-1} + 2k_1a_0^2}{t + a_0^{-1}} - 2k_1\frac{1}{(t + a_0^{-1})^2}, \\ \gamma(t) &= \int_0^t \left[\frac{\gamma_0}{(\tau + a_0^{-1})^{-2}} + \frac{k_1|\mathbf{\Lambda}|^2}{\tau + a_0^{-1}} \int_0^\tau \frac{b(\tau_1)}{\tau_1 + a_0^{-1}} d\tau_1 - \frac{1}{4}|\mathbf{\Lambda}|^2b^2(\tau) \right] d\tau. \end{aligned}$$

The corresponding solution has the form

$$(29) \quad \begin{aligned} \Psi(\mathbf{x}, t) &= \exp(i\gamma(t)) \exp\left(i(b(t)\frac{(\mathbf{x}, \mathbf{\Lambda})}{2} + \frac{|\mathbf{x}|^2}{4(t + a_0^{-1})})\right) \\ &\left(\frac{1}{|t + a_0^{-1}|}\right)^{\frac{n}{2}} A_0\left(\frac{\mathbf{x}}{t + a_0^{-1}} - \mathbf{\Lambda} \int_0^t \frac{b(\tau)}{t + a_0^{-1}} d\tau\right), \end{aligned}$$

with the function $A_0(\mathbf{x})$ satisfying (28). The solution (29) has an interesting feature in the case when $k_1 \neq 0$, $A_0(\mathbf{x}) \in L_2(\mathbb{R}^n)$ and $a_0 < 0$. At a finite time, $T = -a_0^{-1}$, the mass concentrates at a point which escapes to infinity.

If we suppose that $A_0(\mathbf{x})\mathbf{x} \in L_2(\mathbb{R}^n)$, we can express the solution through the functionals $M(t), Q_\Lambda(t)$ and \tilde{P} . Indeed, for the velocity field (24) we have

$$(30) \quad M'(t) = 2a(t)M(t) + b(t)Q_\Lambda(t),$$

$$(31) \quad \tilde{P}_\Lambda = a(t)Q_\Lambda(t) + b(t)N|\mathbf{\Lambda}|^2.$$

Further, from (3), (30), (31) we obtain

$$a'(t) = a(t) \frac{2\tilde{P}_\Lambda Q_\Lambda(t) - M'(t)N|\mathbf{\Lambda}|^2}{M(t)N|\mathbf{\Lambda}|^2 - Q_\Lambda^2(t)} - \frac{\tilde{P}_\Lambda^2 - 8HN|\mathbf{\Lambda}|^2}{M(t)N|\mathbf{\Lambda}|^2 - Q_\Lambda^2(t)}.$$

Then we use (4) and (10) to get

$$(32) \quad a'(t) = -a(t) \frac{F'(t)}{F(t)} + \frac{C}{F(t)},$$

where $F(t) = At^2 + Bt + C$, $A = 8HN|\mathbf{\Lambda}|^2 - \tilde{P}_\Lambda^2$, $B = M'(0)HN|\mathbf{\Lambda}|^2 - 2\tilde{P}_\Lambda Q_\Lambda(0)$, $C = M(0)N|\mathbf{\Lambda}|^2 - Q_\Lambda^2(0)$. Note that by virtue of the Hölder inequality $C \geq 0$.

Further, we find from (32) and (31)

$$a(t) = \frac{Ct}{F(t)} + a_0,$$

$$b(t) = \frac{\tilde{P}_\Lambda(A - C)t^2 + (\tilde{P}_\Lambda(B - a_0) - CQ_\Lambda(0))t + (\tilde{P}_\Lambda C - a_0Q_\Lambda(0))}{F(t)N|\Lambda|^2}.$$

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