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THE THETA DIVISOR AND THE STICKELBERGER THEOREM

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Abstract. This paper is devoted to studying certain trivial correspondences provided by theta divisors and their relation to the Brumer-Stark conjecture.

1. Introduction

In the works [An2], [C] a proof is given of the Brumer-Stark conjecture in the case of abelian extensions of $\mathbb{F}_q(t)$. It is obtained by considering Frobenius correspondences attached to L-series and by proving that they are linearly equivalent to trivial correspondences. In [An2], this result it is proved by considering the theory of theta functions. In this paper we generalize these results for arbitrary curves; we again obtain correspondences attached to L-series and, by using the fact that two theta divisors are linearly equivalent, we prove that these correspondences are linearly equivalent to trivial correspondences. For elliptic curves, we obtain explicit functions, which render these correspondences trivial. These results provide us with another proof for the Brumer part of the Brumer-Stark conjecture in the function field case (cf. [T], Conjecture 6.2, p. 107, and [H], Theorem 1.1). For the extensions $H_{\mathfrak{m}}/K$, K is a global function field with a distinguished place, ∞, of degree 1, and $\mathfrak{m}$ is an effective divisor supported away ∞, where $H_{\mathfrak{m}}$ is the maximal abelian extension of the conductor $\mathfrak{m}$ in which ∞ splits completely. I believe that the interest of this proof lies in the study of the Brumer-Stark conjecture in the non-abelian case for function fields, perhaps because it provides some information about the number field case. All this work is strongly based on the results of [An1] and perhaps these results give some hints to [An3].

2. Preliminaries about the generalized jacobian and the theta divisor

Let X be a projective, smooth and absolutely irreducible curve over $\mathbb{F}_q$ ($q = p^m$) of genus g; ∞ is a rational point in X, $A := H^0(X \setminus \{\infty\}, \mathcal{O}_X)$, $\mathfrak{m}$ is an ideal in A, such that the effective divisor $D(\mathfrak{m})$ associated with $\mathfrak{m}$ is of degree $d+1$, $T = supp(\mathfrak{m})$, $Spec(A_T) = X \setminus T$ and K is the function field for X. R is an $\mathbb{F}_q$-algebra.

Let L be a line bundle over $X \times Spec(R)$. An $\mathfrak{m}$-level structure on L is a pair $(L, f_{\mathfrak{m}})$, where $f_{\mathfrak{m}}$ is an epimorphism of modules

$$f_{\mathfrak{m}} : L \to \mathcal{O}_X/\mathfrak{m} \cdot \mathcal{O}_X \otimes R.$$

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We denote by $f_{\mathfrak{m}}(r)$ the $\mathfrak{m}$-level structure on $L(r)$, given by the natural inclusion $L \hookrightarrow L(r)$, and $f_{\mathfrak{m}}$, $r \in \mathbb{N}$. $L(r)$ denotes $L \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(r \cdot \infty)$.

Given $(L, f_{\mathfrak{m}})$, one defines

$$H^0_{\mathfrak{m}}(X, (L, f_{\mathfrak{m}})) := \{s \in H^0(X, L) : f_{\mathfrak{m}}(s) \equiv \alpha \ mod \ \mathfrak{m}\},$$

$\alpha \in \mathbb{F}_q$. The moduli scheme for the pairs $(L, f_{\mathfrak{m}})$, with $deg(L) = 0$, is given by the generalized jacobian $Pic^0_{X,\mathfrak{m}}$; cf. [Se], chapter V.

By [Se], chapter IV, there exists a singular curve, $X_{\mathfrak{m}}$, that is only singular in a point Q. Its normalized curve is X, and the conductor of $\rho : X \to X_{\mathfrak{m}}$ is given by $A/\mathbb{F}_q + \mathfrak{m}$. Moreover, there exists an equivalence between line bundles on $X_{\mathfrak{m}}$ and line bundles over X with $\mathfrak{m}$-level structures; see [F]. In this way, $Pic^0_{X_{\mathfrak{m}}} = Pic^0_{X,\mathfrak{m}}$ and

$$H^0_{\mathfrak{m}}(X, (L, f_{\mathfrak{m}})) = H^0(X_{\mathfrak{m}}, L'),$$

$(L, f_{\mathfrak{m}})$ being the $\mathfrak{m}$-level structure associated with the line bundle L' on $X_{\mathfrak{m}}$. L' is said to be of degree 0 if $\rho^* L' = L$ is of degree 0.

We denote by $I^r_{\mathfrak{m}}$, $O^{\times}_{\mathfrak{m}}$ the group of ideles of X outside T of degree r, and the group of ideles outside T, without zeroes and poles, respectively. We denote by $K(X)^{\times}_{\mathfrak{m}}$ the functions $a \in K^{\times}$ with $a \equiv \alpha \ mod \ \mathfrak{m}$, $\alpha \in {\mathbb{F}_q}^{\times}$. One notes that the rational points of $Pic^0_{X,\mathfrak{m}}$ are given by $G_{\mathfrak{m}} := I^0_{\mathfrak{m}} / K(X)^{\times}_{\mathfrak{m}} \cdot O^{\times}_{\mathfrak{m}}$.

Given two divisors $N, \bar{N} \in I_{\mathfrak{m}}/O^{\times}_{\mathfrak{m}}$ (= fraction ideals on A_T), we say that they are $\mathfrak{m}$-equivalent (cf. [Se], chapter V, no. 2) when $[N - \bar{N}] = 0 \in G_{\mathfrak{m}}$. $[N]$ denotes the equivalence class of N in $G_{\mathfrak{m}}$. The set of effective divisors $\mathfrak{m}$-equivalent to N is given by the set

$$\mathbb{P}(H^0(X_{\mathfrak{m}}, \mathcal{O}_{X_{\mathfrak{m}}}(N))) \setminus \mathbb{P}(H^0(X, \mathcal{O}_X(N - D(\mathfrak{m})))).$$

$\mathcal{O}_{X_{\mathfrak{m}}}(N)$ denotes the line bundle on $X_{\mathfrak{m}}$ associated with N. We denote by $L_{\mathfrak{m}}(N, 0)$ the cardinal of this last set.

$\mu_{\mathfrak{m}}$ denotes the natural morphism of forgetting level structures

$$\mu_{\mathfrak{m}} : Pic^0_{X,\mathfrak{m}} \to Pic^0_X.$$

One can consider the theta divisor on Pic^0_X, $\Theta(g + d, \mathfrak{m})$ defined by

$$\Theta(g + d, \mathfrak{m}) = \{L \in J \text{ with } H^0(X, L(g + d)(-D(\mathfrak{m}))) \neq 0\}.$$

Definition 2.1. We define the theta divisor, $\Theta_{\mathfrak{m}}$, as the closed subscheme of $Pic^0_{X,\mathfrak{m}}$ defined by the $(L, f_{\mathfrak{m}})$, such that

$$H^0_{\mathfrak{m}}(X, (L(g + d - 1), f_{\mathfrak{m}}(g + d - 1))) \neq 0.$$

In the following proposition we prove that, $\mu^{-1}_{\mathfrak{m}}\Theta(g + d, \mathfrak{m})$ is linearly equivalent to $\Theta_{\mathfrak{m}}$. Owing to a subsequent calculation, we shall assume $d \geq g$, although the result is true for any $d \in \mathbb{N}$. We show which is the meromorphic function, b, on $Pic^0_{X,\mathfrak{m}}$ such that $D(b) = \Theta_{\mathfrak{m}} - \mu^{-1}_{\mathfrak{m}}\Theta(g + d, \mathfrak{m})$. Here, $D(b)$ denotes the Weil divisor associated with b.

Proposition 2.2. *$\gamma^{-1}(\mu^{-1}_{\mathfrak{m}}(\Theta(g + d, \mathfrak{m})))$ is linearly equivalent to $\gamma^{-1}(\Theta_{\mathfrak{m}})$ over S, γ being a morphism of schemes $\gamma : S \to Pic^0_{X,\mathfrak{m}}$.*

Proof. Let $(M, g_{\mathfrak{m}})$ be the $\mathfrak{m}$-level structure given by γ. Let p_2 be the natural projection $X \times S \to S$. As $d \geq g$, $p_{2*}M(g - 1 + d)$ and $p_{2*}M(g + d)$ are locally free over $\mathcal{O}_S$, of ranks d and $d+1$, respectively. We can normalize M such that $p_{2*}(M/M(-\infty))$ is trivial. We thus have the exact sequence of $\mathcal{O}_S$-modules

$$0 \to p_{2*}M(g-1+d) \to p_{2*}M(g+d) \xrightarrow{\eta} \mathcal{O}_S \to 0.$$

Bearing in mind this exact sequence, one can obtain an open covering of S, $\{U_y\}_{y\in S}$, bases $\{s_y^1, \cdots, s_y^d\}$ and $\{s_y^1, \cdots, s_y^d, e_y\}$ for $p_{2*}M(g-1+d)_{U_y}$ and $p_{2*}M(g+d)_{U_y}$ respectively, where, fixing $\xi \in H^0(S, \mathcal{O}_S)^\times$, $\eta_{|U_y}(e_y) = \xi_{|U_y}$.

$\gamma^{-1}(\Theta_{\mathfrak{m}})$ and $\gamma^{-1}(\mu_{\mathfrak{m}}^{-1}(\Theta(g+d, \mathfrak{m})))$ are defined over U_y by the zero locus of the functions of $H^0(U_y, \mathcal{O}_S)$

$$b_y^1 := det(p_{2*}g_{\mathfrak{m}}(g-1+d)(s_y^1), \cdots, p_{2*}g_{\mathfrak{m}}(g-1+d)(s_y^d), 1)$$

and

$$b_y^2 := det(p_{2*}g_{\mathfrak{m}}(g+d)(s_y^1), \cdots, p_{2*}g_{\mathfrak{m}}(g+d)(s_y^d), p_{2*}g_{\mathfrak{m}}(g+d)(e_y)),$$

respectively, where $p_{2*}g_{\mathfrak{m}}(r)(s_y^i)$ are the values of s_y^i over $H^0(U_k, \mathcal{O}_S)\otimes A/\mathfrak{m}$, via the morphism $p_{2*}g_{\mathfrak{m}}(r) : p_{2*}g_{\mathfrak{m}}(r) \to H^0(U_y, \mathcal{O}_S) \otimes A/\mathfrak{m}$ induced by $g_{\mathfrak{m}}$. To calculate this determinant, we choose a basis in $A/\mathfrak{m}$. By the choice of this basis, it is not hard to prove that $b_y^1/b_{\bar{y}}^1 = b_y^2/b_{\bar{y}}^2 \in H^0(U_y \cap U_{\bar{y}}, \mathcal{O}_S)$ for each $y, \bar{y} \in S$. We thus have

$$D(b_y^1/b_y^2) = \gamma^{-1}(\Theta_{\mathfrak{m}}) - \gamma^{-1}(\mu_{\mathfrak{m}}^{-1}(\Theta(g+d, \mathfrak{m}))).$$

□

3. Elliptic sheaves of rank 1

Definition 3.1 ([Dr])**.** An A-elliptic sheaf of rank 1 over R is a diagram of line bundles over $X \times Spec(R)$,

$$\begin{tikzcd} L \arrow[r, "i"] & L(1) \\ F^{\#}L \arrow[ur, "t"] & \end{tikzcd},$$

satisfying:

a) For any $z \in Spec(R)$, $deg(L_z) = 0$.

b) $j_{\infty *}(L(1)/L)$ is a rank-one free module over R. j_∞ is the inclusion $\infty \times Spec(R) \hookrightarrow X \times Spec(R)$.

$F^{\#}$ denotes $(Id \times F)^*$, $F : Spec(R) \to Spec(R)$ being the q-Frobenius morphism.

The latter elliptic sheaf will be denoted by (L, i, t).

Definition 3.2. An $\mathfrak{m}$-level structure for an A-elliptic sheaf of rank 1 over R, (L, i, t), is an $\mathfrak{m}$-level structure for L such that the following diagram is commutative:

$$\begin{tikzcd} F^{\#}L \arrow[r, "t"] \arrow[d, "F^{\#}f_{\mathfrak{m}}"'] & L(1) \arrow[dl, "f_{\mathfrak{m}}(1)"] \\ \mathcal{O}_X/\mathfrak{m}\cdot\mathcal{O}_X \otimes R & \end{tikzcd}.$$

This elliptic sheaf with an $\mathfrak{m}$-level structure is denoted by $(L, i, t, f_{\mathfrak{m}})$.

Let us consider the Lang isogeny $Id - F : Pic^0_{X,\mathfrak{m}} \to Pic^0_{X,\mathfrak{m}}$; $\alpha_\mathfrak{m} : Spec(A_T) \to Pic^0_{X,\mathfrak{m}}$ is the Abel morphism given by the $\mathfrak{m}$-level structure

$$(\mathcal{O}_{X\times Spec(A_T)}(\infty - \Delta_T), f_\mathfrak{m}(1)),$$

where this level structure is obtained from the exact sequence of $\mathcal{O}_{X\times Spec(A_T)}$-modules ($Spec(A_T) = X \setminus T$)

$$0 \to \mathcal{O}_{X\times Spec(A_T)}(-\Delta_T) \to \mathcal{O}_{X\times Spec(A_T)} \overset{\delta_\mathfrak{m}}{\to} j^* j_* A_T \to 0,$$

$j : Spec(A_T) \to X$ being the natural inclusion and $\delta_\mathfrak{m}$ the multiplication morphism.

There exists a fine moduli, $Y_\mathfrak{m}$, for the A-elliptic sheaves of rank 1 with $\mathfrak{m}$-level structures. This moduli is affine; $Y_\mathfrak{m} = Spec(B_\mathfrak{m})$. It is obtained by

$$Y_\mathfrak{m} = Spec(A_T) \mathop{\Large\times}_{Pic^0_{X,\mathfrak{m}}} Pic^0_{X,\mathfrak{m}}.$$

$Pic^0_{X,\mathfrak{m}}$ is a $Pic^0_{X,\mathfrak{m}}$-scheme via the Lang isogeny; cf. [Se].

From the definition of $Y_\mathfrak{m}$, there exists a morphism $\pi_\mathfrak{m} : Y_\mathfrak{m} \to Spec(A_T)$, and we obtain the abelian extension of group $G_\mathfrak{m}$, $K \to H_\mathfrak{m}$. $H_\mathfrak{m}$ denotes the function field for $Y_\mathfrak{m}$, and it is the ray class field for the module $\mathfrak{m}$ at ∞(= the maximal abelian extension of conductor $\mathfrak{m}$ in which ∞ splits completely); cf. [H], [Se], chapter IV. The action of $[N] \in G_\mathfrak{m}$ over $Y_\mathfrak{m}$ is defined by the tensorial product over $(L, i, t, f_\mathfrak{m})$ of the $\mathfrak{m}$-level structure associated with $[N]$. We denote by $\sigma_N := ([N], H_\mathfrak{m}/K)$ the automorphism associated with $[N]$. $(\quad, H_\mathfrak{m}/K)$ denotes the Artin symbol. The Frobenius element for $x \in Spec(A_T)$, σ_x, is given by $[t_x]$, where t_x is a local parameter for x.

We can now obtain the $T(= supp(\mathfrak{m}))$-incomplete L-function evaluator at $s = 0$:

$$\theta_{H_\mathfrak{m}/K,T} = \prod_{x\in X\setminus T} (1 - \sigma_x \cdot t^{deg(x)})^{-1}_{t=1}.$$

Similar to [An1], 4.1.1, bearing in mind that that σ_∞ acts trivially on $Y_\mathfrak{m}$, we obtain

$$\theta_{H_\mathfrak{m}/K,T} = \sum_{[N]\in G_\mathfrak{m}} \sigma_N \cdot \Big(\sum_{i=0}^{2g+d-1} L_\mathfrak{m}(N,i)\cdot t^i + \sum_{j\geq 0} q^{g+j}\cdot t^{2g+d+j}\Big)_{t=1},$$

where $L_\mathfrak{m}(N,i)$ denotes the cardinal of the set of effective divisors on X supported away from T (or equivalently the ideals on A_T) and $\mathfrak{m}$-equivalent to $N + i\cdot\infty$. One can choose $N \in [N]$ supported away from ∞.

3.1. **The theta divisor and elliptic sheaves of rank** 1. Let us consider the morphism defined in [An1], 4.1, this time in the case of level structures, $\beta_\mathfrak{m} : Y_\mathfrak{m} \times Y_\mathfrak{m} \to Pic^0_{X,\mathfrak{m}}$, over the rational points given by

$$\beta_\mathfrak{m}((L, i, t, f_\mathfrak{m}), (M, \bar{i}, \bar{t}, \bar{f}_\mathfrak{m})) := (M^\vee \underset{\mathcal{O}_X}{\otimes} L, \bar{f}^\vee_\mathfrak{m} \otimes f_\mathfrak{m}).$$

Let N be a divisor of degree 0, supported away from T and ∞. For easy notation, we shall denote $H^0_\mathfrak{m}(N) := H^0_\mathfrak{m}(\mathcal{O}_X, \mathcal{O}_X(N)) = H^0_\mathfrak{m}(\mathcal{O}_{X_\mathfrak{m}}, \mathcal{O}_{X_\mathfrak{m}}(N))$ and $H^0(N) := H^0(\mathcal{O}_X, \mathcal{O}_X(N))$. Let us denote by F^N_i the Weil divisor on $Y_\mathfrak{m} \times Y_\mathfrak{m}$ given by the graph of the composition of morphism $F \overset{i}{\cdots} F$ and σ_N. Let us denote by F^i_N the transpose correspondence of F_i^{-N}. Then, by [An1], 4.1.1, $\beta_{\mathfrak{m}}^{-1}(\mu_{\mathfrak{m}}^{-1}(\Theta(g+d,\mathfrak{m})))$ is

$$\sum_{[N]\in G_{\mathfrak{m}}}(\sum_{i=0}^{g+d-1} C(i,N)\cdot F_N^{g+d-i-1} + \sum_{j=1}^{g-1} C(g+d+j-1,N)\cdot F_j^N).$$

Here,

$$C(i,N) = \#\mathbb{P}(H^0(\mathcal{O}_X(i+1)(N-D(\mathfrak{m})))$$

and

$$C(g+d+j-1,N) = \frac{\#\mathbb{P}(H^0(\mathcal{O}_X(g+d+j)(N-D(\mathfrak{m}))) - 1 - \cdots - q^{j-1}}{q^j}.$$

In the case of $\beta_{\mathfrak{m}}^{-1}(\Theta_{\mathfrak{m}})$, one obtains a similar result:

Lemma 3.3. *If* $(L,i,t,f_{\mathfrak{m}})$ *and* $(M,\bar{i},\bar{t},\bar{f}_{\mathfrak{m}})$ *are elliptic sheaves with level structures with* $M(1)/t(F^{\#}M) \simeq L(1)/t(F^{\#}L)$ *and defined over a field* k*, then there exists* $[N]\in G_{\mathfrak{m}}$ *with* $\sigma_N(L,i,t,f_{\mathfrak{m}}) = (M,\bar{i},\bar{t},\bar{f}_{\mathfrak{m}})$.

Proof. From the commutative diagram

$$\begin{tikzcd}
Y_{\mathfrak{m}} \arrow[r] \arrow[d, "\pi_{\mathfrak{m}}"] & Pic^0_{X,\mathfrak{m}} \arrow[d, "Id-F"] \\
Spec(A_T) \arrow[r, "\alpha_{\mathfrak{m}}"] & Pic^0_{X,\mathfrak{m}},
\end{tikzcd}$$

we deduce that $\pi_{\mathfrak{m}}(L,i,t,f_{\mathfrak{m}})$ is given by the point where $L(1)/t(F^{\#}L)$ is concentrated, and we conclude the proof because $\pi_{\mathfrak{m}}$ is a Galois covering of group $G_{\mathfrak{m}}$. ☐

Proposition 3.4. $\beta_{\mathfrak{m}}^{-1}(\Theta_{\mathfrak{m}})$ *is a Weil divisor on* $Y_{\mathfrak{m}}\times Y_{\mathfrak{m}}$*, given by*

$$\sum_{[N]}(\sum_{i=0}^{g+d-1} C_{\mathfrak{m}}(i,N)\cdot F_N^{g+d-i-1} + \sum_{j=1}^{g} C_{\mathfrak{m}}(g+d+j-1,N)\cdot F_j^N).$$

"$C_{\mathfrak{m}}(\quad,\quad)$*"* $\in\mathbb{Z}$ *are calculated in the following proposition.*

Proof. Bearing in ming the last lemma it suffices to prove that if

$$((L,i,t,f_{\mathfrak{m}}),(M,\bar{i},\bar{t},\bar{f}_{\mathfrak{m}}))\in\beta_{\mathfrak{m}}^{-1}(\Theta_{\mathfrak{m}})$$

is defined over a geometric point $Spec(k(s))$, then either there exists i with $0\le i\le g+d-1$ and $(F^{\#})^i(L(1)/F^{\#}L)\simeq M(1)/F^{\#}M$, or there exists $1\le j\le g$ with $(F^{\#})^j(M(1)/F^{\#}M)\simeq L(1)/F^{\#}L$. Recall that $F^{\#}(L,i,t,f_{\mathfrak{m}})$ is again an elliptic sheaf with an $\mathfrak{m}$-level structure.

Because of the definition of $H^0_{\mathfrak{m}}(X_s,\quad)$ and since we have

$$H^0_{\mathfrak{m}}(X_{\mathfrak{m}}\otimes k(s), M^{\vee}\underset{\mathcal{O}_{X_s}}{\otimes} L(g+d-1), \bar{f}_{\mathfrak{m}}^{\vee}\otimes f_{\mathfrak{m}})\neq 0,$$

there exists either a morphism $h : M \to L(g+d-1)(-D(\mathfrak{m}))$ or a commutative diagram

$$\begin{tikzcd}
M \arrow[rr, "h"] \arrow[dr, "\bar{f}_{\mathfrak{m}}"] & & L(g+d-1) \arrow[dl, "f_{\mathfrak{m}}(g+d-1)"'] \\
& \mathcal{O}_X/\mathfrak{m}\cdot\mathcal{O}_X \otimes k(s) & \\
F^{\#}M(-1) \arrow[uu, "\bar{t}"] \arrow[ur, "F^{\#}\bar{f}_{\mathfrak{m}}(-1)"] \arrow[rr, "F^{\#}h(-1)"] & & F^{\#}L(g+d-2). \arrow[uu, "t(g+d-1)"'] \arrow[ul, "F^{\#}f_{\mathfrak{m}}(g+d-2)"']
\end{tikzcd}$$

In the first case, we conclude because of [An1], 3.3.1. In the second case, we obtain a morphism of modules

$$h\cdot\bar{t} - t(g+d-1)\cdot F^{\#}h(-1) : F^{\#}M(-1) \to L(g+d-1)(-D(\mathfrak{m})).$$

If $h\cdot\bar{t} - t(g+d-1)\cdot F^{\#}h(-1) = 0$, it is not hard to prove that there exists i, $0 \le i \le g+d-1$, with $(F^{\#})^i(L(1)/F^{\#}L) \simeq M(1)/F^{\#}M$. If $h\cdot\bar{t} - t(g+d-1)\cdot F^{\#}h(-1) \neq 0$, we again use [An1], 3.3.1. □

To calculate the integers $C_{\mathfrak{m}}(i, N)$ and $C_{\mathfrak{m}}(g+d+j-1, N)$, it suffices to bear in mind in [An1] the following considerations: Let R be an $\mathbb{F}_q$-finite local algebra, $h(z) \in R[z]$. Then, given $[D] \in G_{\mathfrak{m}}$, where D is supported away from ∞, one can obtain in an analogous way to [An1], 1.5, a line bundle, $\mathcal{F}_{\mathfrak{m}}(D, h(z))$, over $X_{\mathfrak{m}} \times Spec(R)$. Let $\mathcal{A}_{\mathfrak{m}}$ be the line bundle over $X_{\mathfrak{m}} \times Spec(A_T)$ defined by the morphism

$$\alpha_{\mathfrak{m}} : Spec(A_T) \to Pic^0_{X,\mathfrak{m}}.$$

Then, as in [An1], 1.6,

$$\mathcal{F}_{\mathfrak{m}}(D, 1-t.z) \simeq (Id \times \iota)^*(\mathcal{A}_T \otimes p_1^*\mathcal{O}_{X_{\mathfrak{m}}}(D)),$$

where $R = \mathbb{F}_q[t]/t^r$ and ι is given by $\mathcal{O}_{X_{\mathfrak{m}}} \to R$, $1/T \to t$; $1/T$ is a local parameter for $\infty \in X_{\mathfrak{m}}$, and p_1 is the natural projection $X_{\mathfrak{m}} \times Spec(R) \to X_{\mathfrak{m}}$. For $\mathcal{O}_{X_{\mathfrak{m}}}(D)$, there exists an $\mathbb{F}_q$-subspace $W \subset \mathbb{F}_q((1/T))$ with

$$H^0_{\mathfrak{m}}(X, \mathcal{O}_X(D)) = H^0(X_{\mathfrak{m}}, \mathcal{O}_{X_{\mathfrak{m}}}(D)) = W \cap \mathbb{F}_q[[1/T]].$$

One can obtain a similar result for the cohomology group H^1. As in [An1], 1.8, by the Riemann-Roch Theorem for $X_{\mathfrak{m}}$ (cf. [Se] chapter IV, no. 6), there exists a basis $\{w_i\}_{i\in\mathbb{N}}$ for $W \subset \mathbb{F}_q((1/T))$, such that w_i is monic,

$$-g-d < deg(w_i) \le deg(w_{i+1}) \text{ and } deg(w_i - T^i) \le g+d$$

for each i. In this way, one can obtain $\tau(D, h(z)) \in R$. Analogously, one can obtain $\tau_W(t, x, y)$ defined in [An1], 2.3. If we consider $Spec(R) \to Pic^0_{X,\mathfrak{m}}$, given by $(Id \times \iota)^*(\mathcal{A}_{\mathfrak{m}} \otimes p_1^*\mathcal{O}_{X_{\mathfrak{m}}}(D)(-deg(D)))$, then $Spec(R) \times_{Pic^0_{X,\mathfrak{m}}} \Theta_{\mathfrak{m}} \subset Spec(R)$ is given by $\tau(D, h(z))$. One finishes by considering in [An1], 3.4.1, the following changes: $R = \mathbb{F}_q[t, x, y]/(t^r, x^r, y^r)$;

$$\iota_{\mathfrak{m}} : Spec(R) \to Spec(A_T) \times Y_{\mathfrak{m}} \times Y_{\mathfrak{m}}$$

is the unique closed immersion such that

$$\iota_{\mathfrak{m}}(t, x, y)_0 = (\infty, (N_1, Y_{\mathfrak{m}}/X)\tilde{\infty}, (N_2, Y_{\mathfrak{m}}/X)\tilde{\infty})$$

and

$$t = \iota_{\mathfrak{m}}^{*} \cdot p_1^{*} T^{-1}, \quad x = \iota_{\mathfrak{m}}^{*} \cdot p_1^{*} \pi_{\mathfrak{m}}^{*} T^{-1}, \quad y = \iota_{\mathfrak{m}}^{*} \cdot p_1^{*} \pi_{\mathfrak{m}}^{*} T^{-1},$$

where D_1 and D_2 are divisors of degree 0 supported away from T; $D_1 - D_2$ is $\mathfrak{m}$-linearly equivalent to D; $\tilde{\infty}$ is defined in [An1], 3.4; and $(\quad , Y_{\mathfrak{m}}/X)$ is the Artin symbol. If we consider

$$\alpha_{\mathfrak{m}} \cdot p_1 + \beta_{\mathfrak{m}} \cdot p_{23} : Spec(A_T) \times Y_{\mathfrak{m}} \times Y_{\mathfrak{m}} \to Pic^0_{X,\mathfrak{m}},$$

then via the morphism $(\alpha_{\mathfrak{m}} \cdot p_1 + \beta_{\mathfrak{m}} \cdot p_{23}) \cdot \iota_{\mathfrak{m}}$, $Spec(R) \times_{Pic^0_{X,\mathfrak{m}}} \Theta_{\mathfrak{m}} \subset Spec(R)$ is given by $\tau_W(t, x, y)$.

From these results, we have, as in [An1], 4.1.1, that

$$C_{\mathfrak{m}}(i, N) = \#\mathbb{P}(H^0_{\mathfrak{m}}(\mathcal{O}_X(i)(N)) \text{ for } 0 \leq i \leq g + d - 1$$

and

$$C_{\mathfrak{m}}(g + d + j - 1, N) = \frac{\#\mathbb{P}(H^0_{\mathfrak{m}}(\mathcal{O}_X(g + d + j - 1)(N))) - 1 - \cdots - q^{j-1}}{q^j}$$

with $1 \leq j \leq g$.

Now, for easy notation let us denote $h_i^N = dim_{\mathbb{F}_q} H^0(\mathcal{O}_X(i)(N - D(\mathfrak{m})))$, then:

Theorem 3.5. *The Weil divisor on* $Y_{\mathfrak{m}} \times Y_{\mathfrak{m}}$

$$\sum_{[N]} \Big(\sum_{i=0}^{g+d-1} L_{\mathfrak{m}}(N, i) \cdot F_N^{g+d-1-i} + \sum_{j=1}^{g} \frac{L_{\mathfrak{m}}(N, g + d + j - 1)}{q^j} \cdot F_j^N \Big)$$

$$+ \sum_{[N]} \Big(- \sum_{i=0}^{g+d-1} \bar{\delta}_{h_i^N, h_{i+1}^N} \cdot q^{h_i^N} \cdot F_N^{g+d-1-i} - \sum_{j=1}^{g} \bar{\delta}_{h^N_{g+d+j-1}, h^N_{g+d+j}} \cdot q^{h^N_{g+d+j-1} - j} \cdot F_j^N \Big)$$

is linearly equivalent to zero. $\bar{\delta} := 1 - \delta$, *with* δ *the Kronecker symbol.*

Proof. From Proposition 2.2, $\beta_{\mathfrak{m}}^{-1}(\mu_{\mathfrak{m}}^{-1}(\Theta(g + d, \mathfrak{m})))$ is linearly equivalent to $\beta_{\mathfrak{m}}^{-1}(\Theta_{\mathfrak{m}})$. We conclude, because $\beta_{\mathfrak{m}}^{-1}(\Theta_{\mathfrak{m}}) - \beta_{\mathfrak{m}}^{-1}(\mu_{\mathfrak{m}}^{-1}(\Theta(g + d, \mathfrak{m})))$ is

$$\sum_{[N]} \Big(\sum_{i=0}^{g+d-1} ((C_{\mathfrak{m}} - C)(i, N)) \cdot F_N^{g+d-1-i} + \sum_{j=1}^{g} ((C_{\mathfrak{m}} - C)(g + d + j - 1, N)) \cdot F_j^N \Big)$$

with

$$\begin{aligned} (C_{\mathfrak{m}} - C)(i, N) = {} & L_{\mathfrak{m}}(N, i) + \#\mathbb{P}(H^0(\mathcal{O}_X(i)(N - D(\mathfrak{m})))) \\ & - \#\mathbb{P}(H^0(\mathcal{O}_X(i + 1)(N - D(\mathfrak{m})))) \end{aligned}$$

and

$$(C_{\mathfrak{m}} - C)(g + d + j - 1, N) = L_{\mathfrak{m}}(N, g + d + j - 1)$$

$$+ \#\mathbb{P}(H^0(\mathcal{O}_X(g + d + j - 1)(N - D(\mathfrak{m})))) - \#\mathbb{P}(H^0(\mathcal{O}_X(g + d + j)(N - D(\mathfrak{m})))).$$

$C(2g + d - 1, N) = 0.$ ☐

Now let us consider the Weil divisors on $Y_{\mathfrak{m}} \times Y_{\mathfrak{m}}$ as correspondences and hence as endomorphisms of $Pic^0_{\tilde{Y}_{\mathfrak{m}}}(\mathbb{F}_q) = Pic^0(H_{\mathfrak{m}})$, where $\tilde{Y}_{\mathfrak{m}}$ is the Riemann variety for $H_{\mathfrak{m}}$. F_N^k and F_k^N give the endomorphisms σ_N and $q^k \cdot \sigma_N$ over $\mathcal{L} \in Pic^0(H_{\mathfrak{m}})$, with $q^k \cdot \sigma_N(\mathcal{L}) := (\sigma_N^{*} \mathcal{L})^{\otimes q^k}$. Recall that $\sigma_N \in G_{\mathfrak{m}}$ acts on $Y_{\mathfrak{m}}$. In this way, by the last theorem the correspondence $\beta_{\mathfrak{m}}^{-1}(\Theta_{\mathfrak{m}}) - \beta_{\mathfrak{m}}^{-1}(\mu_{\mathfrak{m}}^{-1}(\Theta(g+d,\mathfrak{m})))$ acts trivially on $Pic^0(H_{\mathfrak{m}})$, and from this one deduces that

$$\sum_{[N]}(\sum_{k=0}^{2g+d-1} L_{\mathfrak{m}}(N,k) - \sum_{i=0}^{g-1} q^i) \cdot \sigma_N$$

acts trivially over $Pic^0(H_{\mathfrak{m}})$. Apparently, this theorem gives an explicit proof of the Cayley-Hamilton theorem for the Frobenius morphism acting on the $\mathbb{Q}[G_{\mathfrak{m}}]$-module $Corr(Y_{\mathfrak{m}})/\sim=$ correspondences of $Y_{\mathfrak{m}}$ modulo vertical and horizontal ones and linear equivalence. As a consequence of this last theorem we have:

Theorem 3.6 (Brumer). *$(q-1)\theta_{H_{\mathfrak{m}}/K,T}$ annihilates the Picard group $Pic^0(H_{\mathfrak{m}})$.*

Proof. Since

$$\theta_{H_{\mathfrak{m}}/K,T} = \sum_{[N]\in G_{\mathfrak{m}}} (\sum_{i=0}^{2g+d-1} L_{\mathfrak{m}}(N,i) + \sum_{j\geq 0} q^{g+j}) \cdot \sigma_N$$

we have

$$\theta_{H_{\mathfrak{m}}/K,T} - \sum_{[N]}(\sum_{k=0}^{2g+d-1} L_{\mathfrak{m}}(N,k) - \sum_{i=0}^{g-1} q^i) \cdot \sigma_N = (1-q)^{-1} \cdot \sum_{[N]} \sigma_N.$$

We conclude by the existence theorem of class field theory. If $\mathcal{L} \in Pic^0(H_{\mathfrak{m}})$, then we have that $\bigotimes_{[N]} \sigma_N^* \mathcal{L}$ is a principal divisor $D(\alpha)$ with $\alpha \in K^\times$. □

We can say something about the Stark part of the Brumer-Stark conjecture. If $\mathfrak{u}$ is a divisor on $H_{\mathfrak{m}}$ of degree 0, then by the last theorem $(q-1)\theta_{H_{\mathfrak{m}}/K,T}(\mathfrak{u}) = D(h^{q-1}\cdot\alpha)$, with $h \in H_{\mathfrak{m}}$ and $\alpha \in K^\times$, thus $H_{\mathfrak{m}}((h^{q-1}\cdot\alpha)^{1/q-1}) = H_{\mathfrak{m}}(\alpha^{1/q-1})$. $H_{\mathfrak{m}}(\alpha^{1/q-1})/K$ is abelian because $K(\alpha^{1/q-1})/K$ and $H_{\mathfrak{m}}/K$ are abelian extensions. In [H] a complete proof is given for arbitrary divisors by using Drinfeld modules.

3.2. **An explicit computation.** Let consider us $X = C$ an elliptic curve. We take $\mathfrak{m} = m_x^{d+1}$, $d \geq 2$, m_x a maximal ideal in A with x rational and t_x a local parameter. With the above notations we have that the element $\theta_{H_{\mathfrak{m}}/K,T}$ of $\mathbb{Q}[G_{\mathfrak{m}}]$ is

$$(\sum_{i=0}^{d} \sum_{\substack{[N]\in G_{\mathfrak{m}} \\ L_{\mathfrak{m}}(N,i)>0}} t^i \cdot \sigma_N + \sum_{\substack{[N]\in G_{\mathfrak{m}} \\ L_{\mathfrak{m}}(N,d+1)>0}} q^{h_{d+1}^N} t^{d+1} \cdot \sigma_N + \sum_{[N]\in G_{\mathfrak{m}}} \frac{qt^{d+2}}{1-qt} \cdot \sigma_N)_{t=1}.$$

Recall that $h_i^N = dim_{\mathbb{F}_q} H^0(\mathcal{O}_X(i)(N-(d+1)x))$. On $Spec(H_{\mathfrak{m}} \otimes H_{\mathfrak{m}})$ the Weil divisor

$$\sum_{i=0}^{d} \sum_{\substack{[N]\in G_{\mathfrak{m}} \\ L_{\mathfrak{m}}(N,i)>0}} F_N^i + \sum_{\substack{[N]\in G_{\mathfrak{m}} \\ L_{\mathfrak{m}}(N,d+1)>0}} q^{h_{d+1}^N - 1} \cdot F_1^N$$
$$- \sum_{[N]\in G_{\mathfrak{m}}} \bar{\delta}_{h_d^N, h_{d+1}^N} \cdot F_N^0 - \bar{\delta}_{h_{d+1}^N, h_{d+2}^N} \cdot q \cdot F_1^N$$

is trivial (cf. Theorem 3.5). Thus, $(q-1)\theta_{H_{\mathfrak{m}}/K,T}$ acts on $Pic^0(H_{\mathfrak{m}})$ as $\sum_{[N]\in G_{\mathfrak{m}}} \sigma_N$.

Bearing in mind Proposition 2.2, we calculate the function that makes this Weil divisor linearly trivial.

Let us consider the diagram

$$\begin{tikzcd} C \arrow[r, "T_\infty"] \arrow[d, "Id-F+\infty"] & Pic^0_C(\simeq C) \arrow[d, "Id-F"] \\ C \arrow[r, "T_\infty"] & Pic^0_C(\simeq C), \end{tikzcd}$$

where $T_\infty(x) = x - \infty$. C is a fine moduli for the elliptic sheaves of rank 1. Let us denote by $(\mathcal{L}, i, t)$ a universal elliptic sheaf. Let us consider $(L, i, t) := (\mathcal{L}, i, t)_{|Spec(K)}$ and s_1, s_2 and $\bar{s}_1, \bar{s}_2$ a basis of global sections for $L(2)$ and $L^\vee(2)$, respectively, such that s_1 and $\bar{s}_1$ are sections of $L(1)$ and $L^\vee(1)$, respectively. We denote by $p_{ij} : C \times C \times C \to C \times C$ the natural projections and

$$\cup : H^0(p_{12}^* L(i)) \otimes H^0(p_{13}^* L^\vee(j)) \to H^0(p_{12}^* L \otimes_{\mathcal{O}_{C\otimes K\otimes K}} p_{13}^* L^\vee(i+j))$$

the cup product.

Lemma 3.7. *There exists an open subset* $U \subseteq Spec(K \otimes K)$ *where*

$$(p_{12}^* s_1 \cup p_{13}^* \bar{s}_1)_U, (p_{12}^* s_2 \cup p_{13}^* \bar{s}_1)_U, (p_{12}^* s_1 \cup p_{13}^* \bar{s}_2)_U$$

is a basis for $H^0(C \times U, p_{12}^* L \otimes_{\mathcal{O}_{C\otimes K\otimes K}} p_{13}^* L^\vee(3))$.

Proof. Let us consider the elliptic sheaves given by $(\mathcal{O}_C, i, t)$ and $(\mathcal{O}_C(\infty - x), i, t)$, $i = t$ being the natural inclusions. $2\infty - xy$ is linearly equivalent to $y \in C$. We consider $H^0(\mathcal{O}_C(2\infty)) = \langle 1, f_2 \rangle$ and $H^0(\mathcal{O}_C(y + \infty)) = \langle 1, e \rangle$, $e, f_2 \in K$. One checks that $1 \cup 1, 1 \cup f_2, e \cup 1$ is a basis for $H^0(\mathcal{O}_C(y + 2\infty))$.

We are finished, because these elliptic sheaves give a rational point $(z_1, z_2) \in C \times C$ and hence $p_{12}^* \mathcal{L}_{|(z_1,z_2)} = \mathcal{O}_C(\infty - x)$ and $p_{13}^* \mathcal{L}^\vee_{|(z_1,z_2)} = \mathcal{O}_C$, and by using the Grauert theorems the $\mathcal{O}_{C\times C}$-module

$$\frac{p_{23*}(p_{12}^* \mathcal{L} \otimes_{\mathcal{O}_{C\times C\times C}} p_{13}^* \mathcal{L}^\vee)(3)}{p_{23*} p_{12}^* \mathcal{L}(2) \cup p_{23*} p_{13}^* \mathcal{L}^\vee(1) + p_{23*} p_{12}^* \mathcal{L}(1) \cup p_{23*} p_{13}^* \mathcal{L}^\vee(2)}$$

is zero over an open neighbourhood of (z_1, z_2). □

The following lemma is deduced from [Dr], [Mu].

Lemma 3.8. $dim_K H^0(C \otimes K, L(h)) = h$ $(h \geq 0)$ *and if* $H^0(C \otimes K, L(1)) = \langle s_1 \rangle$, *then* $\{s_1, t(F^\# s_1), \cdots, t(F^\# \cdots \overset{h-1}{t}(F^\# s_1) \cdots)\}$ *is a basis for* $H^0(C \otimes K, L(h))$.

If Γ is the pull-back of the diagonal divisor via the morphism

$$Id \times (Id - F + \infty) : Spec(K) \times C \to C \times C,$$

then $L(1) \simeq \mathcal{O}_{C\otimes K}(\Gamma)$. In this way, $L^\vee(1) \simeq \mathcal{O}_{C\otimes K}(2)(-\Gamma)$. Let s_1 be the unique, up to constants, global section on $L(1)$ over $C \otimes K$. Therefore, $div(s_1) = \Gamma$. Let s_h be the global section on $L(h)$ defined by $t(F^\# \cdots \overset{h-1}{t}(F^\# s_1) \cdots)$.

Now let us calculate $\bar{s}_1, \bar{s}_2$. By considering $s_1 = 1$, we bear in mind the isomorphism $\rho : L^\vee(1) \simeq \mathcal{O}_{C\otimes K}(2)(-\Gamma)$ given by $L(1) \simeq \mathcal{O}_{C\otimes K}(\Gamma)$. Therefore, $\rho(\bar{s}_1) = u$ and $u = \lambda + \mu \cdot f_2 \otimes 1$, with $f_2 \in H^0(\mathcal{O}_C(2))$. $\lambda, \mu \in K$ are obtained such that via the epimorphism

$$\mathcal{O}_{C\otimes K} \to \mathcal{O}_{C\otimes K}(2)/\mathcal{O}_{C\otimes K}(-\Gamma),$$

$\lambda + \mu \cdot f_2 \otimes 1$ is 0. In this way, we can take $\mu = 1$ and $\lambda = -1 \otimes f_2'$; f_2' denotes the value of f_2 via the morphism $Id - F + \infty$. By choosing $f_3 \in H^0(\mathcal{O}_C(3))$, one can calculate $\bar{s}_2$ analogously.

We use the same notation for the pull-back of sections "s_h" and $\bar{s}_1, \bar{s}_2$ to $C \otimes H_{\mathfrak{m}}$. By [Al] (Remark 3.1) we have that

$$g_{\mathfrak{m}}(s_h) = \nu_1^{q^{h-1}} + \nu_2^{q^{h-1}} t_x + \cdots + \nu_{d+1}^{q^{h-1}} t_x^d \in H_{\mathfrak{m}}[t_x]/t_x^{d+1}.$$

ν_j is an m_x^j-torsion element for the A-elliptic module, associated with the elliptic sheaf (L, i, t), now considered on $H_{\mathfrak{m}}$ to obtain a level structure $g_{\mathfrak{m}}$; cf. [Dr], [Mu]. We have a commutative diagram

$$\begin{tikzcd}
L \otimes_{\mathcal{O}_{C \otimes H_{\mathfrak{m}}}} L^{\vee}(3) \arrow[r, "\sim"] \arrow[d, "g_{\mathfrak{m}} \otimes g_{\mathfrak{m}}^{\vee}(3)"'] & \mathcal{O}_C(3) \otimes H_{\mathfrak{m}} \arrow[dl, "\pi"] \\
H_{\mathfrak{m}}[t_x]/t_x^{d+1} &
\end{tikzcd}$$

with π being the natural epimorphism $\mathcal{O}_C(3) \to \mathbb{F}_q[t_x]/t_x^{d+1}$. Therefore, we have

$$g_{\mathfrak{m}}^{\vee}(\bar{s}_1).g_{\mathfrak{m}}(s_1) = \pi(f_2) - f_2' \text{ and } g_{\mathfrak{m}}^{\vee}(\bar{s}_2).g_{\mathfrak{m}}(s_1) = \pi(f_3) - f_3'$$

in the ring $H_{\mathfrak{m}}[t_x]/t_x^{d+1}$. Thus, we can obtain $g_{\mathfrak{m}}^{\vee}(\bar{s}_1)$ and $g_{\mathfrak{m}}^{\vee}(\bar{s}_2)$ as elements in $H_{\mathfrak{m}}[t_x]/t_x^{d+1}$.

Bearing in mind Lemma 3.7,

$$(p_{12}^* s_1 \cup p_{13}^* \bar{s}_1)_U, (p_{12}^* s_2 \cup p_{13}^* \bar{s_1})_U, (p_{12}^* s_1 \cup p_{13}^* \bar{s}_2)_U, \cdots, (p_{12}^* s_{d-1} \cup p_{13}^* \bar{s}_2)_U$$

is a basis for $H^0(C \times U, p_{12}^* L \otimes_{\mathcal{O}_{C \otimes K \otimes K}} p_{13}^* L^{\vee}(d+1))$. By using Proposition 2.2, if we denote $s_k^{\mathfrak{m}} = g_{\mathfrak{m}}(s_k) \otimes 1$ and $\bar{s}_k^{\mathfrak{m}} = 1 \otimes g_{\mathfrak{m}}^{\vee}(\bar{s}_k)$ as elements within $H_{\mathfrak{m}} \otimes H_{\mathfrak{m}}[t_x]/t_x^{d+1}$, then

$$b_U^1 = det((s_1^{\mathfrak{m}} \cdot \bar{s}_1^{\mathfrak{m}}), (s_2^{\mathfrak{m}} \cdot \bar{s}_1^{\mathfrak{m}}), (s_1^{\mathfrak{m}} \cdot \bar{s}_2^{\mathfrak{m}}), (s_2^{\mathfrak{m}} \cdot \bar{s}_2^{\mathfrak{m}}), \cdots, (s_{d-2}^{\mathfrak{m}} \cdot \bar{s}_2^{\mathfrak{m}}), 1) \in H_{\mathfrak{m}} \otimes H_{\mathfrak{m}}$$

and

$$b_U^2 = det((s_1^{\mathfrak{m}} \cdot \bar{s}_1^{\mathfrak{m}}), (s_2^{\mathfrak{m}} \cdot \bar{s}_1^{\mathfrak{m}}), (s_1^{\mathfrak{m}} \cdot \bar{s}_2^{\mathfrak{m}}), \cdots, (s_{d-2}^{\mathfrak{m}} \cdot \bar{s}_2^{\mathfrak{m}}), (s_{d-1}^{\mathfrak{m}} \cdot \bar{s}_2^{\mathfrak{m}})) \in H_{\mathfrak{m}} \otimes H_{\mathfrak{m}}.$$

This determinant is obtained considering in $\mathbb{F}_q[t_x]/t_x^{d+1}$ the basis $1, t_x, \cdots, t_x^d$.

It should be noted that for each $z \in Spec(H_{\mathfrak{m}} \otimes H_{\mathfrak{m}})$, $(s_{d-1}^{\mathfrak{m}} \cdot \bar{s}_2^{\mathfrak{m}})_z$ is a global section of $(p_{12}^* L \otimes p_{13}^* L^{\vee})_z(d+1)$ but is not a global section of $(p_{12}^* L \otimes p_{13}^* L^{\vee})_z(d)$. Thus, in Proposition 2.2, $\eta(s_{d-1}^{\mathfrak{m}} \cdot \bar{s}_2^{\mathfrak{m}})$ is a unit in $H_{\mathfrak{m}} \otimes H_{\mathfrak{m}}$. In this way, $D(b_U^1/b_U^2)$ is the trivial Weil divisor given in Theorem 3.5.

For $X = \mathbb{P}_1$, explicit calculations are made in [An2], chapter 6, by considering level structures at ∞.

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