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THE L^2-HARMONIC FORMS ON ROTATIONALLY SYMMETRIC RIEMANNIAN MANIFOLDS REVISITED

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Abstract. We use separation of variables for generalized Dirac operators on rotationally symmetric Riemannian manifolds to recover a theorem of Dodziuk regarding the spaces of L^2-harmonic forms on such manifolds.

The rotationally symmetric Riemannian manifolds play an important role in geometry. As generalizations of surfaces of revolution they form a class of manifolds broad enough to accommodate diversity, yet are sufficiently particular to yield definitive results. Not surprisingly, many papers were devoted to studying their properties: [C, Cr, D, GW, M, Ma].

An n-dimensional Riemannian manifold M, $n \geq 2$, is called *rotationally symmetric* at a point $p \in M$ if the isotropy subgroup at p of the isometry group of M is the orthogonal group $O(n)$. Depending on whether M is compact or not, the simply connected ones are diffeomorphic to $\mathbf{S}^n$ or $\mathbf{R}^n$ [Cr]. Since in this paper we need to consider only the simply connected non-compact class, M will be rotationally symmetric at p if [D, GW]

i) the exponential mapping is a diffeomorphism of T_pM onto M;

ii) every linear isometry of T_pM is induced by an isometry of M.

Consequently, M can be identified with $\mathbf{R}^n$ via the exponential mapping at p and in terms of geodesic polar coordinates $(r, \theta) \in (0, \infty) \times \mathbf{S}^{n-1} \cong M \setminus \{p\}$ the Riemannian metric ds^2 of M can be written as

$$ds^2 = dr^2 + f(r)^2 d\theta^2, \tag{1}$$

where the warping function $f(r)$ is smooth on $[0, \infty)$ and satisfies

$$f(0) = 0, \quad f'(0) = 1, \quad f(r) > 0 \ \text{ for } \ r > 0, \tag{2}$$

and where $d\theta^2$ is the standard metric on $\mathbf{S}^{n-1}$.

Here are two interesting results which hold on simply connected non-compact rotationally symmetric Riemannian manifolds M. The first one is due to Dodziuk [D], and the second one to March [M].

a) The space of L^2-harmonic forms on M is infinite dimensional if and only if

$$n \text{ is even} \quad \text{and} \quad \int_1^\infty \frac{dr}{f(r)} < \infty. \tag{3}$$

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b) M admits non-constant bounded harmonic functions if and only if

$$\int_1^\infty f(r)^{n-3}dr \int_r^\infty f(s)^{1-n}ds < \infty. \tag{4}$$

Dodziuk's proof of *a*) relies on a detailed analysis of the spaces $\mathcal{H}^q(M)$ of L^2-harmonic forms of degree $q = 0, 1, \ldots, n$, via the Hodge $*$-duality and a separation of variables made possible by (1). The separation of variables works fine in every degree q, except for $q = n/2$, n even, in which case Dodziuk employs a conformal identification of the rotationally symmetric manifold with a Euclidean ball in order to finish his proof. Since this exceptional case is the only one which affords an infinite-dimensional supply of L^2-harmonic forms on M a very natural question arises: Can one use separation of variables to prove Dodziuk's result in all cases? In this paper we will show how to do that, and by restating the problem in terms of generalized L^2-harmonic spinors we will also gain some additional benefits.

Let us now briefly review key facts about the generalized Dirac operators on complete Riemannian manifolds [GL] and, in the particular case of rotationally symmetric manifolds, their separation of variables [A].

If M is a complete (non-compact) n-dimensional Riemannian manifold let $Cl(M)$ be the Clifford bundle of algebras induced by the tangent bundle TM and the Riemannian metric $\langle\cdot,\cdot\rangle$. $Cl(M)$ comes naturally equipped with a metric and a connection, which extend the metric and the Levi-Cività connection of M. Let S be a bundle of Clifford modules over M equipped with a Hermitian metric $\langle\cdot,\cdot\rangle$ and a metric connection ∇ such that unit vectors in TM act isometrically on S and such that the connection ∇ is compatible with that of $Cl(M)$. In other words, S is a *generalized Dirac bundle* over M [GL].

Any generalized Dirac bundle S generates a first-order elliptic differential operator $D : C^\infty(S) \to C^\infty(S)$, the *generalized Dirac operator*, defined as follows. Denote by $\cdot$ the Clifford multiplication of $Cl(M)$ on S and by $\mu : TM \otimes S \to S$ its restriction to TM. Then the generalized Dirac operator D is the composite $\mu \circ \nabla$. Locally, D admits the representation

$$D = \sum_{i=0}^{n-1} v_i \cdot \nabla_{v_i},$$

where $(v_0, v_1, \ldots, v_{n-1})$ is a local orthonormal frame in TM.

D is an essentially self-adjoint first-order elliptic differential operator and for the square of D the following Bochner-Weitzenböck formula holds true [GL],

$$D^2 = \nabla^*\nabla + \mathcal{R},$$

where $\mathcal{R}$ is a Hermitian curvature bundle morphism acting on S according to the formula

$$\mathcal{R} = \sum_{i<j} v_i \cdot v_j \cdot R_{v_i,v_j}, \quad R_{v_i,v_j} = [\nabla_{v_i}, \nabla_{v_j}] - \nabla_{[v_i,v_j]}. \tag{5}$$

When M is rotationally symmetric at p the restriction of D to $M \setminus \{p\} \cong (0,\infty) \times_f \mathbf{S}^{n-1}$ admits a natural separation of variables [A]. In order to state it in the simplest form we need to perturb the warping function $f(r)$ slightly near $r = 1$ so that $f(1) = 1$ and $f''(1) \neq 0$. This deformation will not affect the generality of our result. Then let Σ be the restriction of S to $\{1\} \times \mathbf{S}^{n-1} \equiv \mathbf{S}^{n-1}$. The bundle $\Sigma \longrightarrow \mathbf{S}^{n-1}$ inherits a canonical structure of a generalized Dirac bundle, with associated Dirac operator $\not\partial$, and the parallel transport along radial geodesics in S allows us to trivialize $S|_{M\setminus\{p\}}$ in the radial direction. More precisely, if $\pi : (0,\infty)\times \mathbf{S}^{n-1} \longrightarrow \mathbf{S}^{n-1}$ is the projection, then $\pi^*(\Sigma)$ is canonically isomorphic to $S_{|M\setminus\{p\}}$. When $S_{|M\setminus\{p\}}$ is viewed as $\pi^*(\Sigma)$ any section in $C^\infty(M\setminus\{p\}, S)$ can be viewed as an element of $C^\infty((0,\infty), C^\infty(\mathbf{S}^{n-1},\Sigma))$, and the following *separation of variables* formula holds:

$$Ds = \mathbf{n}\cdot s' + \frac{\not\partial s}{f} + \frac{f'}{f}\Xi s, \quad s\in C^\infty((0,\infty), C^\infty(\mathbf{S}^{n-1},\Sigma)). \tag{6}$$

In (6) $'$ represents ordinary differentiation with respect to the radial variable r, $\mathbf{n}$ is the tangent vector $\partial/\partial r$ restricted to $\mathbf{S}^{n-1}$, and Ξ is the bundle morphism on Σ given by

$$\Xi\sigma = \frac{1}{f''(1)}\sum_{i=1}^{n-1} e_i\cdot R_{\mathbf{n},e_i}\sigma, \quad \sigma\in\Sigma, \tag{7}$$

where $(e_1, e_2, \cdots, e_{n-1})$ is a local orthonormal frame in $T\mathbf{S}^{n-1}$. We want to mention for later use the following two commutation formulas:

$$\not\partial(\mathbf{n}\cdot\sigma) + \mathbf{n}\cdot\not\partial\sigma = 0, \quad \Xi(\mathbf{n}\cdot\sigma) + \mathbf{n}\cdot\Xi\sigma = -(n-1)\sigma, \quad \sigma\in C^\infty(\mathbf{S}^{n-1},\Sigma). \tag{8}$$

As in [A] let us specialize the above discussion to the particular situations when $S = Cl(M)$, the Clifford bundle itself, or, for a spin manifold M, when $S = Spin(M)$, the classical spinor bundle.

a) $S = Cl(M)$. In this case $\Sigma = Cl(\mathbf{S}^{n-1})\oplus \mathbf{n}\cdot Cl(\mathbf{S}^{n-1})$, $\not\partial$ can be identified with the direct sum of two copies of the Gauss-Bonnet/Hodge-de Rham operator of $\mathbf{S}^{n-1}$, and on forms $\omega\in Cl(\mathbf{S}^{n-1})$ of degree q, $\Xi\omega = q\mathbf{n}\cdot\omega$.

b) $S = Spin(M)$, for a spin manifold M. Depending on whether $n = \dim M$ is odd or even, Σ can be identified with $Spin(\mathbf{S}^{n-1})$ or $Spin(\mathbf{S}^{n-1})\oplus\mathbf{n}\cdot Spin(\mathbf{S}^{n-1})$, $\not\partial$ with the classical Dirac operator on $\mathbf{S}^{n-1}$ or a direct sum of two copies of it, and $\Xi\sigma = \frac{n-1}{2}\mathbf{n}\cdot\sigma$, for $\sigma\in\Sigma$.

We will now turn our attention to the space of *generalized L^2-harmonic spinors* of D, that is, the space of $s\in L^2(M,S)$ such that (distributionally) $Ds = 0$. Since on a complete manifold $Ds = 0$ if and only if $D^2 s = 0$ for every $s\in L^2(M,S)$ [GL], the above terminology is justified. By elliptic regularity any L^2-harmonic spinor belongs to $C^\infty(M,S)$.

In the rotationally symmetric case let us denote by $\langle\cdot,\cdot\rangle$, $|\cdot|$, the pointwise inner product, respectively the norm, in S, and by $\langle\cdot,\cdot\rangle_1$, $|\cdot|_1$, the corresponding objects in Σ. The global (integrated) inner product on M will be denoted by $(\cdot,\cdot)$, while the one on $\mathbf{S}^{n-1}$ will be denoted by $(\cdot,\cdot)_1$. Therefore, if $s\in C^\infty(M,S)$ is an L^2-harmonic spinor, then for $s|_{M\setminus\{p\}} = s(r,\theta)\in C^\infty\big((0,\infty), C^\infty(\mathbf{S}^{n-1},\Sigma)\big)$ we have

$$\mathbf{n}\cdot s' + \frac{\not\partial s}{f} + \frac{f'}{f}\Xi s = 0 \quad \text{and} \tag{9}$$

$$(s,s) = \int_0^\infty f^{n-1}(r)dr\int_{\mathbf{S}^{n-1}} |s(r,\theta)|_1^2 d\theta < \infty. \tag{10}$$

Conversely, by elliptic regularity any element $s(r,\theta)\in C^\infty\big((0,\infty), C^\infty(\mathbf{S}^{n-1},\Sigma)\big)$ satisfying equations (9) and (10) is the restriction to $M\setminus\{p\}$ of some L^2-harmonic spinor.

It is also useful to notice that harmonic spinors are bounded near $r = 0$; that is,

$$|s(r,\theta)|_1^2 \leq C, \quad r \in (0,1], \theta \in \mathbf{S}^{n-1}. \tag{11}$$

Lemma. *Let $s \in C^\infty(M,S)$ be a non-trivial L^2-harmonic spinor for some generalized Dirac operator on a rotationally symmetric Riemannian manifold. With the above notation, assume there is an orthogonal bundle decomposition $\Sigma = \Sigma_+ \oplus \Sigma_-$ satisfying*

$$\mathbf{n}\cdot \not\partial \left(C^\infty(\mathbf{S}^{n-1}, \Sigma_\pm)\right) \subseteq C^\infty(\mathbf{S}^{n-1}, \Sigma_\mp), \tag{12}$$

such that when $s|_{M\setminus\{p\}}$ is viewed as an element of $C^\infty\left((0,\infty), C^\infty(\mathbf{S}^{n-1},\Sigma)\right)$ then

$$s|_{M\setminus\{p\}} = s_+ + s_-,$$

where $s_\pm \in C^\infty\left((0,\infty), C^\infty(\mathbf{S}^{n-1},\Sigma_\pm)\right)$ and for some $\alpha_\pm \in \mathbf{R}$, $\varXi s_\pm = \alpha_\pm \mathbf{n}\cdot s_\pm$. Then the following hold true:

i) If $\alpha_+ + \alpha_- > 0$ and $s_\pm \neq 0$, then correspondingly

$$\int_1^\infty f^{n-1-2\alpha_\pm}(r)dr < \infty. \tag{13}$$

ii) If $s_\pm = 0$, then correspondingly $\alpha_\mp \leq 0$ and $\not\partial s_\mp = 0$. Moreover, there is $\sigma_\mp \in C^\infty(\mathbf{S}^{n-1},\Sigma_\mp)$ such that $\not\partial \sigma_\mp = 0$ and $s_\mp = f^{-\alpha_\mp}\sigma_\mp$.

Proof. *i*) Under the hypotheses of the lemma, equation (9) applied to $s_+ + s_-$ yields

$$\mathbf{n}\cdot s_+' + \frac{\not\partial s_+}{f} + \frac{f'}{f}\alpha_+\mathbf{n}\cdot s_+ + \mathbf{n}\cdot s_-' + \frac{\not\partial s_-}{f} + \frac{f'}{f}\alpha_-\mathbf{n}\cdot s_- = 0,$$

or equivalently,

$$\left(s_+' - \frac{\mathbf{n}\cdot\not\partial s_-}{f} + \frac{f'}{f}\alpha_+ s_+\right) + \left(s_-' - \frac{\mathbf{n}\cdot\not\partial s_+}{f} + \frac{f'}{f}\alpha_- s_-\right) = 0. \tag{14}$$

By (12), equation (14) is subsequently equivalent to the system

$$\begin{cases} s_+' - (\mathbf{n}\cdot\not\partial s_-)/f + (f'/f)\alpha_+ s_+ = & 0, \\ s_-' - (\mathbf{n}\cdot\not\partial s_+)/f + (f'/f)\alpha_- s_- = & 0, \end{cases}$$

which can be rewritten as

$$\begin{cases} (f^{\alpha_+}s_+)' = & f^{\alpha_+-1}\mathbf{n}\cdot\not\partial s_-, \\ (f^{\alpha_-}s_-)' = & f^{\alpha_--1}\mathbf{n}\cdot\not\partial s_+. \end{cases} \tag{15}$$

Now after applying $\mathbf{n}\cdot\not\partial$ to the first equation (15) we get

$$(f^{\alpha_+}\mathbf{n}\cdot\not\partial s_+)' = f^{\alpha_+-1}\left(\mathbf{n}\cdot\not\partial\right)^2 s_-,$$

which coupled with the second equation (15) gives

$$\left(f^{\alpha_+-\alpha_-+1}\left(f^{\alpha_-}s_-\right)'\right)' = f^{\alpha_+-1}\left(\mathbf{n}\cdot\not\partial\right)^2 s_-.$$

As a result, for an arbitrary $r \in (0,\infty)$ we have

$$\begin{aligned} &\left(\left(f^{\alpha_+-\alpha_-+1}\left(f^{\alpha_-}s_-\right)'\right)', f^{\alpha_-}s_-\right)_1 \\ &\quad = \left(f^{\alpha_+-1}\left(\mathbf{n}\cdot\not\partial\right)^2 s_-, f^{\alpha_-}s_-\right)_1 = f^{\alpha_++\alpha_--1}\left(\left(\mathbf{n}\cdot\not\partial\right)s_-, \left(\mathbf{n}\cdot\not\partial\right)s_-\right)_1 \geq 0. \end{aligned} \tag{16}$$

Equation (16) implies immediately

$$\left(f^{\alpha_+-\alpha_-+1}\left(f^{\alpha_-}s_-\right)', f^{\alpha_-}s_-\right)_1' \geq 0, \quad \text{for every} \quad r > 0. \tag{17}$$

Since by (11) $s_\pm = O(1)$ near $r = 0$, we conclude that

$$\left(f^{\alpha_+ - \alpha_- + 1}\left(f^{\alpha_-} s_-\right)', f^{\alpha_-} s_-\right)_1 = O\left(r^{\alpha_+ + \alpha_-}\right),$$

and so equation (17) and the hypothesis $\alpha_+ + \alpha_- > 0$ imply that

$$\left(f^{\alpha_+ - \alpha_- + 1}\left(f^{\alpha_-} s_-\right)', f^{\alpha_-} s_-\right)_1 \geq 0,$$

or equivalently

$$\left(\left(f^{\alpha_-} s_-\right)', f^{\alpha_-} s_-\right)_1 \geq 0, \quad \text{for every } r > 0.$$

It follows that $\left(f^{\alpha_-} s_-, f^{\alpha_-} s_-\right)_1$ is a non-decreasing function of r and since

$$(s_-, s_-) = \int_0^\infty f^{n-1-2\alpha_-}\left(f^{\alpha_-} s_-, f^{\alpha_-} s_-\right)_1 \, dr \; < \infty$$

we conclude that $\int_0^\infty f^{n-1-2\alpha_-}(r)dr < \infty$ if $s_- \neq 0$, as stated. An identical argument also shows that $\int_0^\infty f^{n-1-2\alpha_+}(r)dr < \infty$ if $s_+ \neq 0$.

ii) Suppose, for instance, $s_+ = 0$. Then $s|_{M\setminus\{p\}} = s_-$, and since s is assumed non-trivial, $s_- \neq 0$. The system (15) then gives

$$\mathbf{n} \cdot \not{\partial} s_- = 0 \quad \text{and} \quad \left(f^{\alpha_-} s_-\right)' = 0.$$

Therefore, $f^{\alpha_-} s_-$ is independent of r and $\not{\partial} s_- = 0$. As a result, there is $\sigma_- \in C^\infty\left(\mathbf{S}^{n-1}, \Sigma_-\right)$, $\not{\partial}\sigma_- = 0$, such that $s_- = f^{-\alpha_-}\sigma_-$. We conclude that for r near 0, $s_- = O\left(r^{-\alpha_-}\right)$. However, $s_- = O(1)$, by equation (11), and so $\alpha_- \leq 0$. An identical argument applies when s_-, instead of s_+, vanishes.

Theorem. *a*) [D] *The space of L^2-harmonic forms on a simply connected non-compact rotationally symmetric n-dimensional Riemannian manifold M with warping function f is infinite dimensional if and only if*

$$n \text{ is even} \quad \text{and} \quad \int_1^\infty \frac{dr}{f(r)} < \infty.$$

More precisely, there can be non-trivial L^2-harmonic forms on M only in degrees 0 *and n* (*only if $\int_1^\infty f^{n-1}(r)dr < \infty$, in which case both these spaces have dimension* 1) *or in degree $n/2$, n even* (*only if $\int_1^\infty 1/f(r)dr < \infty$, in which case this latter space is infinite dimensional*).

b) *There are no non-trivial L^2-harmonic spinors for the classical Dirac operator on a simply connected non-compact rotationally symmetric spin manifold M.*

Proof. *a*) Let s be a non-trivial L^2-harmonic form of degree q, $q = 0, 1, \ldots, n$, on such a manifold M. By identifying s with a Clifford form of degree q and using parallel transport in the radial direction in $S|_{M\setminus\{p\}}$, we can write $s|_{M\setminus\{p\}} = a + \mathbf{n}\cdot b$, where a and b are Clifford forms on $\mathbf{S}^{n-1}$ of degree q, respectively $q-1$, depending smoothly on the radial parameter r (evidently, when $q = 0$, $b = 0$, and when $q = n$, $a = 0$). Notice now that the form $a + \mathbf{n}\cdot b$ satisfies the hypotheses of the previous lemma if we take $\Sigma_+ = Cl(\mathbf{S}^{n-1})$, $\Sigma_- = \mathbf{n}\cdot Cl(\mathbf{S}^{n-1})$, $\not{\partial}$ the Gauss-Bonnet operator on $Cl(\mathbf{S}^{n-1})$, $s_+ = a$, and $s_- = \mathbf{n}\cdot b$. Indeed, as noted earlier in such a case, $\Xi s_+ = q\mathbf{n}\cdot s_+$ and, by (8), $\Xi s_- = (n-q)\mathbf{n}\cdot s_-$. Therefore, $\alpha_+ = q$, $\alpha_- = n - q$, and so $\alpha_+ + \alpha_- = n > 0$.

If $q = 0$ or $q = n$, then both cases *i*) and *ii*) of the lemma apply simultaneously. By case *i*), in order to have non-trivial L^2-harmonic forms it is necessary that $\int_1^\infty f^{n-1}(r)dr < \infty$. Case *ii*) and equation (10) show that this is also sufficient.

Since the spaces of harmonic forms of degree 0 and $n-1$ on $\mathbf{S}^{n-1}$ are 1-dimensional we reach the desired conclusion for these values of q.

If $0 < q < n$, then case *ii*) of the lemma cannot happen because $\alpha_\pm > 0$. Thus both s_+ and s_- are non-vanishing and by case *i*) we have

$$\int_1^\infty f^{n-1-2q}(r)dr < \infty \quad \text{and} \quad \int_1^\infty f^{2q-n-1}(r)dr < \infty.$$

Obviously, the above integrals cannot both converge when the exponents $n-1-2q$ and $2q-n-1$ have opposite signs or one of them vanishes. This happens precisely when $2q \neq n$. In conclusion, there are no L^2-harmonic forms for $0 < q < n$, $q \neq n/2$.

It remains to analyse the case n even, $q = n/2$. In this situation $\alpha_+ = \alpha_- = n/2$ and so case *i*) of the lemma requires $\int_1^\infty 1/f(r)dr < \infty$ for the existence of L^2 forms. The actual existence of an infinite-dimensional supply of such forms will be shown below through a constructive argument, based on the structure of mid-degrees forms on $\mathbf{S}^{n-1}$.

To this end denote by $x_0, x_1, \ldots, x_{n-1}$ the variables in $\mathbf{R}^n$, and by $v_0, v_1, \ldots, v_{n-1}$ the Clifford elements $v_j = \partial/(\partial x_j)$, $j = 0, 1, \ldots, n-1$, in $Cl(\mathbf{R}^n)$. For each integer k, $k \geq 0$, define the complex Clifford form of degree $n/2$, $\omega_k \in C^\infty(\mathbf{R}^n, Cl(\mathbf{R}^n))$, by

$$\omega_k = (x_0 + ix_1)^k (v_0 + iv_1) \cdot (v_2 + iv_3) \cdot \ldots \cdot (v_{n-2} + iv_{n-1}).$$

Since $v_0, v_1, \ldots, v_{n-1}$ are obtained by a parallel transport in the radial direction in $Cl(\mathbf{R}^n)$ of their restrictions to $\mathbf{S}^{n-1}$, we see that when written in polar coordinates in $\mathbf{R}^n \setminus \{0\}$, ω_k becomes

$$\omega_k(r, \theta) = r^k \left(a_k(\theta) + \mathbf{n} \cdot b_k(\theta)\right),$$

where $a_k, b_k \in C^\infty\left(\mathbf{S}^{n-1}, Cl(\mathbf{S}^{n-1})\right)$ are (non-zero) Clifford forms of degree $n/2$, respectively $n/2 - 1$. It is obvious that $D\omega_k = 0$, where now D is the Gauss-Bonnet operator in $\mathbf{R}^n$. The separation of variables formula (6) in $\mathbf{R}^n$ ($f(r) = r$) then gives

$$\begin{cases} \not{\partial} a_k = (n/2 + k) b_k, \\ \not{\partial} b_k = (n/2 + k) a_k. \end{cases} \tag{18}$$

It is easy now to check that under the hypothesis $\int_1^\infty 1/f(r)dr < \infty$, $s_k \in C^\infty\left((0, \infty), C^\infty(\mathbf{S}^{n-1}, \Sigma)\right)$, defined by

$$s_k(r, \theta) = f(r)^{-n/2} e^{(n/2+k)\int_1^r ds/f(s)} \left(a_k(\theta) + \mathbf{n} \cdot b_k(\theta)\right),$$

satisfies equations (9) and (10), thus generating an L^2-harmonic form on M. Since the family s_k, $k = 0, 1, \ldots,$ is clearly linearly independent, the proof of case *a*) of the theorem is complete.

b) This case is considerably simpler. If M is a rotationally symmetric spin manifold of dimension n and $S = Spin(M)$, then for n even the splitting of $\Sigma = Spin(\mathbf{S}^{n-1}) \oplus \mathbf{n} \cdot Spin(\mathbf{S}^{n-1})$ as required by the lemma is similar to that in case *a*) of this theorem. If n is odd, then $\Sigma = Spin(\mathbf{S}^{n-1})$ and we can take $\Sigma_\pm = \pm 1$ eigenspaces of the Clifford element $\epsilon = i^{(n-1)/2} e_1 \cdot e_2 \cdot \ldots \cdot e_{n-1}$ acting on $Spin(\mathbf{S}^{n-1})$, where $e_1, e_2, \ldots, e_{n-1}$ in an oriented local orthonormal frame in $T\mathbf{S}^{n-1}$. As a result, the lemma applies in case *b*) regardless of the parity of $\dim M$. Since we always have $\alpha_\pm = (n-1)/2$, equation (13) cannot hold and so there are no non-trivial L^2 classical harmonic spinors.

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