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WEAK L^1 NORMS OF RANDOM SUMS

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ABSTRACT. Let $\{g_j\}$ denote a sequence of measurable functions on $\mathbf{R}^n$, and let $\|\cdot\|_{WL^1}$ denote the weak L^1 norm. It is shown that

$$\left\| \mathbb{E}\left(\left| \sum_{j=1}^{N} \epsilon_j g_j \right| \right) \right\|_{WL^1} \lesssim \sum_{j=1}^{N} \|g_j\|_{WL^1} ,$$

where $\{\epsilon_j\}$ is a sequence of independent random variables taking on values $+1$ and -1 with equal probability. Moreover, it is shown that

$$\left\| \mathbb{E}\left(\left| \sum_{j=1}^{N} \epsilon_j g_j \right| \right) \right\|_{WL^1} \lesssim \mathbb{E}\left(\left\| \sum_{j=1}^{N} \epsilon_j g_j \right\|_{WL^1} \right) .$$

The paper concludes by providing an example indicating that, if $\|g_1\|_{WL^1} = \cdots = \|g_N\|_{WL^1} = 1$, then the estimate

$$\mathbb{E}\left(\left\| \sum_{j=1}^{N} \epsilon_j g_j \right\|_{WL^1} \right) \lesssim N \log N$$

is the best possible.

1. Introduction

The weak L^1 norm of a measurable function f supported on $\mathbf{R}^n$ is defined by

$$\|f\|_{WL^1} = \sup_{\alpha>0} \alpha \left|\{x \in \mathbf{R}^n : |f(x)| > \alpha\}\right| .$$

This norm is of frequent occurrence in modern-day mathematics, finding use in probability theory, harmonic analysis, and ergodic theory. In spite of its many applications, however, the weak L^1 norm has one decidedly inconvenient drawback: it is not a “norm” in the strictest sense of the word as it does not satisfy the triangle inequality. For example, the functions $\frac{1}{x}\chi_{(0,1)}(x)$ and $\frac{1}{1-x}\chi_{(0,1)}(x)$ both have weak L^1 norms of 1, but their sum has a weak L^1 norm of 4.

The best-known result relating the weak L^1 norm of a sum of functions to their respective individual weak L^1 norms is the following due to E. M. Stein and N. J. Weiss [1].

Theorem 1. *Let $\{g_j\}$ denote a sequence of measurable functions on $\mathbf{R}^n$ such that $\|g_j\|_{WL^1} = 1$ for each j. Let $\{c_j\}$ be a sequence of positive numbers with $\sum_j c_j = 1$,*

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let $s > 0$*, and set* $K = \sum_j c_j \log\left(\frac{1}{c_j}\right)$*. Then* $\left|\left\{x \in \mathbf{R}^n : \sum_j c_j g_j(x) > s\right\}\right| < \frac{2(K+2)}{s}$.

The bound of $\|\sum c_j g_j\|_{WL^1}$ in this theorem is essentially optimal: the log term may not be removed. For example, letting $c_1 = \cdots = c_N = \frac{1}{N}$ and $g_j(x) = \frac{1}{2}\frac{1}{\left|x - \frac{j}{N}\right|}$, one may readily check that $\|g_j\|_{WL^1} = 1$ for each j and that

$$\left\|\sum_{j=1}^{N} c_j g_j(x)\right\|_{WL^1} \sim \sum_{j=1}^{N} c_j \log\left(\frac{1}{c_j}\right) \sim \log N.$$

Given the prevalence of randomization techniques in modern harmonic analysis, it is natural to conduct the admittedly Baconian experiment of determining the expected value of $\left\|\sum_{j=1}^{N} \epsilon_j c_j g_j(x)\right\|_{WL^1}$, where the c_j and $g_j(x)$ are as above and $\{\epsilon_j\}$ is a sequence of independent random variables taking on the values of $+1$ and -1 with equal probability. It turns out that a large majority of the 2^N associated weak L^1 norms are close to one, exhibiting significant improvement over the $\log N$ bound above because of cancellation between the $\epsilon_j c_j g_j$'s. In fact, one can calculate that $\mathbb{E}\left(\left\|\sum_{j=1}^{N} \epsilon_j c_j g_j(x)\right\|_{WL^1}\right) \sim 1$. Although not difficult, the techniques involved in this calculation are not immediately transparent and are accordingly given below.

Example 1. Let $c_1 = \cdots = c_N = \frac{1}{N}$ and $g_j(x) = \frac{1}{2}\frac{1}{\left|x - \frac{j}{N}\right|}$. Then

$$\mathbb{E}\left(\left\|\sum_{j=1}^{N} \epsilon_j c_j g_j(x)\right\|_{WL^1}\right) \sim \sum_{j=1}^{N} \frac{1}{N} = 1.$$

We first observe that

$$2^{2^N}\left|\left\{x : \sum_{j=1}^{N} |\epsilon_j c_j g_j(x)| > 2^{2^N}\right\}\right| \sim 2^{2^N} \cdot N \cdot N^{-1} \cdot 2^{-2^N} = 1$$

for any sequence $\{\epsilon_j\}$ consisting of -1's and $+1$'s, and hence the result that

$$\mathbb{E}\left(\left\|\sum_{j=1}^{N} \epsilon_j c_j g_j(x)\right\|_{WL^1}\right) \gtrsim \sum_{j=1}^{N} \frac{1}{N} = 1$$

easily holds. We accordingly turn our attention to the opposite inequality.

Let $\tilde{g}_j(x) = g_j(x)\,\chi_{\{x : |g_j(x)| > 2N\}}(x)$. Let $h_j(x) = g_j(x) - \tilde{g}_j(x)$. Now,

$$\left\|\sum_{j=1}^{N} \epsilon_j c_j g_j\right\|_{WL^1} \leq 2\left[\left\|\sum_{j=1}^{N} \epsilon_j c_j \tilde{g}_j\right\|_{WL^1} + \left\|\sum_{j=1}^{N} \epsilon_j c_j h_j\right\|_{WL^1}\right].$$

As the $\tilde{g}_j$ have disjoint supports,

$$\left\|\sum_{j=1}^{N} \epsilon_j c_j \tilde{g}_j\right\|_{WL^1} \leq \sum_{j=1}^{N} \|c_j \tilde{g}_j\|_{WL^1} \leq 1.$$

So it suffices to show that $\mathbb{E}\left(\left\|\sum_{j=1}^{N} \epsilon_j c_j h_j\right\|_{WL^1}\right) \lesssim 1$.

Note that, letting $E=[-1,2]$ and $\tilde{E}=(-\infty,-1)\cup(2,\infty)$, we have that

$$\begin{aligned}\mathbb{E}\left(\left\|\sum_{j=1}^{N}\epsilon_j c_j h_j\right\|_{WL^1}\right)&\\ \leq&\mathbb{E}\left(\left\|\left(\sum_{j=1}^{N}\epsilon_j c_j h_j\right)\chi_E\right\|_{WL^1}\right)\\ &+\mathbb{E}\left(\left\|\left(\sum_{j=1}^{N}\epsilon_j c_j h_j\right)\chi_{\tilde{E}}\right\|_{WL^1}\right).\end{aligned}$$

Now, for $x\in\tilde{E}$, $\left|\sum_{j=1}^{N}\epsilon_j c_j h_j(x)\right|\sim\left|\left(\sum_{j=1}^{N}\epsilon_j\right)\cdot\frac{1}{|Nx|}\right|\leq\frac{1}{|x|}$, and so

$$\mathbb{E}\left(\left\|\left(\sum_{j=1}^{N}\epsilon_j c_j h_j\right)\chi_{\tilde{E}}\right\|_{WL^1}\right)\lesssim 1.$$

Hence it suffices to show that

$$\mathbb{E}\left(\left\|\sum_{j=1}^{N}\epsilon_j c_j h_j\chi_{[-1,2]}\right\|_{WL^1}\right)\lesssim 1.$$

To see this, note that

$$\begin{aligned}\mathbb{E}\left(\left\|\sum_{j=1}^{N}\epsilon_j c_j h_j\chi_{[-1,2]}\right\|_{WL^1}\right)&\leq\mathbb{E}\left(\left\|\sum_{j=1}^{N}\epsilon_j c_j h_j\chi_{[-1,2]}\right\|_{L^1}\right)\\ &=\left\|\mathbb{E}\left|\sum_{j=1}^{N}\epsilon_j c_j h_j\chi_{[-1,2]}\right|\right\|_{L^1}\\ &\lesssim\left(\sum_{j=1}^{N}|c_j h_j(1)|^2\right)^{1/2}\\ &\lesssim\left(\sum_{j=1}^{N}\left(\frac{1}{j}\right)^2\right)^{1/2}\sim 1,\end{aligned}$$

as desired.

It is natural at this point to consider whether or not this example is indicative of more general inequalities involving the expectation operator and the weak L^1 norm. The purpose of this paper is to show that the techniques involved in the above example may indeed be generalized to prove that, for $\{g_j\}$ and $\{c_j\}$ satisfying the hypotheses of Theorem 1, $\|\mathbb{E}(|\sum c_j g_j|)\|_{WL^1}\lesssim\sum c_j$ and $\|\mathbb{E}(|\sum\epsilon_j c_j g_j|)\|_{WL^1}\lesssim\mathbb{E}\left(\|\sum\epsilon_j c_j g_j\|_{WL^1}\right)$. It will also be shown, however, that the *a priori* Stein-Weiss estimate $\mathbb{E}\|\sum\epsilon_j c_j g_j\|_{WL^1}\lesssim\sum c_j\log\left(\frac{1}{c_j}\right)$ is the best possible.

2. Weak L^1 norms of expectations of random sums

This section is devoted to the proof of the following.

Theorem 2. *Let $\{g_j\}$ denote a sequence of measurable functions on $\mathbf{R}^n$, and let $\{\epsilon_j\}$ be a sequence of independent random variables taking on values $+1$ and -1 with equal probability. Then*

$$\left\|\mathbb{E}\left(\left|\sum_{j=1}^{N}\epsilon_j g_j(x)\right|\right)\right\|_{WL^1} \lesssim \sum_{j=1}^{N}\|g_j\|_{WL^1}.$$

Proof. We first note that

$$\mathbb{E}\left(\left|\sum_{j=1}^{N}\epsilon_j g_j(x)\right|\right) \sim \left(\sum_{j=1}^{N}|g_j(x)|^2\right)^{1/2},$$

as follows from elementary probability theory. So it suffices to show that

$$\left\|\left(\sum_{j=1}^{N}|g_j(x)|^2\right)^{1/2}\right\|_{WL^1} \lesssim \sum_{j=1}^{N}\|g_j\|_{WL^1}.$$

Let $\alpha > 0$. Let $g_{j,\alpha}(x) = g_j(x)\,\chi_{\{x:|g_j(x)|\leq\alpha\}}(x)$. Note that

$$\left\{x : \left(\sum_{j=1}^{N}|g_j(x)|^2\right)^{1/2} > \alpha\right\} \subset \left(\bigcup_j \{x : |g_j(x)| > \alpha\}\right) \cup \left\{x : \sum_{j=1}^{N}|g_{j,\alpha}(x)|^2 > \alpha^2\right\}. \tag{1}$$

By the definition of the weak L^1 norm, we have

$$|\{x : |g_j(x)| > \alpha\}| \leq \frac{1}{\alpha}\|g_j\|_{WL^1}. \tag{2}$$

Also,

$$\left|\left\{x : \sum_{j=1}^{N}|g_{j,\alpha}(x)|^2 > \alpha^2\right\}\right| \leq \frac{1}{\alpha^2}\sum_{j=1}^{N}\int_{\mathbf{R}^n}|g_{j,\alpha}|^2\;dx$$

and

$$\begin{aligned}\int_{\mathbf{R}^n}|g_{j,\alpha}(x)|^2\,dx &\leq \int_{y=\frac{1}{\alpha}\|g_j\|_{WL^1}}^{\infty}\left(\frac{\|g_j\|_{WL^1}}{y}\right)^2\;dy\\ &\lesssim \|g_j\|_{WL^1}^2\cdot\left[-y^{-1}\right]_{\frac{1}{\alpha}\|g_j\|_{WL^1}}^{\infty}\\ &\sim \alpha\,\|g_j\|_{WL^1}.\end{aligned}$$

Hence $\left|\left\{x : \sum_{j=1}^{N}|g_{j,\alpha}(x)|^2 > \alpha^2\right\}\right| \lesssim \frac{1}{\alpha^2}\cdot\alpha\sum_{j=1}^{N}\|g_j\|_{WL^1} = \frac{1}{\alpha}\sum_{j=1}^{N}\|g_j\|_{WL^1}$. This inequality together with (1), (2) implies that $\left|\left\{x : \left(\sum_{j=1}^{N}|g_j(x)|^2\right)^{1/2} > \alpha\right\}\right| \lesssim \frac{1}{\alpha}\sum_{j=1}^{N}\|g_j\|_{WL^1}$. So $\left\|\left(\sum_{j=1}^{N}|g_j|^2\right)^{1/2}\right\|_{WL^1} \lesssim \sum_{j=1}^{N}\|g_j\|_{WL^1}$, as desired. □

3. Expectations of weak L^1 norms of random sums

The theory involving the relationship between the expectation operator and the weak L^1 norm would be very nice indeed if the comparability $\left\|\mathbb{E}\left(\left|\sum_{j=1}^N \epsilon_j g_j\right|\right)\right\|_{WL^1} \sim \mathbb{E}\left(\left\|\sum_{j=1}^N \epsilon_j g_j\right\|_{WL^1}\right)$ held. In this section we shall see that the inequality $\left\|\mathbb{E}\left(\left|\sum_{j=1}^N \epsilon_j g_j\right|\right)\right\|_{WL^1} \lesssim \mathbb{E}\left(\left\|\sum_{j=1}^N \epsilon_j g_j\right\|_{WL^1}\right)$ does hold. However, afterwards we shall provide an example indicating that the opposite inequality is false.

Theorem 3. *Let $\{g_j\}$ denote a sequence of measurable functions on $\mathbf{R}^n$, and let $\{\epsilon_j\}$ be a sequence of independent random variables taking on values $+1$ and -1 with equal probability. Then*

$$\left\|\mathbb{E}\left(\left|\sum_{j=1}^N \epsilon_j g_j\right|\right)\right\|_{WL^1} \lesssim \mathbb{E}\left(\left\|\sum_{j=1}^N \epsilon_j g_j\right\|_{WL^1}\right).$$

Proof. It suffices to show that there exist universal constants $0 < c, C < \infty$ such that, for any $\alpha > 0$,

$$\left|\left\{x : \mathbb{E}\left(\left|\sum_{j=1}^N \epsilon_j g_j(x)\right|\right) > \alpha\right\}\right| \leq C\, \mathbb{E}\left(\left|\left\{x : \left|\sum_{j=1}^N \epsilon_j g_j(x)\right| > c\alpha\right\}\right|\right). \tag{3}$$

To see this, suppose (3) held. Let β be such that $\beta \cdot \left|\left\{x : \mathbb{E}\left(\left|\sum_{j=1}^N \epsilon_j g_j(x)\right|\right) > \beta\right\}\right| \sim \left\|\mathbb{E}\left(\left|\sum_{j=1}^N \epsilon_j g_j\right|\right)\right\|_{WL^1}$. (We may assume that $\left\|\mathbb{E}\left(\left|\sum_{j=1}^N \epsilon_j g_j\right|\right)\right\|_{WL^1} < \infty$ without loss of generality.) Then

$$\begin{aligned}
\left\|\mathbb{E}\left(\left|\sum_{j=1}^N \epsilon_j g_j\right|\right)\right\|_{WL^1} &\sim \beta \cdot \left|\left\{x : \mathbb{E}\left(\left|\sum_{j=1}^N \epsilon_j g_j(x)\right|\right) > \beta\right\}\right| \\
&\leq C\beta\, \mathbb{E}\left(\left|\left\{x : \left|\sum_{j=1}^N \epsilon_j g_j(x)\right| > c\beta\right\}\right|\right) \\
&= C\, \mathbb{E}\left(\beta\left|\left\{x : \left|\sum_{j=1}^N \epsilon_j g_j(x)\right| > c\beta\right\}\right|\right) \\
&\leq C\, \mathbb{E}\left(\frac{1}{c}\left\|\sum_{j=1}^N \epsilon_j g_j\right\|_{WL^1}\right) \\
&\leq \frac{C}{c}\, \mathbb{E}\left(\left\|\sum_{j=1}^N \epsilon_j g_j\right\|_{WL^1}\right),
\end{aligned}$$

the desired result.

We accordingly turn our attention to (3). To prove (3), it suffices to show that there exist universal constants $0 < c, C < \infty$ such that

$$\left|\left\{x : \mathbb{E}\left(\left|\sum_{j=1}^N \epsilon_j g_j(x)\right|\right) > 1\right\}\right| \leq C\, \mathbb{E}\left(\left|\left\{x : \left|\sum_{j=1}^N \epsilon_j g_j(x)\right| > c\right\}\right|\right). \tag{4}$$

The proof of (4) will be facilitated by the use of Rademacher functions.

Definition 1. Let $r_n(t)$ denote the Rademacher functions on $\mathbf{R}$, defined by

$$r_n(t) = r_0(2^n t),$$

where $r_0(t) = 1$ if $0 \le t \le \frac{1}{2}$, -1 if $\frac{1}{2} < t < 1$, and $r_0(t+1) = r_0(t)$.

We will also need the following lemma:

Lemma 1 ([2])**.** *Let $\sum_{j=0}^{\infty} |a_j|^2 < \infty$, and let $F(t) = \sum_{j=0}^{N} a_j r_j(t)$ be a Rademacher series. Let $0 < p < \infty$. Then there exist finite, positive constants $A(p)$, $B(p)$ such that*

$$A(p)\|F\|_{L^p([0,1])} \le \left(\sum_{j=0}^{N} |a_j|^2\right)^{1/2} \le B(p)\|F\|_{L^p([0,1])}.$$

We now fix x and let $a_j = g_j(x)$. To prove (4), it suffices to show that

$$\mathbb{E}\left(\left|\sum_{j=1}^{N} \epsilon_j a_j\right|\right) = 1 \text{ implies that } \left|\left\{t \in [0,1] : \left|\sum_{j=1}^{N} a_j r_j(t)\right| > \frac{1}{2}\right\}\right| \ge \tilde{c} \tag{5}$$

for some universal constant $0 < \tilde{c} < \infty$. Well, suppose $\mathbb{E}\left(\left|\sum_{j=1}^{N} \epsilon_j a_j\right|\right) = 1$. Then

$$\left\|\sum_{j=1}^{N} a_j r_j(t)\right\|_{L^1([0,1])} = 1. \tag{6}$$

By the above lemma we then realize that for some universal constant $0 < C_2 < \infty$ we have

$$\left\|\sum_{j=1}^{N} a_j r_j(t)\right\|_{L^2([0,1])} \le C_2. \tag{7}$$

Equations (6) and (7) imply that $\left|\sum_{j=1}^{N} a_j r_j(t)\right| > \frac{1}{2}$ on a set of measure at least $\tilde{c} > 0$, where $\tilde{c}$ is a universal constant, and so (5) holds, as desired. □

We now provide an example indicating that the inequality opposite to that of Theorem 3 is false. Let $\phi_1, \phi_2, \ldots, \phi_{2^N}$ denote the 2^N functions from $\{1, \ldots, N\}$ to $\{-1, +1\}$. For $j = 1, \ldots, N$, let

$$h_j(x) = \sum_{k=1}^{2^N} \phi_k(j)\, 2^k \chi_{[2^{-(k+1)}, 2^{-k})}(x).$$

Note that

$$\mathbb{E}\left(\left|\sum_{k=1}^{N} \epsilon_k h_k(x)\right|\right) \sim N^{1/2} \sum_{k=1}^{2^N} 2^k \chi_{[2^{-(k+1)}, 2^{-k})}(x),$$

and so $\left\|\mathbb{E}\left(\left|\sum_{k=1}^{N} \epsilon_k h_k\right|\right)\right\|_{WL^1} \sim N^{1/2}$. However, if $\{\epsilon_j\}_{j=1}^{N}$ is a sequence of -1's and $+1$'s, $\phi_\ell(j) = \epsilon_j$ for some $\ell \in \{1, \ldots, 2^N\}$. Hence $\sum_{j=1}^{N} \epsilon_j h_j(x) = N \cdot 2^\ell$ on $[2^{-(\ell+1)}, 2^{-\ell})$, and so $\left\|\sum_{k=1}^{N} \epsilon_k h_k\right\|_{WL^1} \gtrsim N$. As $\{\epsilon_k\}_{k=1}^{N}$ is an arbitrary sequence of -1's and $+1$'s, we see that $\mathbb{E}\left(\left\|\sum_{k=1}^{N}\epsilon_k h_k\right\|_{WL^1}\right)\gtrsim N$. As N may be arbitrarily large, we see that the inequality opposite to that of Theorem 3 cannot hold.

4. A counterexample regarding expectations of weak L^1 norms of random sums

Perhaps the most interesting inequality we could wish for regarding expectations of weak L^1 norms of random sums would be that $\mathbb{E}\left(\left\|\sum_{j=1}^{N}\epsilon_j g_j\right\|_{WL^1}\right)\lesssim \sum_{j=1}^{N}\|g_j\|_{WL^1}$, providing via Theorem 3 an improvement to Theorem 2. Note that the functions h_j constructed at the end of the previous section, although not satisfying $\mathbb{E}\left(\left\|\sum_{j=1}^{N}\epsilon_j h_j\right\|_{WL^1}\right)\lesssim\left\|\mathbb{E}\left(\left|\sum_{j=1}^{N}\epsilon_j h_j\right|\right)\right\|_{WL^1}$, do satisfy such an estimate. However, we shall now construct a counterexample indicating that the inequality $\mathbb{E}\left(\left\|\sum_{j=1}^{N}\epsilon_j g_j\right\|_{WL^1}\right)\lesssim\sum_{j=1}^{N}\|g_j\|_{WL^1}$ in general does not hold, and that, if $\{c_j\}$ is a sequence of positive numbers such that $\sum c_j = 1$ and $\{g_j\}$ is a sequence of functions each having a weak L^1 norm of 1, then the estimate $\mathbb{E}\left(\|\sum\epsilon_j c_j g_j\|_{WL^1}\right)\lesssim\sum c_j\log\left(\frac{1}{c_j}\right)$ following from Theorem 1 is the best possible.

Theorem 4. *Let $N\geq 2$ be a positive integer. Then there exist functions $g_1, g_2, \ldots, g_N$ supported on $[0,1]$ such that $\|g_j\|_{WL^1}\sim 1$ for each j and such that*

$$\mathbb{E}\left(\left\|\sum_{j=1}^{N}\epsilon_j g_j\right\|_{WL^1}\right)\sim N\log N. \tag{8}$$

Proof. Define the function $h_{1,1}(x)$ on $[0,1]$ by

$$h_{1,1}(x) = \begin{cases}\left(\frac{1}{N}+\left(\frac{1-\frac{1}{N}}{N}\right)\cdot j\right)^{-1} & \text{if } \frac{1}{N}+\frac{1-\frac{1}{N}}{N}\cdot(j-1)<x\leq\frac{1}{N}+\frac{\left(1-\frac{1}{N}\right)}{N}j \\ & \text{for some } j=1,\ldots,N, \\ 0 \text{ otherwise.} & \end{cases}$$

For $m=2,\ldots,N$ we define the functions $h_{m,1}(x)$ by

$$h_{m,1}(x) = \begin{cases}\left(\frac{1}{N}+\left(\frac{1-\frac{1}{N}}{N}\right)\cdot(N-m+j+1)\right)^{-1} & \text{if } \frac{1}{N}+\frac{1-\frac{1}{N}}{N}\cdot(j-1)<x \\ & \leq\frac{1}{N}+\frac{\left(1-\frac{1}{N}\right)}{N}j \\ & \text{for some } j=1,\ldots,m-1, \\ \left(\frac{1}{N}+\left(\frac{1-\frac{1}{N}}{N}\right)\cdot(j-m+1)\right)^{-1} & \text{if } \frac{1}{N}+\frac{1-\frac{1}{N}}{N}\cdot(j-1)<x \\ & \leq\frac{1}{N}+\frac{\left(1-\frac{1}{N}\right)}{N}j \\ & \text{for some } j=m,\ldots,N, \\ 0 \text{ otherwise.} & \end{cases}$$

For $k\geq 2$ and $m=1,\ldots,N$, let $h_{m,k}(x)=N^{k-1}h_{m,1}\left(N^{k-1}x\right)$. Note that $h_{m,k}$ is supported on $\left[\left(\frac{1}{N}\right)^k,\left(\frac{1}{N}\right)^{k-1}\right]$.

Now, we again let $\phi_1, \phi_2, \ldots, \phi_{2^N}$ denote the 2^N functions from $\{1, \ldots, N\}$ to $\{-1, 1\}$. For $k = 1, \ldots, N$ we define the functions $g_k(x)$ by

$$g_k(x) = \sum_{j=1}^{2^N} \phi_j(k)\, h_{k,j}(x).$$

Note that $\|g_j\|_{WL^1} \sim \left\|\frac{1}{x}\right\|_{WL^1} \sim 1$ for $j = 1, \ldots, N$. Now, if $\{\epsilon_j\}_{j=1}^N$ is a sequence of -1's and $+1$'s, $\{\epsilon_j\}_{j=1}^N = \{\phi_\ell(j)\}_{j=1}^N$ for some $1 \leq \ell \leq 2^N$. So for $x \in \left(N^{-\ell}, N^{-\ell+1}\right)$ we have

$$\sum_{j=1}^N \epsilon_j g_j(x) \sim N^\ell \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N}\right) \sim N^\ell \log N.$$

So

$$\left\| \left(\sum_{j=1}^N \epsilon_j g_j \right) \chi_{[N^{-\ell}, N^{-\ell+1}]} \right\|_{WL^1} \sim N^{-\ell+1} \left(N^\ell \log N\right) = N \log N.$$

Since $\{\epsilon_j\}_{j=1}^N$ is an arbitrary sequence of -1's and $+1$'s, we may conclude that $\mathbb{E}\left(\left\|\sum_{j=1}^N \epsilon_j g_j\right\|_{WL^1}\right) \sim N \log N$, as desired.

□

References

[1] E. M. Stein and N. J. Weiss, *On the convergence of Poisson integrals*, Trans. Amer. Math. Soc. **140** (1969), 35-54. MR0241685 (39:3024)

[2] A. Zygmund, *Trigonometric Series*, Cambridge University Press, 1959. MR0107776 (21:6498)

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