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POINTWISE UNIFORMLY ROTUND NORMS

JAN RYCHTÁŘ

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ABSTRACT. It is shown that some properties of compact spaces K, such as carrying a strictly positive measure or being descriptive, are closely related to renormings of $C(K)$ or $C(K)^*$, respectively, by pointwise uniformly rotund norms.

Let X be a Banach space. If F is a closed, weak* dense subspace of X^*, then a norm $\|.\|$ on X is said to be *F-uniformly rotund* (UR^F) if $\lim_{n\to\infty} f(x_n - y_n) = 0$ for every $f \in F$ and every $x_n, y_n \in X$ such that

$$\|x_n\| = \|y_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|x_n + y_n\| = 2.$$

The norm is called *pointwise uniformly rotund* (*p-UR*) if it is UR^F for some weak* dense $F \subset X^*$ (see [20], [19]). In particular, the norm on $X = Y^*$ is called *weak* uniformly rotund* if it is UR^Y with the canonical embedding $Y \subset X^* = Y^{**}$. The norm $\|.\|$ is called *uniformly rotund in every direction* (*URED*) if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for every $x_n, y_n \in X$ such that $\|x_n\| = \|y_n\| = 1, \lim_{n\to\infty} \|x_n + y_n\| = 2$, and $x_n - y_n \in \operatorname{span}\{z_0\}$ for some $z_0 \in X$.

A measure μ on a compact space K is said to be *strictly positive* if $\mu(U) > 0$ for every nonempty open set $U \subset K$. A compact space K is called a *uniform Eberlein compact* if K is homeomorphic to a weakly compact set in a Hilbert space [3]. A family $\mathfrak{N}$ of subsets of a compact space K is said to be a *network* if every open set in K is a union of members of $\mathfrak{N}$. A compact space K is *descriptive* if there are closed sets $A_n \subset K$ and a network $\mathfrak{N} = \bigcup_n \mathfrak{N}_n$ such that every $\mathfrak{N}_n$ consists of relatively open and pairwise disjoint sets in A_n [18, Lemma 3.1]. A compact space (K, τ) is *fragmentable*, if there is a metric ϱ on K such that for every $\varepsilon > 0$ and every nonempty subset $M \subset K$ there exists an τ-open set $\Omega \subset K$ such that $M \cap \Omega$ is nonempty and has ϱ-diameter less than ε ([7], [16]). A Banach space X is *weakly compactly generated* if there is a weakly compact set $K \subset X$ such that $X = \overline{\operatorname{span}} K$. For unexplained terms used in this paper we refer to [7] and [9].

Clearly, every p-UR norm is URED. URED norms are used in fixed point theory; see e.g. [5]. It turned out that p-UR norms can be used in characterizing some properties of compact spaces as follows.

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Theorem 1. *The space $C(K)$ of continuous functions on a compact space K admits an equivalent pointwise uniformly rotund norm if and only if K carries a strictly positive Radon probability.*

Theorem 2. (a) *For a compact space K, the space $C(K)^*$ admits a pointwise uniformly rotund (in general nondual) norm if and only if the space $L_1(\mu)$ is separable for every Radon probability on K.*

(b) *If K is a descriptive compact space, then $C(K)^*$ admits an equivalent dual pointwise uniformly rotund norm.*

(c) *There is a nondescriptive (fragmentable) compact space K such that $C(K)^*$ admits an equivalent dual pointwise uniformly rotund norm.*

(d) *If K is a fragmentable compact space, then $C(K)^*$ admits an equivalent pointwise uniformly rotund norm. Consequently, the space $L_1(\mu)$ is separable for every Radon probability μ on a fragmentable compact K.*

Theorem 3. *Let μ be a finite measure. Then $L_1(\mu)$ admits an equivalent pointwise uniformly rotund norm if and only if $L_1(\mu)$ is separable.*

Theorem 4. *If a Banach space X admits an equivalent pointwise uniformly rotund norm, then every weakly compact subset of X is a uniform Eberlein compact.*

For any finite measure μ, the space $L_1(\mu)$ admits an equivalent URED norm by [13]; see also [5, Theorem 2.7.16]. Consequently, by Theorem 3, nonseparable $L_1(\mu)$ admits an equivalent URED norm and no p-UR norm. This is connected to [20, Problem 1]. Moreover, every weakly compact subset of $L_1(\mu)$ is a uniform Eberlein compact [1, Section 4]. Thus the converse of Theorem 4 does not hold even in WCG spaces. This is connected to [1, Problem 2.9].

There are fragmentable compact spaces such that $C(K)^*$ admits no dual strictly convex norm (e.g. $[0,\omega_1]$; see [5, Theorem 7.5.2]) and thus no dual p-UR norm (cf. Theorem 2(b) and (d)). It was proved in [21, Theorem 2] that $L_1(\mu)$ is separable for every Radon probability μ on a compact subset of the first Baire class. Thus split interval $S(I)$ is a nonfragmentable compact satisfying the conclusion of Theorem 2(d). By [13] and Kakutani's Theorem, $C(K)^*$ admits an equivalent URED norm for every compact K. The space $C([0,1]^{[0,1]})^*$ does not admit an equivalent p-UR norm, as $L_1(\lambda)$ is nonseparable, where λ is a product of Lebesgue measures on [0,1].

By [18], if $C(K)^*$ admits a dual weak* locally uniformly rotund norm, then K is descriptive. Thus by Theorem 2(b), $C(K)^*$ admits an equivalent dual p-UR norm. By [7, Theorem 5.3.1], if $C(K)^*$ admits a dual strictly convex norm, then K is fragmentable and thus, by Theorem 2(d), $C(K)^*$ admits an equivalent p-UR norm. We do not know if in this case $C(K)^*$ admits an equivalent dual p-UR norm.

As shown in [14], there is a reflexive Banach space that does not admit any equivalent norm that is uniformly rotund in every direction. Thus this space does not admit any equivalent p-UR norm, although it admits an equivalent dual locally uniformly rotund norm.

Proof of Theorem 4. By Šmulyan's type theorem [5, Theorem 2.6.7], if the norm $\|.\|$ on a Banach space X is UR^F, then the limit

$$\lim_{t\to 0}\frac{\|f+tg\|^*-\|f\|^*}{t} \tag{1}$$

exists for every $g\in X^*, \|g\|^*=1$ and is uniform in $f\in F, \|f\|^*=1$, where $\|.\|^*$ is the dual norm to $\|.\|$. In particular, the norm $\|.\|^*$ is uniformly Gâteaux smooth on

F. By [8], the dual unit ball B_{F^*} is a uniform Eberlein compact in weak* topology of F^*. Hence, by [2], F is a subspace of weakly compactly generated space $C(B_{F^*})$.

For a given weak* dense subspace $F \subset X^*$, let an operator $T : X \to F^*$ be given by $T = r \circ i$, where $i : X \to X^{**}$ is the canonical inclusion and $r : X^{**} \to F^*$ is the canonical restriction. The operator T is one-to-one and $\sigma(X, X^*) - \sigma(F^*, F)$ continuous. Since B_{F^*} is a uniform Eberlein compact in $\sigma(F^*, F)$ topology, $T(K)$ is a uniform Eberlein compact for every weakly compact set $K \subset X$. Hence K is a uniform Eberlein compact, and the proof of Theorem 4 is finished.

Note that if F admits a uniformly Gâteaux smooth norm, then F^* admits a weak* uniformly rotund norm (see [5, Theorem 2.6.7]), and thus the norm $|||.|||$ on X defined by

$$|||x|||^2 = \|x\|^2 + \|Tx\|^2$$

is an equivalent UR^F norm.

Proof of Theorem 1. Let μ be a strictly positive Radon probability measure on K. Then the identity map $I : C(K) \to L_2(\mu)$ is one-to-one and with a dense range. Thus the norm $|||.|||$ defined on $C(K)$ by

$$|||f|||^2 = \|f\|^2 + \|If\|^2_{L_2(\mu)}$$

is an equivalent UR^F norm, where $F = \overline{\text{span}}\, I^*\big(L_2(\mu)\big) \subset C(K)^*$.

Conversely, if $C(K)$ admits an equivalent UR^F norm, then F is a subspace of a weakly compactly generated space (see the proof of Theorem 4). Thus $\ell_1(\Gamma)$ is not a subspace of F for any uncountable set Γ; see [9, Chapter 11]. By [17, Lemma 1.3], there is a Radon probability μ on K such that $F \subset L_1(\mu) \subset C(K)^*$. Note that the measure μ is strictly positive as F is weak* dense in $C(K)^*$. This concludes the proof of Theorem 1.

Proof of Theorem 3. If $L_1(\mu)$ is separable, then it admits an equivalent p-UR norm with the same proof as that of [5, Corollary 2.6.9]. Assume that $L_1(\mu)$ is nonseparable and admits an equivalent UR^F norm. We claim that F is norm separable. This means that $L_1(\mu)^*$ is weak* separable, which is a contradiction with [9, Theorem 11.3].

To prove our claim, let us identify $L_1(\mu)^* \cong L_\infty(\mu)$ with $C(\Omega)$, where Ω is a Stonian space for measure μ (see [4, Appendix B] for details). Since the measure μ is finite, the space $L_1(\mu)^*$ admits an equivalent weak* uniformly rotund norm. By Theorem 1, Ω carries a strictly positive probability measure. In particular, Ω has a property ccc, that is, every collection of pairwise disjoint open sets of Ω is countable. Thus we only need to prove the following fact, which is a version of [17, Theorem 4.5(a) and Proposition 4.7].

Fact 5. *Let Ω be a compact space with property ccc and let $X \subset C(\Omega)$ be isomorphic to a subspace of a weakly compactly generated space. Then X is separable.*

Proof. By [7, Theorem 7.2.2], there exists a Markushevich basis of X, i.e. a biorthogonal system $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma} \subset X \times X^*$ such that $\overline{\text{span}}\{x_\gamma; \gamma \in \Gamma\} = X$ and $\{f_\gamma; \gamma \in \Gamma\}$ separates points of X. We may and do assume that $\|x_\gamma\| = 3$. By [10], there exists a decomposition of $\Gamma = \bigcup_{n=1}^\infty \Gamma_n$ such that, for every $n \in \mathbb{N}$,

$$\emptyset \neq \overline{\{x_\gamma; \gamma \in \Gamma_n\}}^{\sigma(X^{**},X^*)} \setminus \{x_\gamma; \gamma \in \Gamma_n\} \subset B_{X^{**}}. \tag{2}$$

Take n such that Γ_n is uncountable and define open sets $U_\gamma \subset \Omega$ by $U_\gamma = \{\omega \in \Omega; |x_\gamma(\omega)| > 2\}$ for $\gamma \in \Gamma_n$. Since Ω has ccc, there is a sequence $\{\gamma_i\}_{i=1}^\infty \subset \Gamma_n$ such that $\bigcap_{i=1}^\infty U_{\gamma_i} \neq \emptyset$ [17, Lemma 4.2]. Thus there is $\omega \in \bigcap_{i=1}^\infty U_{\gamma_i}$ such that $|x_{\gamma_i}(\omega)| > 2$ for every $i \in \mathbb{N}$, a contradiction with (2). Thus X is separable. This concludes the proof of Fact 5 and the proof of Theorem 3 is complete.

Remark. After submission, we learned that Theorem 3 was proved by a different method in [6, Theorem 2.11].

Proof of Theorem 2(a). Theorem follows easily from Theorem 3 and the Kakutani's Theorem; see e.g. [15].

Proof of Theorem 2(b). Let $\|.\|_1$ be the canonical dual norm on $C(K)^*$. Fix the family $\mathfrak{N} = \bigcup_{n=1}^\infty \mathfrak{N}_n$ given by the definition of descriptivity of K. Consider $\mathfrak{N} \subset C(K)^{**}$ by the action $N(\mu) = \mu(N)$ for $N \in \mathfrak{N}$ and $\mu \in C(K)^*$. Let $F = \overline{\text{span}}\mathfrak{N}$. We will show that there is an equivalent dual UR^F norm on $C(K)^*$.

We claim that F is weak* dense in $C(K)^{**}$. Indeed, $\mu(G) = 0$ for all open $G \subset K$ whenever $\mu(N) = 0$ for all $N \in \mathfrak{N}$, as $\mathfrak{N}$ is a σ-isolated network consisting of relatively open pairwise disjoint sets.

Define a norm $\|.\|$ on $C(K)^*$ in four steps, similarly as in [18, Proof of Theorem 3.3]. First, for every $n \in \mathbb{N}$, define a convex function F_n on $C(K)^*$ by

$$F_n(\mu)^2 = \sum_{N \in \mathfrak{N}_n} |\mu|(N)^2.$$

The function F_n is weak* lower semi-continuous on $C(A_n)^*$. Second, for every $n, m \in \mathbb{N}$, define a weak* lower semi-continuous seminorm $\|.\|_{m,n}$ on $C(K)^*$ by

$$\|\mu\|_{m,n}^2 = \inf\Big\{\|\mu - u\|_1^2 + m^{-1}F_n(u)^2; u \in C(A_n)^*\Big\}.$$

Third, define an equivalent dual norm on $C(K)^*$ by

$$\|\mu\|_+^2 = \|\mu\|_1^2 + \sum_{m,n \in \mathbb{N}} 2^{-m-n}\|\mu\|_{m,n}^2.$$

Claim 6.

$$\lim_{\omega\to\infty} (\mu_\omega - \nu_\omega)(N) = 0, \tag{3}$$

for all $n \in \mathbb{N}$, $N \in \mathfrak{N}_n$ *and all positive measures* $\mu_\omega, \nu_\omega \in C(K)^*, \omega \in \mathbb{N}$, *such that* $\|\mu_\omega\|_1 \leq 1, \|\nu_\omega\|_1 \leq 1$, *and*

$$\lim_{\omega\to\infty} 2\|\mu_\omega\|_+^2 + 2\|\nu_\omega\|_+^2 - \|\mu_\omega + \nu_\omega\|_+^2 = 0. \tag{4}$$

Once the claim is proved, finally define a norm $\|.\|$ by

$$\|\mu\|^2 = \inf\Big\{\|\mu_1\|_+^2 + \|\mu_2\|_+^2 \ ; \ \mu_i \in M(K), \mu_i \geq 0, \mu = \mu_1 - \mu_2\Big\}. \tag{5}$$

Using the compactness argument, it follows from the weak* lower semicontinuity of $\|.\|_+$ that the infimum in (5) is attained for every $\mu \in C(K)^*$ and that the norm $\|.\|$ is an equivalent dual norm on $C(K)^*$. Thus (3) holds whenever $\|\mu_\omega\| = 1 = \|\nu_\omega\|$ and $\lim_{\omega\to\infty} \|\mu_\omega + \nu_\omega\| = 2$. Hence the norm $\|.\|$ is UR^F.

Proof of Claim 6. Fix $n \in \mathbb{N}$ and $N \in \mathfrak{N}_n$. From (4) and a convexity argument,

$$\lim_{\omega\to\infty} 2\|\mu_\omega\|_{m,n}^2 + 2\|\nu_\omega\|_{m,n}^2 - \|\mu_\omega + \nu_\omega\|_{m,n}^2 = 0, \tag{6}$$

for every $m \in \mathbb{N}$. From a compactness argument, for every $\omega, m \in \mathbb{N}$, there are positive measures $u_\omega^{m,n}, v_\omega^{m,n} \in C(A_n)^*$ such that

$$\|\mu_\omega\|_{m,n}^2 = \|\mu_\omega - u_\omega^{m,n}\|_1^2 + m^{-1}F_n(u_\omega^{m,n})^2 \text{ and} \tag{7}$$

$$\|\nu_\omega\|_{m,n}^2 = \|\nu_\omega - v_\omega^{m,n}\|_1^2 + m^{-1}F_n(v_\omega^{m,n})^2. \tag{8}$$

Consequently,

$$F_n(u_\omega^{m,n}) \le m\|\mu_\omega\|_{m,n} \le m\|\mu_\omega\|_1 \le m$$

and similarly $F_n(v_\omega^{m,n}) \le m$. By passing to a subsequence, we may assume that

$$\lim_{\omega\to\infty} \|\mu_\omega\|_{m,n} = d_{m,n} = \lim_{\omega\to\infty} \|\nu_\omega\|_{m,n}.$$

The sequence $\{\|\mu\|_{m,n}\}_{m=1}^\infty$ is nonincreasing for every measure $\mu \in C(K)^*$. Thus there is $d_n = \lim_{m\to\infty} d_{m,n}$. Choose $\varepsilon > 0$ and let $m_0 \in \mathbb{N}$ be such that $d_{m_0,n} < d_n + \varepsilon$. We will estimate $|(\mu_\omega - \nu_\omega)(N)|$ by

$$\big|(\mu_\omega - u_\omega^{m_0,n})(N)\big| + \big|(u_\omega^{m_0,n} - v_\omega^{m_0,n})(N)\big| + \big|(v_\omega^{m_0,n} - \nu_\omega)(N)\big|.$$

By a convexity argument and (6), (7), (8),

$$\lim_{\omega\to\infty} 2F_n(u_\omega^{m_0,n})^2 + 2F_n(v_\omega^{m_0,n})^2 - F_n(u_\omega^{m_0,n} + v_\omega^{m_0,n})^2 = 0.$$

Since $u_\omega^{m_0,n}$ and $v_\omega^{m_0,n}$ are positive measures, by a convexity argument again

$$\lim_{\omega\to\infty}\big|(u_\omega^{m_0,n} - v_\omega^{m_0,n})(N)\big| = 0.$$

In order to estimate $|(\mu_\omega - u_\omega^{m_0,n})(N)|$, consider a measure

$$u = \mu_\omega\restriction_N + u_\omega^{m_0,n}\restriction_{K\setminus N}$$

in the definition of $\|\mu_\omega\|_{m,n}$, where $\mu\restriction_A$ means the restriction of μ on $A \subset K$. We get

$$\begin{aligned}
\|\mu_\omega\|_{m,n}^2 &\le \|(\mu_\omega - u_\omega^{m_0,n})\restriction_{K\setminus N}\|_1^2 + m^{-1}F_n(\mu_\omega\restriction_N + u_\omega^{m_0,n}\restriction_{K\setminus N})^2 \\
&\le \|(\mu_\omega - u_\omega^{m_0,n})\restriction_{K\setminus N}\|_1^2 + m^{-1}\big(F_n(\mu_\omega\restriction_N) + F_n(u_\omega^{m_0,n}\restriction_{K\setminus N})\big)^2 \\
&\le \|(\mu_\omega - u_\omega^{m_0,n})\restriction_{K\setminus N}\|_1^2 + m^{-1}\big(\mu_\omega(N) + F_n(u_\omega^{m_0,n})\big)^2 \\
&\le \|(\mu_\omega - u_\omega^{m_0,n})\restriction_{K\setminus N}\|_1^2 + m^{-1}(1+m_0)^2.
\end{aligned}$$

Thus, for all $m \in \mathbb{N}$,

$$\begin{aligned}
\limsup_{\omega\to\infty} \|(\mu_\omega - u_\omega^{m_0,n})\restriction_{K\setminus N}\|_1^2 &\ge \lim_{\omega\to\infty}\|\mu_\omega\|_{m,n}^2 - m^{-1}(1+m_0)^2, \\
\limsup_{\omega\to\infty} \|(\mu_\omega - u_\omega^{m_0,n})\restriction_{K\setminus N}\|_1^2 &\ge d_{m,n}^2 - m^{-1}(1+m_0)^2, \text{ and} \\
\limsup_{\omega\to\infty} \|(\mu_\omega - u_\omega^{m_0,n})\restriction_{K\setminus N}\|_1^2 &\ge d_n^2.
\end{aligned}$$

For all $\omega \in \mathbb{N}$ we have

$$\begin{aligned}
|(\mu_\omega - u_\omega^{m_0,n})(N)| &\le \|(\mu_\omega - u_\omega^{m_0,n})\restriction_N\|_1 \\
&= \|\mu_\omega - u_\omega^{m_0,n}\|_1 - \|(\mu_\omega - u_\omega^{m_0,n})\restriction_{K\setminus N}\|_1 \\
&\le \|\mu_\omega\|_{m_0,n} - \|(\mu_\omega - u_\omega^{m_0,n})\restriction_{K\setminus N}\|_1.
\end{aligned}$$

Thus

$$\liminf_{\omega\to\infty}\left|(\mu_\omega - u_\omega^{m_0,n})(N)\right| \le d_{m_0,n} - d_n \le \varepsilon.$$

The same estimate holds for $|(\nu_\omega - v_\omega^{m_0,n})(N)|$. The proof of Claim 6 is complete.

Proof of Theorem 2(c). First, we prove the following claim, which is a version of [11, Theorem 7.1].

Claim 7. *Suppose that on a tree T there is an increasing function $\varrho : T \to \mathbb{R}$ which is constant on no strictly increasing sequence in T. Then there is an equivalent dual p-UR norm on $C_0(T)^*$.*

Proof of Claim 7. The space $C_0(T)^*$ can be identified with $\ell_1(T)$ with the canonical dual norm $\|\mu\|_1 = \sum_{t\in T}|\mu(t)|$. Let us define T^+ as the set of successors and T_0 as the set of all $t \in T^+$ such that $\varrho(t) > \varrho(t^-)$. We may modify the function ϱ so that it takes rational values at all points of T_0.

We will show that there is an equivalent dual UR^F norm where

$$F = \overline{\operatorname{span}}\Big\{\big\{\{s\}; s\in T^+\big\} \cup \big\{[s,\infty); s \in T_0\big\} \cup \{T\}\Big\} \subset \ell_\infty(T).$$

We claim that F is weak*-dense in $C_0(T)^{**}$. To prove it, let $\mu \in C(T)^*$ be such that $\mu(f) = 0$ for all $f \in F$. We want to show that $\mu(\{t\}) = 0$ for all $t \in T$. Choose $t \in T$ and put $A(t) = \{u; u \in (t,\infty), \varrho(u) = \varrho(t)\}$ and $B(t) = \min\{u \in (t,\infty); \varrho(u) > \varrho(t)\}$. We have

$$(t,\infty) = \bigcup_{u\in A(t)}\{u\} \cup \bigcup_{u\in B(t)}[u,\infty).$$

The union above is a union of disjoint open sets and $|\mu|$ is nonzero at most on countably many of them. Hence $\mu\big((t,\infty)\big) = 0$. Thus $\mu\big([t,\infty)\big) = 0$ for all $t \in T^+$. Since

$$(0,t] = T \setminus \bigcup_{s\le t}\Big(\bigcup_{r\in s^+\setminus(0,t]}[r,\infty)\Big),$$

we have that $\mu\big((0,t]\big) = 0$ for all $t \in T$. Every limit element $t \in T$ is a limit of a sequence (of elements of T_0), thus $\mu((0,t)) = 0$ for all $t \in T$. Hence $\mu\big(\{t\}\big) = 0$ for all $t \in T$.

For every $q \in \mathbb{Q}$, the wedges $[s,\infty)$ with $s \in T_0$ and $\varrho(s) = q$ are disjoint, so we can define an equivalent dual norm on $C(T)^*$ by

$$\|\mu\|_+^2 = \|\mu\|_1^2 + \sum_{s\in T^+}\|\mu\restriction_{\{s\}}\|_1^2 + \sum_{q\in\mathbb{Q}} c_q\Big(\sum_{s\in T_0\cap\varrho^{-1}(q)}\|\mu\restriction_{[s,\infty)}\|_1^2\Big),$$

where c_q are some positive constants.

Let $\mu_n, \nu_n \in C_0(T)^*$ be positive elements such that $\|\mu_n\| \le 1, \|\nu_n\| \le 1$ and

$$\lim_{n\to\infty} 2\|\mu_n\|_+^2 + 2\|\nu_n\|_+^2 - \|\mu_n + \nu_n\|_+^2 = 0.$$

A standard convexity argument shows that

$$\lim_{n\to\infty}(\mu_n - \nu_n)(T) = 0, \lim_{n\to\infty}(\mu_n - \nu_n)(s) = 0,$$

for all $s \in T^+$, and

$$\lim_{n\to\infty}(\mu_n - \nu_n)\big([s,\infty)\big) = 0,$$

for any $s \in T_0$. Thus the norm $\|.\|$ defined by (5) is UR^F. This concludes the proof of Claim 7.

Now, let Λ be a tree defined in [11, Section 10] and let K be its Alexandroff compactification. Then $C(K)^*$ admits an equivalent dual p-UR norm by Claim 7. The space $C(K)^*$ does not admit any equivalent dual locally uniformly rotund norm, since $C(K)$ does not admit an equivalent Fréchet smooth norm [11, Corollary 10.9]. Thus K is not a descriptive compact space by [18, Corollary 4.9]. The proof of Theorem 2(c) is complete.

Proof of Theorem 2(d). Let K be a fragmentable compact. By [7, Theorem 5.1.9 and Proof of Theorem 5.1.12(iii)], there is a family $\mathfrak{U} = \bigcup_{n=1}^{\infty} \mathfrak{U}_n$ of subsets of K such that

(1) $\mathfrak{U}$ is a separating family, i.e. if $x \neq y \in K$, then there is $U \in \mathfrak{U}$ such that $\#U \cap \{x, y\} = 1$;
(2) $\mathfrak{U}$ is a network;
(3) for every $n \in \mathbb{N}$, $\mathfrak{U}_n$ is an open partitioning, i.e. $\mathfrak{U}_n = \{U_\xi; \xi < \xi_n\}$ is well ordered such that U_ξ is contained and is relatively open in $K \setminus (\bigcup_{\eta<\xi} U_\eta)$ for every $\xi < \xi_n$ and $K = \bigcup_{\xi<\xi_n} U_\xi$;
(4) for every $U \in \mathfrak{U}_{n+1}$ there is $V \subset \mathfrak{U}_n$ such that $\overline{U} \subset V$.

As $\mathfrak{U}_n$ is an open partitioning, it follows that

$$\sum_{U\in\mathfrak{U}_n} \mu(U) = \mu(K).$$

Define equivalent norms on $C(K)^*$

$$\|\mu\|_+^2 = |\mu|^2(K) + \sum_{n=1}^{\infty} 2^{-n} \sum_{U\in\mathfrak{U}_n} |\mu|^2(U)$$

and

$$\|\mu\|^2 = \inf\{\|\mu_1\|_+^2 + \|\mu_2\|_+^2; \mu_i \in C(K)^*, \mu_i \geq 0, \mu = \mu_1 - \mu_2\}. \tag{9}$$

From a definition of a norm $\|.\|_+$ it follows that $\|\mu\|^2 = \|\mu^+\|_+^2 + \|\mu^-\|_+^2$. Let $F = \overline{\text{span}}\{U; U \in \mathfrak{U}\} \subset C(K)^{**}$. We will show that the norm $\|.\|$ is UR^F. Note that $F \subset C(K)^{**}$ is weak* dense. Indeed, assume $\mu(U) = 0$ for all $U \in \mathfrak{U}$ and let $G \subset K$ be an open set. Since $\mathfrak{U}$ is a network, we have $G = \bigcup_n(\bigcup \mathfrak{U}_n')$, where for every $n \in \mathbb{N}$, $\mathfrak{U}_n'$ is a subfamily of $\mathfrak{U}_n$. Moreover, by condition (4), we may assume that $\mathfrak{U}_n' \cap \mathfrak{U}_m' = \emptyset$ for $m \neq n$. Thus

$$\mu(G) = \mu(\bigcup_n(\bigcup \mathfrak{U}_n')) = \sum_n \mu(\bigcup \mathfrak{U}_n') = \sum_n \sum_{U\in\mathfrak{U}_n'} \mu(U) = 0,$$

where the third equality hold as $\mathfrak{U}_n'$'s are relatively open partitioning. Thus, by a convexity argument, the norm $\|.\|$ is UR^F.

The proof of Theorem 2(d) is complete.

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Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

E-mail address: jrychtar@math.ualberta.ca

Current address: Department of Mathematical Sciences, University of North Carolina at Greensboro, Greensboro, North Carolina 27402

E-mail address: rychtar@uncg.edu