SHARPNESS OF THE KORÁNYI APPROACH REGION

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(Communicated by Andreas Seeger)

Abstract. We prove a Littlewood-type theorem which shows the sharpness of the Korányi approach region for the boundary behavior of Poisson-Szegö integrals on the unit ball of $\mathbb{C}^n$. Our result is stronger than Hakim and Sibony (1983).

1. Introduction

Let $\mathbb{C}^n$ be the $n$-dimensional complex space with inner product

$$\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w_j},$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$, and the associated norm $|z| = \sqrt{\langle z, z \rangle}$. We denote by $B$ the unit ball of $\mathbb{C}^n$ and by $S$ its boundary. Let $\sigma$ be the normalized surface measure on $S$. For an integrable function $f$ on $S$, the Poisson-Szegö integral of $f$ is defined by

$$\mathcal{P}[f](z) = \int_{S} \frac{(1 - |z|^2)^n}{|1 - \langle z, \zeta \rangle|^{2n}} f(\zeta) \, d\sigma(\zeta) \quad \text{for } z \in B.$$

In [4], Korányi investigated the boundary behavior of Poisson-Szegö integrals. For $\alpha > 1$ and $\xi \in S$, the Korányi approach region at $\xi$ is given by

$$\mathcal{A}_\alpha(\xi) = \left\{ z \in B : |1 - \langle z, \xi \rangle| < \frac{\alpha}{2} |1 - |z|^2| \right\}.$$

Theorem A. Let $\alpha > 1$. If $f$ is an integrable function on $S$, then the Poisson-Szegö integral $\mathcal{P}[f](z)$ has the limit $f(\xi)$ as $z \to \xi$ within $\mathcal{A}_\alpha(\xi)$ at almost every point $\xi$ of $S$.

When $n = 1$, this theorem is well known as Fatou’s theorem. In this case, $\mathcal{A}_\alpha(\xi)$ is a non-tangential approach region at $\xi$. The best possibility of this approach region was first proved by Littlewood [2] in the following sense: Let $C_0$ be a tangential curve in the unit disc $D$ which ends at $z = 1$, and let $C_\theta$ be the curve $C_0$ rotated about the origin through an angle $\theta$, so that $C_\theta$ touches the unit circle internally at $e^{i\theta}$. Then there exists a bounded harmonic function on $D$ which admits no limits as $z \to e^{i\theta}$ along $C_\theta$ for almost every $\theta$, $0 \leq \theta \leq 2\pi$. Aikawa [1] improved this result.
by showing that there exists a bounded holomorphic function on $D$ which admits no limits as $z \to e^{i\theta}$ along $C_\theta$ for every $\theta$.

In [3], Nagel and Stein proved that the Poisson integral on the upper half space of $\mathbb{R}^{n+1}$ has the boundary limit at almost every point of $\mathbb{R}^n$ within a certain approach region which is not contained in any non-tangential approach regions. Sueiro [8] extended Nagel and Stein’s result to $\mathbb{C}^n$ and proved that the Poisson-Szegö integral has the boundary limit at almost every point of $S$ within a certain approach region which is not contained in any Korányi approach regions.

The purpose of the present paper is to prove a Littlewood-type theorem in higher dimensions. Let $\gamma$ be a curve in $B$ which ends at $e_1 = (1, 0, \cdots, 0)$ and satisfies

$$\lim_{z \to e_1} \frac{|1 - \langle z, e_1 \rangle|}{1 - |z|^2} = \infty.$$  

(1.1)

This means that, for each $\alpha > 1$, points of $\gamma$ near $e_1$ lie outside $A_\alpha(e_1)$. Let $U$ denote the group of unitary transformations of $\mathbb{C}^n$. We write $U\gamma$ for the image of $U$ through $\gamma$. Since $U$ preserves inner products, $U\gamma$ touches $S$ internally at $Ue_1$ and lies outside $A_\alpha(Ue_1)$ near $Ue_1$ for every $\alpha > 1$.

Our main result is as follows.

**Theorem.** Let $\gamma$ be a curve in $B$ which ends at $e_1$ and satisfies (1.1). Then there exists a bounded function $f$ on $S$ of which Poisson-Szegö integral $P[f](z)$ admits no limits as $|z| \to 1$ along $U\gamma$ for every $U \in U$, that is,

$$\liminf_{|z| \to 1} P[f](z) \neq \limsup_{|z| \to 1} P[f](z) \quad \text{for every } U \in U.$$  

**Remark 1.** Since $U$ acts transitively on $S$, for each $\xi \in S$ there is $U\xi \in U$ such that $\xi = U\xi_1$. Therefore, the Theorem implies that there exists a bounded Poisson-Szegö integral which admits no limits as $z \to \xi$ along $U\xi\gamma$ at every point $\xi$ of $S$. Moreover, we can make $f$ satisfy

$$\liminf_{|z| \to 1} P[f](z) = \inf_{\zeta \in S} f(\zeta) \quad \text{and} \quad \limsup_{|z| \to 1} P[f](z) = \sup_{\zeta \in S} f(\zeta)$$

for every $U \in U$.

**Remark 2.** By Sueiro’s result, the limit in (1.1) cannot be replaced by the upper limit.

As a related topic in higher dimensions, there is the following result due to Hakim and Sibony [3].

**Theorem B.** Suppose $n > 1$. Let $\alpha > 1$ and $h : (0, 1] \to [\alpha, \infty)$ be a decreasing function such that

$$\lim_{x \to 0^+} h(x) = \infty,$$

and let

$$D_{\alpha, h}(\xi) = \left\{ z \in B : |1 - \langle z, \xi \rangle| \leq \alpha(1 - |\langle z, \xi \rangle|) \quad \text{and} \quad |1 - \langle z, \xi \rangle| \leq h(1 - |\langle z, \xi \rangle|)(1 - |z|) \right\}.$$  

Then there exists a bounded holomorphic function on $B$ which admits no limits as $z \to \xi$ within $D_{\alpha, h}(\xi)$ at almost every point $\xi$ of $S$. 

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We note that the approach region $D_{\alpha,h}(\xi)$ is wider than any Korányi approach regions in the complex tangential directions, but is the same in the special real direction. Our theorem is stronger than Theorem B in the following points:

- It improves no convergence “almost everywhere” to “everywhere”.
- It establishes that a tangential approach in the special real direction cannot be allowed in Theorem A.
- The existence of a bounded Poisson-Szegő integral which fails to have a boundary limit is ensured even if we replace $D_{\alpha,h}(e_1)$ by much smaller curve $\gamma$ satisfying (1.1).

Also, our method is different from Hakim and Sibony’s. Theorem B is proved by constructing a higher-dimensional Blaschke product. However, we will prove the Theorem in Section 3 by constructing a bounded function on $S$ and using lower and upper estimates of Poisson-Szegő integrals in Section 2. In the proofs we adapt ideas from [1, 2]. Whereas the polar and the euclidean coordinates were used to construct a bounded function on the unit circle and on $\mathbb{R}^n$ in [1, 2], they are not applicable in our case. This is an important difference between [1, 2] and our case.

Throughout the paper we use the symbols $A_0, A_1, A_2, \ldots$ to denote absolute positive constants depending only on the dimension $n$.

2. Estimates of Poisson-Szegő Integrals

In this section we give lower and upper estimates for Poisson-Szegő integrals. To this end, we start by introducing a non-isotropic ball in $S$. We observe that the function $d(z,w) = |1 - \langle z, w \rangle|^{1/2}$ satisfies the triangle inequality on $B \cup S$, and defines a metric on $S$. See [7, Lemma 7.3]. For $\xi \in S$ and $r > 0$, we write

$$Q(\xi, r) = \{ \zeta \in S : d(\zeta, \xi) < r \},$$

the non-isotropic ball of center $\xi$ and radius $r$. Note that, to emphasize the metric $d$, we use the slightly different definition from Stoll’s book. We observe that $\sigma(Q(U\xi, r)) = \sigma(Q(\xi, r))$ for any unitary transformations $U$ and that

$$\lim_{r \to 0} \frac{\sigma(Q(\xi, r))}{r^{2n}} = \frac{2^n}{4\sqrt{\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}.$$

See [4, p. 84]. Moreover, there is a constant $A_0 > 1$ depending only on the dimension $n$ such that

$$A_0^{-1} r^{2n} \leq \sigma(Q(\xi, r)) \leq A_0 r^{2n}$$

for $\xi \in S$ and $0 \leq r \leq \text{diam } S = \sqrt{2}$. Here $\text{diam } F = \sup\{d(\eta, \zeta) : \eta, \zeta \in F\}$ for $F \subset S$.

Let $T > 0$ and $\xi \in S$. For an integrable function $g$ on $S$, we define the truncated maximal function at $\xi$ by

$$M_T[g](\xi) = \sup_{r \geq T} r^{-2n} \int_{Q(\xi, r)} |g(\zeta)| d\sigma(\zeta).$$

By the argument in [7, Theorem 7.8], we obtain the following estimate for the Poisson-Szegő integral. For completeness we give the proof.
Lemma 1. There exists a positive constant $A_1$ depending only on the dimension $n$ such that if $g$ is an integrable function on $S$ and $C > 0$, then

$$|P[g](tξ)| \leq A_1 \left( (1 - t)^{-n} \int_{Q(ξ,C\sqrt{1 - t})} |g(ζ)| \, dσ(ζ) + C^{-2n} M_{C \sqrt{1 - t}} [g](ξ) \right)$$

for $ξ ∈ S$ and $0 < t < 1$.

Proof. Let $ξ ∈ S$ and $0 < t < 1$ be fixed, and let

$$V_0 = Q(ξ,C\sqrt{1 - t}),$$

$$V_j = Q(ξ,2jC\sqrt{1 - t}) \setminus Q(ξ,2j^{-1}C\sqrt{1 - t}) \quad (j = 1, \cdots, N),$$

where $N$ is the smallest integer such that $2^N C \sqrt{1 - t} > \sqrt{2}$. Then

$$|P[g](tξ)| \leq \sum_{j=0}^N \int_{V_j} \frac{(1 - t^2)^n}{1 - (tξ, ζ)^2n} |g(ζ)| \, dσ(ζ).$$

Since $|1 - (tξ, ζ)| \geq 1 - t$ for $ζ ∈ S$, it follows that

$$\int_{V_0} \frac{(1 - t^2)^n}{1 - (tξ, ζ)^2n} |g(ζ)| \, dσ(ζ) \leq \frac{2n}{(1 - t)^n} \int_{Q(ξ, C\sqrt{1 - t})} |g(ζ)| \, dσ(ζ).$$

Let $j = 1, \cdots, N$. By the triangle inequality, we have for $ζ ∈ V_j$,

$$2^{j-1} C \sqrt{1 - t} \leq d(ξ, ζ) \leq d(ξ, tξ) + d(tξ, ζ) \leq 2d(tξ, ζ) = 2|1 - (tξ, ζ)|^{1/2}.$$

Hence it follows that

$$\int_{V_j} \frac{(1 - t^2)^n}{1 - (tξ, ζ)^2n} |g(ζ)| \, dσ(ζ) \leq \frac{2^{-2n}}{2^{2n} C^{2n}(1 - t)^n} \int_{Q(ξ, 2jC\sqrt{1 - t})} |g(ζ)| \, dσ(ζ)$$

$$\leq \frac{2^{2n}}{2^{2n} C^{2n}} M_{C \sqrt{1 - t}} [g](ξ).$$

Noting that $\sum_{j=1}^N 2^{-2nj} < 1$, we obtain the lemma with $A_1 = 2^{9n}$. □

As a consequence of Lemma 1 we obtain the following upper and lower estimates.

Lemma 2. The following statements hold:

(i) If $g$ is an integrable function on $S$, then

$$|P[g](tξ)| \leq A_2 M_{C \sqrt{1 - t}} [g](ξ)$$

for $ξ ∈ S$ and $0 < t < 1$,

where $A_2$ is a positive constant depending only on the dimension $n$.

(ii) Let $ξ ∈ S$, $0 < r < 1$ and $C > 0$. If $g$ is a measurable function on $S$ such that $g = 1$ on $Q(ξ, C\sqrt{1 - r})$ and $|g| \leq 1$ on $S$, then

$$P[g](tξ) \geq 1 - \frac{A_3}{C^{2n}}$$

for $t < 1$,

where $A_3$ is a positive constant depending only on the dimension $n$.

Proof. Putting $C = 1$ in Lemma 1 we obtain (i) with $A_2 = 2A_1$. Let us show (ii). We put $h = (1 - g)/2$. Then $h = 0$ on $Q(ξ, C\sqrt{1 - r})$ and $|h| \leq 1$ on $S$. Applying Lemma 1 to $h$, we obtain from (2.2) that for $r < t < 1$,

$$P[h](tξ) \leq \frac{A_1}{C^{2n}} M_{C \sqrt{1 - t}} [h](ξ) \leq \frac{A_1}{C^{2n}} \sup_{ρ ≥ C \sqrt{1 - r}} \frac{σ(Q(ξ, ρ))}{ρ^{2n}} \leq \frac{A_0 A_1}{C^{2n}}.$$

Since $P[g] = 1 - 2P[h]$, we obtain (ii) with $A_3 = 2A_0 A_1$. □
3. Proof of the Theorem

Let \( \pi \) be the radial projection to \( S \) defined by \( \pi(z) = z/|z| \) for \( z \neq 0 \). We note that (1.1) implies

\[
\lim_{z \to \gamma_1} \frac{d(z, e_1)}{d(z, \pi(z))} = \infty,
\]

since \( 1 - |z|^2 \geq 1 - |z| = d(z, \pi(z))^2 \) for \( z \in B \setminus \{0\} \). Recall that

\[
\text{diam } F = \sup_{\eta, \zeta \in F} d(\eta, \zeta) \quad \text{for } F \subset S.
\]

Lemma 3. Let \( \gamma \) be the curve as in the Theorem. Then there exist sequences of positive numbers \( \{a_j\}_{j=1}^\infty, \{b_j\}_{j=1}^\infty \) and subcurves \( \{\gamma_j\}_{j=1}^\infty \) of \( \gamma \) with the following properties:

(i) \( 0 < a_j < b_j < a_{j+1} < b_{j+1} < 1 \) and \( \lim_{j \to \infty} a_j = 1 \);

(ii) \( a_j \leq |z| \leq b_j \) for \( z \in \gamma_j \);

(iii) \( \text{diam } \pi(\gamma_j) \leq \sqrt{1 - b_{j-1}} \) if \( j \geq 2 \);

(iv) \( \lim_{j \to \infty} \frac{\text{diam } \pi(\gamma_j)}{\sqrt{1 - a_j}} = \infty \).

Proof. Let \( \alpha_j > 1 \) be such that \( \alpha_j \to \infty \) as \( j \to \infty \). We shall choose \( \{a_j\}, \{b_j\} \) and \( \{\gamma_j\} \), inductively. By (1.1), we can find \( a_1 \) with \( \inf_{z \in \gamma} |z| < a_1 < 1 \) and

\[
d(z, e_1) \geq \alpha_1 d(z, \pi(z)) \quad \text{for } z \in \gamma \cap \{|z| \geq a_1\}.
\]

Let \( \gamma' \) be the connected component of \( \gamma \cap \{|z| \geq a_1\} \) which ends at \( e_1 \). Since there is \( z_0 \in \gamma' \cap \{|z| = a_1\} \), we have from the triangle inequality that

\[
\text{diam } \pi(\gamma') \geq d(\pi(z_0), e_1)
\]

\[
\geq d(z_0, e_1) - d(z_0, \pi(z_0))
\]

\[
\geq (\alpha_1 - 1) d(z_0, \pi(z_0))
\]

\[
= (\alpha_1 - 1) \sqrt{1 - a_1}.
\]

Let \( \gamma'' \) be a subcurve of \( \gamma' \) connecting a point in \( \{|z| = a_1\} \) and a point near \( e_1 \) such that

\[
\text{diam } \pi(\gamma'') \geq \frac{1}{2} \text{diam } \pi(\gamma').
\]

We take \( b_1 \) so that \( \sup_{z \in \gamma''} |z| < b_1 < 1 \), and let \( \gamma_1 \) be the connected component of \( \gamma \cap \{a_1 \leq |z| \leq b_1\} \) containing \( \gamma'' \). Then

\[
\text{diam } \pi(\gamma_1) \geq \text{diam } \pi(\gamma'') \geq \frac{\alpha_1 - 1}{2} \sqrt{1 - a_1}.
\]

We next choose \( a_2, b_2 \) and \( \gamma_2 \) as follows. Let \( a_2 \) be such that \( b_1 < a_2 < 1 \) and

\[
\frac{1}{4} \sqrt{1 - b_1} \geq d(z, e_1) \geq \alpha_2 d(z, \pi(z)) \quad \text{for } z \in \gamma \cap \{|z| \geq a_2\}.
\]

By repeating the above procedure, we can find \( b_2 \) and \( \gamma_2 \) with \( a_2 < b_2 < 1 \) and \( a_2 \leq |z| \leq b_2 \) for \( z \in \gamma_2 \), and

\[
\text{diam } \pi(\gamma_2) \geq \frac{\alpha_2 - 1}{2} \sqrt{1 - a_2}.
\]
It also follows from (3.2) and \( \alpha_2 > 1 \) that
\[
d(\pi(z), e_1) \leq d(z, e_1) + d(z, \pi(z)) \leq \frac{1}{2}\sqrt{1 - b_1}
\quad \text{for } z \in \gamma_2,
\]
and so \( \text{diam } \pi(\gamma_2) \leq \sqrt{1 - b_1} \). Hence \( \gamma_2 \) satisfies (iii). Continuing this procedure, we obtain the required sequences.

In the rest of this section, we suppose that \( \{a_j\}, \{b_j\} \) and \( \{\gamma_j\} \) are as in Lemma 3 and put
\[
\ell_j = \frac{\text{diam } \pi(\gamma_j)}{4}, \quad c_j = \left(\frac{\text{diam } \pi(\gamma_j)}{\sqrt{1 - a_j}}\right)^{1/2} \quad \text{and} \quad \rho_j = c_j\sqrt{1 - a_j}
\]
to simplify the notation. Note from Lemma 3 that
\[
\lim_{j \to \infty} \ell_j = 0, \quad \lim_{j \to \infty} \frac{\rho_j}{\ell_j} = 0 \quad \text{and} \quad \lim_{j \to \infty} c_j = \infty.
\]
Therefore, in the argument below, we may assume that \( \rho_j < \ell_j \) for every \( j \in \mathbb{N} \).

For each \( j \in \mathbb{N} \), let us choose finitely many points \( \{\eta_j^\nu\}_\nu \) in \( S \) such that
\[
(P1) \quad S = \bigcup_\nu Q(\eta_j^\nu, \ell_j),
\]
\[
(P2) \quad \{Q(\eta_j^\nu, \ell_j/2)\}_\nu \text{ are mutually disjoint}.
\]
This is possible. In fact, we first take an arbitrary \( \eta_j^1 \in S \), and take \( \eta_j^\nu \in S \setminus \bigcup_{\nu=1}^{\mu-1} Q(\eta_j^\nu, \ell_j) \) inductively as long as \( S \setminus \bigcup_{\nu=1}^{\mu-1} Q(\eta_j^\nu, \ell_j) \neq \emptyset \). Since \( S \) is compact, we can get finitely many points \( \{\eta_j^\nu\}_\nu \) satisfying (P1). It also fulfills that \( d(\eta_j^\nu, \eta_j^\mu) \geq \ell_j \) if \( \nu \neq \mu \) by the definition of the non-isotropic ball. Hence (P2) follows from the triangle inequality.

We put
\[
M_j = \bigcup_\nu \{\zeta \in S : d(\zeta, \eta_j^\nu) = \ell_j\}.
\]
Then \( \pi(U\gamma_j) \cap M_j \neq \emptyset \) for any unitary transformations \( U \). In fact, there is \( \nu \) such that \( \pi(U\gamma_j) \cap Q(\eta_j^\nu, \ell_j) \neq \emptyset \) by (P1). Since \( \text{diam } \pi(U\gamma_j) = 4\ell_j \) and \( \text{diam } Q(\eta_j^\nu, \ell_j) \leq 2\ell_j \), we have \( \pi(U\gamma_j) \cap \{\zeta \in S : d(\zeta, \eta_j^\nu) = \ell_j\} \neq \emptyset \), and so \( \pi(U\gamma_j) \cap M_j \neq \emptyset \). Let \( G_j \) be the subset of \( B \) given by
\[
G_j = \{z \in B : a_j \leq |z| \leq b_j \text{ and } \pi(z) \in M_j\}.
\]
Since \( U\gamma_j \subset \{a_j \leq |z| \leq b_j\} \) by Lemma 3, it follows that \( U\gamma_j \cap G_j \neq \emptyset \). We also put
\[
E_j = \bigcup_\nu R_j^\nu,
\]
where \( R_j^\nu = \{\zeta \in S : \ell_j - \rho_j < d(\zeta, \eta_j^\nu) < \ell_j + \rho_j\} \) is the non-isotropic ring. Since the value \( \sigma(R_j^\nu) \) is independent of \( \eta_j^\nu \) by unitary invariance, we write \( \kappa_j \) for this value. We note that
\[
\lim_{j \to \infty} \frac{\kappa_j}{\ell_j} = 0.
\]
Proof. Let (3.5)
\[
\frac{\kappa_j}{\ell_j^{2n}} = \frac{\sigma(Q(\eta, \ell_j + \rho_j)) - \sigma(Q(\eta, \ell_j - \rho_j))}{\ell_j^{2n}}
\]
\[
= \left(\frac{\ell_j + \rho_j}{\ell_j}\right)^2 \frac{\sigma(Q(\eta, \ell_j + \rho_j))}{(\ell_j + \rho_j)^{2n}} - \left(\frac{\ell_j - \rho_j}{\ell_j}\right)^2 \frac{\sigma(Q(\eta, \ell_j - \rho_j))}{(\ell_j - \rho_j)^{2n}}
\]
\[
\rightarrow 0 \quad \text{as } j \rightarrow \infty.
\]

Lemma 4. Let \(\{E_j\}\) be as above, and let \(\chi_{E_j}\) denote the characteristic function of \(E_j\). Then the following properties hold:

(i) \(\lim_{j \to \infty} \left(\sup_{|z| \leq b_{j-1}} \mathcal{P}[\chi_{E_j}](z)\right) = 0.\)

(ii) \(\lim_{j \to \infty} \sigma(E_j) = 0.\)

Proof. Let \(z \in B\) be such that \(|z| \leq b_{j-1}\). By Lemma 2(i), we have
\[
\mathcal{P}[\chi_{E_j}](z) \leq A_2 M \sqrt{1 - |z|} \mathcal{P}[\chi_{E_j}](\pi(z))
\]
\[
\leq A_2 \sup_{r \geq \sqrt{1 - |z|}} r^{-2n} \sum_{\nu} \sigma(R_{\nu}^j \cap Q(\pi(z), r))
\]
\[
\leq A_2 \sup_{r \geq \sqrt{1 - |z|}} r^{-2n} N_j(z, r)\kappa_j,
\]
where \(N_j(z, r)\) is the number of \(\eta^j_i\) such that \(R_{\nu}^j \cap Q(\pi(z), r) \neq \emptyset\). Since \(\sqrt{1 - |z|} \geq \text{diam } \pi(\gamma_j)\) by Lemma 3(iii), we observe from \(\rho_j < \ell_j \leq r/4\) that if \(R_{\nu}^j \cap Q(\pi(z), r) \neq \emptyset\), then \(Q(\eta^j_i, \ell_j/2) \subset Q(\pi(z), 2r)\). Therefore it follows from (2.2) and (P2) that \(N_j(z, r) \leq A_4 (r/\ell_j)^{2n}\) with a positive constant \(A_4\) depending only on the dimension \(n\). Hence we obtain
\[
\mathcal{P}[\chi_{E_j}](z) \leq A_2 A_4 \frac{\kappa_j}{\ell_j^{2n}},
\]
so that (i) follows from (3.4).

Taking \(z = 0\) in (i), we obtain
\[
\sigma(E_j) = \mathcal{P}[\chi_{E_j}](0) \rightarrow 0 \quad \text{as } j \rightarrow \infty.
\]
Thus (ii) follows. \(\square\)

We now construct a bounded function \(f\) on \(S\) satisfying the property in the Theorem.

Proof of the Theorem. In view of Lemma 4, taking a subsequence of \(j\) if necessary, we may assume that
\[
(3.5) \quad \mathcal{P}[\chi_{E_j}](z) \leq 2^{-j} \quad \text{for } |z| \leq b_{j-1},
\]
and \(\sigma(E_j) \leq 2^{-j}\). Then \(\sigma(\bigcap_{k} \bigcup_{j=k}^{\infty} E_j) = 0.\) Let
\[
f_j(\zeta) = \begin{cases} (-1)^{j}(\zeta) & \text{if } \zeta \in \bigcup_{i=1}^{j} E_i, \\ 0 & \text{if } \zeta \notin \bigcup_{i=1}^{j} E_i, \end{cases}
\]
where $I_j(\zeta)$ is the maximum integer $i$ such that $\zeta \in E_i$ for $\zeta \in \bigcup_{i=1}^j E_i$. Then we observe that $f_j$ converges almost everywhere on $S$ to

$$f(\zeta) = \begin{cases} (-1)^{I(\zeta)} & \text{if } \zeta \in \bigcup_{i=1}^\infty E_i \setminus \bigcap_k \bigcup_{j=k}^\infty E_j, \\
0 & \text{if } \zeta \not\in \bigcup_{i=1}^\infty E_i \text{ or } \zeta \in \bigcap_k \bigcup_{j=k}^\infty E_j,
\end{cases}$$

where $I(\zeta)$ is the maximum integer $i$ such that $\zeta \in E_i$ for $\zeta \in \bigcup_{j=1}^\infty E_j \setminus \bigcap_k \bigcup_{j=k}^\infty E_j$.

We also see that

(a) $f_j = (-1)^j$ on $E_j$ and $|f_j| \leq 1$ on $S$,
(b) $|f_{j+1} - f_j| \leq 2\chi_{E_{j+1}},$
(c) $\mathcal{P}[f_j]$ converges to $\mathcal{P}[f]$ on $B$.

Let $U$ be a unitary transformation. Since $U\gamma$ intersects $G_j$ for every $j$ as stated in the paragraph defining $G_j$, we can take $z_j \in U\gamma \cap G_j$. Note that $a_j \leq |z_j| \leq b_j$ and $Q(\pi(z_j), c_j \sqrt{1 - a_j}) \subset E_j$. If $j$ is even, then it follows from Lemma 2(ii), Lemma 3(i) and (3.5) that

$$\mathcal{P}[f](z_j) = \mathcal{P}[f_j](z_j) + \sum_{k=j}^\infty \mathcal{P}[f_{k+1} - f_k](z_j)$$

$$\geq \mathcal{P}[f_j](z_j) - \sum_{k=j}^\infty \mathcal{P}[|f_{k+1} - f_k|](z_j)$$

$$\geq 1 - \frac{A_3}{c_j^{2n}} - 2 \sum_{k=j}^\infty \mathcal{P}[\chi_{E_{k+1}}](z_j)$$

$$\geq 1 - \frac{A_3}{c_j^{2n}} - 2^{k-1} \sum_{k=j}^\infty 2^{-k-1}$$

$$= 1 - \frac{A_3}{c_j^{2n}} - 2^{1-j}.$$

Similarly, if $j$ is odd, then

$$\mathcal{P}[f](z_j) \leq -1 + \frac{A_3}{c_j^{2n}} + 2^{1-j}.$$ 

Hence we obtain

$$\liminf_{|z| \to 1 \atop z \in U\gamma} \mathcal{P}[f](z) = -1 < 1 = \limsup_{|z| \to 1 \atop z \in U\gamma} \mathcal{P}[f](z)$$

by (3.3). Thus the Theorem is proved.

Acknowledgment

The author expresses his deep gratitude to Professor Hiroaki Aikawa for his valuable advice and encouragement.

References


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