

PIECEWISE LINEAR SPECTRAL SEQUENCES

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ABSTRACT. We study a class of orthonormal exponential bases for the space $L^2[0, 1]$ and introduce the concept of spectral sequences. We characterize piecewise linear spectral sequences with the knot at $1/2$ and investigate the non-continuity of the piecewise linear spectral sequences. From a special construction of a piecewise constant spectral sequence, the classical Walsh system is recovered.

1. INTRODUCTION

The classical Fourier functions $\mathcal{F} := \{e_n : n \in Z\}$, where $Z := \{\dots, -1, 0, 1, \dots\}$ and $e_n(t) := e^{2\pi i n t}$, $t \in I := [0, 1]$, form an orthonormal basis for the space $L^2(I)$. This basis plays an important role in many areas of mathematics and engineering. However, since the Fourier basis functions have constant frequencies, it has serious limitations in some applications. In particular, when a signal is non-linear and non-stationary, the classical Fourier basis does *not* give a satisfactory representation of such a signal [4]. Finding a better basis has been a main research subject in harmonic analysis, approximation theory and signal analysis, such as window Fourier bases, Gabor bases, wavelet bases (cf., [2, 3, 6, 10]) and *intrinsic mode functions* recently studied in [1, 4].

The main purpose of this paper is to study a class of orthonormal exponential bases having non-constant frequencies. Specifically, we develop orthonormal bases

$$(1.1) \quad f_n := e^{2\pi i g_n}, \quad n \in Z,$$

for $L^2(I)$, where the *frequency* functions g_n are piecewise linear on I . We anticipate that the non-constant frequency functions g_n will allow us to better represent certain signals. In fact, as we will show in the last section of this paper, when g_n are chosen as some piecewise constant functions, the basis f_n , $n \in Z$, turns out to be the well-known Walsh system (cf., [8]). In other words, the classical Walsh system is an orthonormal exponential basis having piecewise constant frequency functions.

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Therefore, these bases unify the classical Fourier basis and the Walsh system, as well as many other interesting bases.

We now introduce the concept of spectral sequences. A sequence of real-valued functions g_n , $n \in Z$, defined on I , is called a *spectral sequence* of I if the exponential function system f_n , $n \in Z$, defined by (1.1) in terms of g_n is an orthonormal basis for $L^2(I)$. In some sense, the notion of spectral sequences is a generalization of the spectral sets, which attracted significant attention in recent years (see, for example, [5, 9] and the references cited therein).

We organize this paper into five sections. In Section 2, we describe two classes of piecewise linear spectral sequences. We establish in Section 3 characterizations of piecewise linear spectral sequences with the knot at $1/2$. Can we construct *continuous* non-trivial piecewise linear spectral sequences? We answer this question in Section 4 by providing a negative result. In Section 5, we construct a special piecewise constant spectral sequence and identify it with the classical Walsh system.

2. TWO CONSTRUCTIONS

We describe two constructions of piecewise linear spectral sequences g_n with the knot at $1/2$. We first present a simple technical lemma that will be used frequently in this paper.

Lemma 2.1. *Let $c \in R$ be a fixed number. For any $a \in Z$, let $I(a, c) := \int_c^{c+\frac{1}{2}} e^{2\pi i a t} dt$. Then, the integral $I(a, c) = 0$ if and only if $a \in 2Z \setminus \{0\}$.*

Proof. A direct computation confirms that $I(a, c) = \frac{e^{2\pi i a c}}{2\pi i a} (e^{\pi i a} - 1)$, for $a \neq 0$, from which this lemma follows. \square

Throughout this paper, we are interested in frequency functions having the form

$$(2.1) \quad g_n(t) = \begin{cases} a_n t + b_n, & t \in [0, \theta), \\ c_n t + d_n, & t \in [\theta, 1], \end{cases}$$

where $a_n, b_n, c_n, d_n \in R$ and $\theta \in (0, 1)$. These frequency functions define basis functions f_n by equation (1.1).

Next, we present two classes of spectral sequences having the form given by (2.1) in which $a_n = c_n$ and $\theta = 1/2$. The first class takes the form that

$$(2.2) \quad g_{2n}(t) = \begin{cases} 2nt + b_n, & t \in [0, \frac{1}{2}), \\ 2nt + d_n, & t \in [\frac{1}{2}, 1], \end{cases} \quad \text{and} \quad g_{2n+1}(t) = \begin{cases} 2nt + b_n, & t \in [0, \frac{1}{2}), \\ 2nt + d_n + \frac{1}{2}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Our first theorem ensures that the set of functions g_n , $n \in Z$, is a spectral sequence.

Theorem 2.2. *If g_n , $n \in Z$ is defined by (2.2), then it is a spectral sequence of I .*

Proof. We prove that the corresponding system f_n , $n \in Z$, is an orthonormal basis for $L^2(I)$. The orthonormality of the functions f_n , $n \in Z$, follows directly from $I(2n, 0) = 0$ and $I(2n, 1/2) = 0$ for each $n \in Z \setminus \{0\}$, which are ensured by Lemma 2.1.

It remains to prove the completeness of the system. To do this, we assume that $f \in L^2(I)$ such that

$$(2.3) \quad \langle f, f_n \rangle = 0, \quad \text{for all } n \in Z,$$

and we wish to show that $f = 0$. Using assumption (2.3) and the definition of g_n , it can be verified that $\int_0^{\frac{1}{2}} f(t)e^{-2\pi i(2nt)} dt = 0$ and $\int_{\frac{1}{2}}^1 f(t)e^{-2\pi i(2nt)} dt = 0$. By changes of variables, we obtain that for each $n \in Z$,

$$(2.4) \quad \int_0^1 f\left(\frac{t}{2}\right) e^{-2\pi i n t} dt = 0 \quad \text{and} \quad \int_1^2 f\left(\frac{t}{2}\right) e^{-2\pi i n t} dt = 0.$$

Since the system $e_n, n \in Z$, forms an orthonormal basis for both spaces $L^2(I)$ and $L^2[1, 2]$, it follows from (2.4) that $f = 0$. \square

Next, we study systems $f_n, n \in Z$, associated with a different choice of exponents g_n . Specifically, for $a_n, b_n \in R$ and for a fixed $c \in R$, we set

$$(2.5) \quad g_{2n}(t) := 2nt + a_n \quad \text{and} \quad g_{2n+1}(t) = \begin{cases} (2n + c)t + b_n, & t \in [0, \frac{1}{2}), \\ (2n + c)t + b_n + \frac{1-c}{2}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

The next theorem ensures that $g_n, n \in Z$, defined by (2.5) is a spectral sequence of I .

Theorem 2.3. *If $g_n, n \in Z$, is defined by (2.5), then it is a spectral sequence of I .*

Proof. Again, by using Lemma 2.1, it can be easily verified that the functions f_n associated with the exponents g_n defined by equations (2.5) are orthonormal on interval I .

To prove the completeness of the system, we let $f \in L^2(I)$ satisfy (2.3). A simple computation leads to the equation $\langle f, f_{2n} \rangle = e^{-2\pi i a_n} \langle f, e_{2n} \rangle$. Now let c_n denote the Fourier coefficients of function f , i.e., $c_n := \langle f, e_n \rangle, n \in Z$. Thus, (2.3) and the equation above imply that $c_{2n} = 0$ for all $n \in Z$. The Fourier series of f then becomes $f = \sum_{n \in Z} c_{2n+1} e_{2n+1}$, which, with the well-known Carleson theorem, implies that

$$f\left(t + \frac{1}{2}\right) = \sum_{n \in Z} c_{2n+1} e^{-2\pi i(2n+1)(t+\frac{1}{2})}$$

for almost every $t \in I$. Using the simple fact that $e^{-i\pi(2n+1)} = -1$, we derive from the last formula that for almost every $t \in I$

$$(2.6) \quad f\left(t + \frac{1}{2}\right) = -f(t).$$

On the other hand, by the definition of g_{2n+1} we conclude that

$$(2.7) \quad \langle f, f_{2n+1} \rangle = \int_0^{\frac{1}{2}} f(t)e^{-2\pi i[(2n+c)t+b_n]} dt + \int_{\frac{1}{2}}^1 f(t)e^{-2\pi i[(2n+c)t+b_n+\frac{1-c}{2}]} dt.$$

In the second term, we replace variable t by $t + 1/2$ and it reduces to

$$(2.8) \quad \begin{aligned} & \int_0^{\frac{1}{2}} f\left(t + \frac{1}{2}\right) e^{-2\pi i[(2n+c)t+\frac{2n+c}{2}+b_n+\frac{1-c}{2}]} dt \\ &= -e^{-2\pi i b_n} \int_0^{\frac{1}{2}} f\left(t + \frac{1}{2}\right) e^{-2\pi i(2n+c)t} dt. \end{aligned}$$

Substituting equations (2.6) and (2.8) into (2.7) we conclude that $\langle f, f_{2n+1} \rangle = 2e^{-2\pi i b_n} \int_0^{\frac{1}{2}} f(t)e^{-2\pi i(2n+c)t} dt$. It follows from this equation and (2.3) that

$$\int_0^1 f\left(\frac{t}{2}\right) e^{-2\pi i(n+\frac{c}{2})t} dt = 0.$$

We introduce a new function defined by $\tilde{f}(t) := f\left(\frac{t}{2}\right) e^{-2\pi i \frac{c}{2}t}$, $t \in I$, and conclude from the completeness of the classical Fourier basis and from the fact that $\tilde{f} \in L^2(I)$ that $\tilde{f} = 0$ on I . Thus, $f = 0$ on $[0, \frac{1}{2}]$. Again, by (2.6), we have $f = 0$ on I . This completes the proof. \square

We remark that these two classes of bases include the classical Fourier basis as a special case. In fact, if we choose $a_n = b_n = 0$ and $c = 1$ in (2.5), then the classical Fourier basis e_n is recovered.

To close this section, we consider the construction of real trigonometric bases with non-constant frequency functions. The classical Fourier basis e_n , $n \in \mathbb{Z}$, has its real trigonometric polynomial bases for $L^2(I)$. We are interested in the question of whether the two families of exponential bases constructed in this section have their real trigonometric counterparts which form bases for space $L^2(I)$. We shall show that the real trigonometric form of the first family forms a basis for $L^2(I)$ but that of the second family does not in general. Given a sequence of functions g_n , $n \in \mathbb{Z}$, we define their real trigonometric form T_n by

$$(2.9) \quad \begin{aligned} T_0(t) &:= 1, & T_{2n-1}(t) &:= \sqrt{2} \sin 2\pi g_n(t), \\ T_{2n}(t) &:= \sqrt{2} \cos 2\pi g_n(t), & n \in \mathbb{N} &:= \{1, 2, \dots\}. \end{aligned}$$

Theorem 2.4. *Let g_n , $n \in \mathbb{Z}$, be defined by (2.2). Then, the system T_n , $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, defined by (2.9), forms an orthonormal basis for $L^2(I)$.*

Proof. Using the definition of the functions g_n , it is easy to verify that for each $n \in \mathbb{N}_0$, T_n is normal. The orthogonality of T_n follows from the fact that for any $c, d \in \mathbb{R}$ and $n \in \mathbb{N}$ the following equations hold:

$$\int_c^{c+\frac{1}{2}} \cos 2\pi(2nt + d) dt = 0$$

and

$$\int_c^{c+\frac{1}{2}} \sin 2\pi(2nt + d) dt = 0.$$

To show the completeness of T_n , $n \in \mathbb{N}_0$, we assume that $\int_0^1 f(t) \cos 2\pi g_{2n}(t) dt = 0$ and $\int_0^1 f(t) \cos 2\pi g_{2n+1}(t) dt = 0$ for $f \in L^2(I)$. These with the definition of g_n imply that

$$(2.10) \quad \int_0^{1/2} f(t) \cos 2\pi(2nt + b_n) dt = \int_{1/2}^1 f(t) \cos 2\pi(2nt + d_n) dt = 0.$$

Likewise, equations $\int_0^1 f(t) \sin 2\pi g_{2n}(t) dt = 0$ and $\int_0^1 f(t) \sin 2\pi g_{2n+1}(t) dt = 0$ imply that

$$(2.11) \quad \int_0^{1/2} f(t) \sin 2\pi(2nt + b_n) dt = \int_{1/2}^1 f(t) \sin 2\pi(2nt + d_n) dt = 0.$$

On the other hand, we note that the classical system $1, \cos 2\pi nt, \sin 2\pi nt$, $n \in \mathbb{N}$, is complete in the space $L^2[\alpha, \alpha+1]$ for any $\alpha \in \mathbb{R}$. Hence, system $1, \cos 4\pi nt, \sin 4\pi nt$, $n \in \mathbb{N}$, is complete in the space $L^2[\alpha, \alpha + 1/2]$. Furthermore since for each $\beta_n \in \mathbb{R}$ and $n \in \mathbb{Z}$, there hold the formulas

$$\cos 4\pi nt = \cos(2\pi\beta_n) \cos 2\pi(2nt + \beta_n) + \sin(2\pi\beta_n) \sin 2\pi(2nt + \beta_n)$$

and

$$\sin 4\pi nt = \cos(2\pi\beta_n) \sin 2\pi(2nt + \beta_n) - \sin(2\pi\beta_n) \cos 2\pi(2nt + \beta_n).$$

We conclude that $1, \cos 2\pi(2nt + \beta_n), \sin 2\pi(2nt + \beta_n), n \in \mathbb{N}$, is complete in the space $L^2[\alpha, \alpha + 1/2]$. Therefore, by this argument with (2.10) and (2.11), we find that $f = 0$ on both $[0, 1/2]$ and $[1/2, 1]$. Finally, we conclude that $f = 0$ and the completeness is proved. \square

In the next theorem, we show that if the trigonometric system $T_n, n \in \mathbb{N}_0$, is constructed in terms of the frequency functions g_n defined by (2.5), the system is not orthogonal in general.

Theorem 2.5. *Let $g_n, n \in \mathbb{N}$, be the sequence of functions defined by (2.5). Then, the system $T_n, n \in \mathbb{N}_0$, is not orthogonal unless $c \in Z$.*

Proof. It suffices to show that T_n with $g_{2n+1}, n \in \mathbb{N}_0$ is not orthogonal unless $c \in Z$. For $n, m \in \mathbb{N}$ with $n \neq m$, it can be verified that $\langle \cos 2\pi g_{2n+1}, \cos 2\pi g_{2m+1} \rangle = 0$ only if $c \in Z$. \square

3. CHARACTERIZATIONS

In the last section, we present two constructions of piecewise linear spectral sequences $g_n, n \in Z$, with the knot at the half. Our next task is to investigate necessary and sufficient conditions on such a piecewise linear spectral sequence. We need the following technical lemma for the proof of necessary conditions.

Lemma 3.1. *Suppose that $g_n, n \in Z$, defined by (2.1) with $a_n, c_n \in 2Z$ and $\theta = 1/2$ is a spectral sequence. For each $p \in 2Z$, let $\mathcal{A}_p := \{n \in Z : a_n = p\}$ and $\mathcal{C}_p := \{n \in Z : c_n = p\}$. Then, the cardinality $\#\mathcal{A}_p = \#\mathcal{C}_p = 2$.*

Proof. By hypotheses, $f_n, n \in Z$, is an orthonormal basis for $L^2(I)$. We then have the Parseval identity, i.e., for any $f \in L^2(I)$,

$$(3.1) \quad \sum_{n \in Z} \left| \int_0^1 f(t) e^{2\pi i g_n(t)} dt \right|^2 = \int_0^1 |f(t)|^2 dt.$$

In particular, for any $p \in 2Z$, we choose $f(t) := e^{2\pi i p t} \chi_{[0, \frac{1}{2}]}(t), t \in I$, where χ_A denotes the characteristic function of set A and identity (3.1) reduces to

$$(3.2) \quad \sum_{n \in Z} \left| \int_0^{\frac{1}{2}} e^{2\pi i(a_n - p)t} dt \right|^2 = \frac{1}{2}.$$

Noting that $a_n \in 2Z, p \in 2Z$ and $a_n - p \in 2Z$, by using Lemma 2.1, we conclude that

$$(3.3) \quad \int_0^{\frac{1}{2}} e^{2\pi i(a_n - p)t} dt = \begin{cases} 0, & a_n \neq p, \\ \frac{1}{2}, & a_n = p. \end{cases}$$

It follows from (3.2) and (3.3) that $\#\mathcal{A}_p = 2$.

Likewise, if for any $p \in 2Z$ we choose $f(t) := e^{2\pi i p t} \chi_{[\frac{1}{2}, 1]}(t), t \in I$, in (3.1), we conclude that $\#\mathcal{C}_p = 2$. \square

We now consider our first case and set

$$(3.4) \quad g_n(t) := \begin{cases} a_n t + b_n, & t \in [0, \frac{1}{2}), \\ a_n t + d_n, & t \in [\frac{1}{2}, 1], \end{cases}$$

where $a_n, b_n, d_n \in R$. Note that given real-valued functions g_n , the system $e^{2\pi i g_n}$, $n \in Z$, constitutes an orthonormal basis of $L^2(I)$ if and only if $e^{2\pi i(g_n - g_0)}$, $n \in Z$, does. Without loss of generality, we will assume $g_0 = 0$ in our discussion.

The following theorem shows that the two constructions presented in Section 2 are essentially all that we can have, if we assume that g_n has the form given in (3.4).

Theorem 3.2. *Suppose that $g_n, n \in Z$, are defined by (3.4) and let $G := \{g_n : n \in Z\}$. Then, $g_n, n \in Z$, is a spectral sequence of I if and only if $G = \{u_n, v_n : n \in Z\}$, where u_n and $v_n, n \in Z$, are defined by*

$$(3.5) \quad u_n(t) := \begin{cases} 2nt + b_n, & t \in [0, \frac{1}{2}), \\ 2nt + d_n, & t \in [\frac{1}{2}, 1], \end{cases} \quad \text{and} \quad v_n(t) := \begin{cases} (2n + c)t + b'_n, & t \in [0, \frac{1}{2}), \\ (2n + c)t + d'_n, & t \in [\frac{1}{2}, 1], \end{cases}$$

with b_n, d_n, b'_n, d'_n and c satisfying one of the following conditions:

- (i) $(b_n - b'_n) - (d_n - d'_n) \in Z + \frac{1}{2}$ and $c = 0$.
- (ii) $b_n - d_n \in Z, b'_n - d'_n \in Z + \frac{1-c}{2}$, and $c \in R \setminus 2Z$.

Proof. The sufficient condition is proved by a similar argument used in the proof of Theorems 2.2 and 2.3.

We now prove the necessary condition. Because the system $f_n, n \in Z$, is an orthonormal basis for $L^2(I)$ and $g_0 = 0$, we have that $\langle f_n, f_0 \rangle = 0$, for $n \in Z \setminus \{0\}$. Using (3.4), we conclude for $n \in Z \setminus \{0\}$ that $\int_0^{\frac{1}{2}} e^{2\pi i(a_n t + b_n)} dt + \int_{\frac{1}{2}}^1 e^{2\pi i(a_n t + d_n)} dt = 0$. Thus, $(e^{2\pi i b_n} + e^{2\pi i(\frac{a_n}{2} + d_n)}) I(a_n, 0) = 0$. This implies that either $I(a_n, 0) = 0$, which by Lemma 2.1 is equivalent to $a_n \in 2Z \setminus \{0\}$, or $e^{2\pi i b_n} + e^{2\pi i(\frac{a_n}{2} + d_n)} = 0$, which is equivalent to $d_n \in Z - \frac{a_n}{2} + b_n + \frac{1}{2}$.

Associated with sequence a_n , we introduce two sets of exponent functions $G_1 := \{g_n \in G : a_n \in 2Z\}$ and $G_2 := \{g_n \in G : a_n \notin 2Z, d_n \in Z - \frac{a_n}{2} + b_n + \frac{1}{2}\}$. We observe that $G = G_1 \cup G_2$ and $G_1 \neq \emptyset$ due to $g_0 \in G_1$.

If $G_2 = \emptyset$, then $G = G_1$ and by Lemma 3.1, for each $p \in 2Z, \#A_p = 2$. In this case, we define the set G by (3.5) with $c = 0$. By the hypothesis of this theorem, we have that for $n \in Z, \langle e^{2\pi i u_n(\cdot)}, e^{2\pi i v_n(\cdot)} \rangle = 0$. On the other hand, by using (3.5) with $c = 0$, a direct computation leads to $\langle e^{2\pi i u_n(\cdot)}, e^{2\pi i v_n(\cdot)} \rangle = \frac{1}{2}[e^{2\pi i(b_n - b'_n)} + e^{2\pi i(d_n - d'_n)}]$. Hence, $e^{2\pi i(b_n - b'_n)} + e^{2\pi i(d_n - d'_n)} = 0$, and thus we have that $(b_n - b'_n) - (d_n - d'_n) \in Z + \frac{1}{2}$, which is condition (i).

If $G_2 \neq \emptyset$, fixed $g_n \in G_2$, then for each $g_m \in G_1$, we have that $a_m \in 2Z, a_n \notin 2Z$ and $d_n \in Z - \frac{a_n}{2} + b_n + \frac{1}{2}$. Furthermore we have

$$(3.6) \quad \int_0^1 e^{2\pi i(g_m(t) - g_n(t))} dt = \int_0^{\frac{1}{2}} e^{2\pi i[(a_m - a_n)t + (b_m - b_n)]} dt + \int_{\frac{1}{2}}^1 e^{2\pi i[(a_m - a_n)t + (d_m - d_n)]} dt.$$

Since $a_m \in 2Z$ and $d_n \in Z - \frac{a_n}{2} + b_n + \frac{1}{2}$, the second term in the right-hand side of (3.6) becomes

$$\int_0^{\frac{1}{2}} e^{2\pi i[(a_m - a_n)t + \frac{a_m - a_n}{2} + (d_m - d_n)]} dt = -e^{2\pi i(d_m - b_n)} I(a_m - a_n, 0).$$

Hence, it follows from formula (3.6) and the hypothesis for the spectral sequence that $[e^{2\pi i(b_m-b_n)} - e^{2\pi i(d_m-b_n)}]I(a_m - a_n, 0) = 0$.

Again, since $a_m \in 2Z, a_n \notin 2Z$, we have that $a_m - a_n \notin 2Z$ and thus, by Lemma 2.1, $I(a_m - a_n, 0) \neq 0$. Thus, $e^{2\pi i b_m} = e^{2\pi i d_m}$. This is equivalent to $b_m - d_m \in Z$ for each $g_m \in G_1$. By the definition of G_1 , we have that $G_1 = \{g_n : a_n \in 2Z, b_n - d_n \in Z\}$. It follows from this equation that for each $n \in Z, e^{2\pi i(2n+1)t}$ is orthogonal to $e^{2\pi i g_m(t)}$ for every $g_m \in G_1$. Hence, we conclude that G_2 contains infinitely many elements. Now for fixed $g_l \in G_2$ and for every $g_n \in G_2$ with $n \neq l$, because $d_n \in Z - \frac{a_n}{2} + b_n + \frac{1}{2}$ and $d_l \in Z - \frac{a_l}{2} + b_l + \frac{1}{2}$, the orthogonality of f_n and f_l implies $2e^{2\pi i(b_n-b_l)}I(a_n - a_l, 0) = 0$. This ensures that $I(a_n - a_l, 0) = 0$, which, by Lemma 2.1, is equivalent to $a_n - a_l \in 2Z \setminus \{0\}$. Consequently, $a_n \in 2Z + c$ with $c =: a_l \in R \setminus 2Z$ for each $n \in Z_2$. Recalling the definition of G_2 , we obtain that

$$(3.7) \quad G_2 := \left\{ g_n \in G : a_n \in 2Z + c, d_n \in Z + b_n + \frac{1-c}{2} \right\}.$$

The desired condition (ii) follows from (3.8) and (3.9). □

In previous cases, the functions g_n are defined in the way that $a_n = c_n$. It is easy to construct examples in which $a_n \neq c_n$. Note that for any real valued function $\tilde{g}, g_n + \tilde{g}, n \in Z$, is a spectral sequence of I if and only if $g_n, n \in Z$, is. For example, if we choose

$$\tilde{g}(t) = \begin{cases} at, & t \in [0, \frac{1}{2}), \\ (2+c)t, & t \in [\frac{1}{2}, 1], \end{cases}$$

with those g_n constructed in the last two sections, we can construct many orthonormal bases $f_n, n \in Z$, for $L^2(I)$. To study this general case, we consider the form

$$(3.8) \quad g_n(t) = \begin{cases} a_n t + b_n, & t \in [0, \frac{1}{2}), \\ c_n t + d_n, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Theorem 3.3. *Suppose that $g_n, n \in Z$, is defined by (3.8) with $g_0 = 0, a_1 = 0$, and let $G := \{g_n : n \in Z\}$. Then $g_n, n \in Z$, is a spectral sequence of I if and only if $G = \{u_n, v_n : n \in Z\}$, where u_n, v_n are defined by*

$$(3.9) \quad u_n(t) = \begin{cases} 2nt + b_n, & t \in [0, \frac{1}{2}), \\ c_n t + d_n, & t \in [\frac{1}{2}, 1], \end{cases} \quad v_n(t) = \begin{cases} 2nt + b'_n, & t \in [0, \frac{1}{2}), \\ c_n t + d'_n, & t \in [\frac{1}{2}, 1], \end{cases}$$

with constants b_n, c_n, d_n, b'_n and d'_n satisfying the conditions that $\{c_n : n \in Z\} = 2Z, c_n \neq c_m$ for $n \neq m$ and $(b'_n - d'_n) - (b_n - d_n) \in Z + \frac{1}{2}$.

Proof. We first prove the sufficiency. Due to (3.9) the orthogonality of $e^{2\pi i u_n}, e^{2\pi i v_n}, n \in Z$, follows from Lemma 2.1 with the assumption $\{c_n : n \in Z\} = 2Z, c_n \neq c_m$ for $n \neq m$. The proof of the completeness for the system is similar to Theorem 2.2. Suppose that $\langle f, e^{2\pi i u_n} \rangle = \langle f, e^{2\pi i v_n} \rangle = 0, n \in Z$, and we wish to derive $f = 0$. In fact, by (3.9), we have that $e^{2\pi i b_n} A_n + e^{2\pi i d_n} B_n = 0$ and $e^{2\pi i b'_n} A_n + e^{2\pi i d'_n} B_n = 0$, where $A_n := \int_0^{\frac{1}{2}} f(t) e^{-2\pi i 2nt} dt$ and $B_n := \int_{\frac{1}{2}}^1 f(t) e^{-2\pi i c_n t} dt$. Because of the given assumption $(b'_n - d'_n) - (b_n - d_n) \in Z + \frac{1}{2}$, the second equation reduces to $e^{2\pi i [d'_n + (b_n - d_n) + \frac{1}{2}]} A_n + e^{2\pi i d'_n} B_n = 0$ or equivalently, $-e^{2\pi i b_n} A_n + e^{2\pi i d'_n} B_n = 0$. Combining this with the first equation, we obtain that $A_n = B_n = 0$. Since $\{c_n : n \in Z\} = 2Z$, we conclude that $f = 0$.

Now, we prove the necessity. Since $f_n, n \in Z$, is an orthonormal basis of $L^2(I)$, for $f \in L^2(I)$ we have the Parseval identity (3.1). In particular, we choose

$f = \chi_{[0, \frac{1}{2}]}$, the characteristic function on the interval $[0, \frac{1}{2}]$ in (3.1). Note that the hypotheses $g_0 = 0$ and $a_1 = 0$ imply $\left| \int_0^{\frac{1}{2}} e^{2\pi i(a_0 t + b_0)} dt \right| = \frac{1}{2}$ and $\left| \int_0^{\frac{1}{2}} e^{2\pi i(a_1 t + b_1)} dt \right| = \frac{1}{2}$. Hence, in this case, identity (3.1) reduces to $\sum_{n \in Z, n \neq 0, 1} \left| \int_0^{\frac{1}{2}} e^{2\pi i(a_n t + b_n)} dt \right|^2 = 0$. That is, $I(a_n, 0) = 0$, for $n \in Z \setminus \{0, 1\}$. It follows from Lemma 2.1 that $a_n \in 2Z \setminus \{0\}$ for $n \in Z \setminus \{0, 1\}$. By $g_0 = 0$, the last equation and the orthogonality of f_n , $n \in Z$, we obtain that

$$\begin{aligned} \langle e^{2\pi i g_n(t)}, e^{2\pi i g_0(t)} \rangle &= \int_0^{\frac{1}{2}} e^{2\pi i(a_n t + b_n)} dt + \int_{\frac{1}{2}}^1 e^{2\pi i(c_n t + d_n)} dt \\ &= \int_{\frac{1}{2}}^1 e^{2\pi i(c_n t + d_n)} dt = 0. \end{aligned}$$

Furthermore, we have that $I(c_n, 1/2) = 0$, for $n \in Z \setminus \{0, 1\}$. Again, by Lemma 2.1, this implies that $c_n \in 2Z \setminus \{0\}$ for $n \in Z \setminus \{0, 1\}$.

Next we prove $c_1 \in 2Z$. Since $a_3 \in 2Z \setminus \{0\}$ and $a_1 = 0$, we have that

$$\int_0^{\frac{1}{2}} e^{2\pi i[(a_3 - a_1)t + (b_3 - b_1)]} dt = 0,$$

by Lemma 2.1. Using the orthogonality of f_n , $n \in Z$, we conclude that

$$\int_{\frac{1}{2}}^1 e^{2\pi i[(c_3 - c_1)t + (d_3 - d_1)]} dt = 0.$$

This is equivalent to $c_3 - c_1 \in 2Z \setminus \{0\}$, due to Lemma 2.1. Moreover, since $c_3 \in 2Z$, we have that $c_1 \in 2Z$.

Summarizing the discussion above with the assumption that $g_0 = 0$, we conclude that $a_n, c_n \in 2Z$, for $n \in Z$. Moreover, it follows from Lemma 3.1 that $\#\mathcal{A}_p = \#\mathcal{C}_p = 2$, for each $p \in 2Z$. Therefore, we can assume that $G = \{u_n, v_n : n \in Z\}$, where u_n is as defined in (3.9) and v_n is as defined in (3.9) with c_n replaced by c'_n and with $c_n, c'_n \in 2Z$. Note that $c_n - c'_n \in 2Z$ and

$$(3.10) \quad I(c_n - c'_n, 1/2) = \begin{cases} \frac{1}{2}, & c_n = c'_n, \\ 0, & c_n \neq c'_n. \end{cases}$$

Again, by the orthogonality of f_n , $n \in Z$, we obtain that

$$(3.11) \quad \langle e^{2\pi i u_n}, e^{2\pi i v_n} \rangle = \frac{1}{2} e^{2\pi i(b_n - b'_n)} + e^{2\pi i(d_n - d'_n)} I(c_n - c'_n, 1/2) = 0.$$

Consequently, we must have that $I(c_n - c'_n, 1/2) \neq 0$. Employing (3.10) we find that $c_n = c'_n$. In this case, equation (3.11) implies that $e^{2\pi i(b_n - b'_n)} + e^{2\pi i(d_n - d'_n)} = 0$ and hence, $(b'_n - d'_n) - (b_n - d_n) \in Z + \frac{1}{2}$.

For $n \neq m$, by assumption, $\langle e^{2\pi i u_n}, e^{2\pi i u_m} \rangle = 0$. Using the definition of function u_n , we have that

$$\langle e^{2\pi i u_n}, e^{2\pi i u_m} \rangle = \int_0^{\frac{1}{2}} e^{2\pi i[(2n-2m)t + (b_n - b_m)]} dt + \int_{\frac{1}{2}}^1 e^{2\pi i[(c_n - c_m)t + (d_n - d_m)]} dt.$$

By Lemma 2.1, we see that the first integral in the right-hand side of the above equation vanishes. Therefore, by this equation and our assumption, we obtain that $e^{2\pi i(d_n - d_m)} I(c_n - c_m, 1/2) = 0$, from which we conclude that $c_n \neq c_m$ again by Lemma 2.1. □

4. NON-CONTINUITY OF SPECTRAL SEQUENCES

We will prove in this section that the spectral sequences $g_n, n \in Z$, of form (2.1), cannot be continuous except for the classical case. We first develop a technical lemma.

Lemma 4.1. *For $\theta \in (0, 1)$ let*

$$g(t) = \begin{cases} at + b, & t \in [0, \theta), \\ ct + d, & t \in [\theta, 1] \end{cases}$$

be continuous and assume that $\int_0^1 e^{2\pi ig(t)} dt = 0$. Then, either $a = c$ or $\theta a \in Z, (1 - \theta)c \in Z$.

Proof. Note that the continuity of g implies $d = (a - c)\theta + b$. Using this relation and the hypothesis of the lemma, we have that $\int_0^\theta e^{2\pi iat} dt + e^{2\pi i(a-c)\theta} \int_\theta^1 e^{2\pi ict} dt = 0$.

We next show that if either a or c is zero, then both of them are zero. To this end, we suppose $a = 0$ and $c \neq 0$. Integration of the equation above yields that

$$\theta + e^{-2\pi ic\theta} \frac{e^{2\pi ic} - e^{2\pi ic\theta}}{2\pi ic} = \frac{1}{2\pi ic} [2\pi ic\theta + e^{2\pi i(1-\theta)c} - 1] = 0.$$

However,

$$|2\pi ic\theta + e^{2\pi i(1-\theta)c} - 1| \geq |2\pi ic\theta - 1| - |e^{2\pi i(1-\theta)c}| = \sqrt{1 + (2\pi c\theta)^2} - 1 > 0,$$

a contradiction. Likewise, $c = 0$ implies $a = 0$.

We next consider the case when $a \neq 0$ and $c \neq 0$. In this case, we have that

$$\frac{e^{2\pi ia\theta} - 1}{2\pi ia} + \frac{e^{2\pi i(a-c)\theta}}{2\pi ic} [e^{2\pi ic} - e^{2\pi ic\theta}] = 0.$$

Multiplying both sides by $2\pi iace^{-2\pi ia\theta}$, we have that

$$c(1 - e^{-2\pi ia\theta}) = a[1 - e^{2\pi i(1-\theta)c}],$$

which is equivalent to

$$(4.1) \quad c(1 - \cos 2\pi a\theta) = a[1 - \cos 2\pi(1 - \theta)c]$$

and

$$(4.2) \quad c \sin 2\pi a\theta = -a \sin 2\pi(1 - \theta)c.$$

We conclude from (4.1) that $\theta a \in Z$ if and only if $(1 - \theta)c \in Z$ and from (4.2) that $\theta a \in Z/2$ if and only if $(1 - \theta)c \in Z/2$.

To complete the proof of this lemma, it suffices to prove $a = c$ under the assumptions $\theta a \notin Z$ or $(1 - \theta)c \notin Z$. By the conclusion stated in the last paragraph we know that $\theta a \notin Z$ is equivalent to $(1 - \theta)c \notin Z$. Hence, in the rest of this proof, we assume that $\theta a \notin Z$. If $\theta a \in \frac{Z}{2}$, then $\theta a \in \frac{Z}{2} \setminus Z$ and $(1 - \theta)c \in \frac{Z}{2} \setminus Z$. That is, both $2\theta a$ and $2(1 - \theta)c$ are odd integers. Thus, by (4.1), $a = c$. If $\theta a \notin Z/2$, then $(1 - \theta)c \notin Z/2$. Therefore, (4.1) and (4.2) imply

$$\frac{c(1 - \cos 2\pi a\theta)}{c \sin 2\pi a\theta} = \frac{a[1 - \cos 2\pi(1 - \theta)c]}{-a \sin 2\pi(1 - \theta)c}.$$

Now we have $\tan \pi a\theta = -\tan \pi(1 - \theta)c$, which ensures that $\pi a\theta = -\pi(1 - \theta)c + \pi k$ with $k \in Z$. Substituting it into (4.2), we conclude that $a = c$ because $(1 - \theta)c \notin Z/2$. □

With this lemma, we are ready to show our main result in this section. To state the theorem, we define

$$(4.3) \quad g_n(t) := \begin{cases} a_n t + b_n, & t \in [0, \theta), \\ c_n t + d_n, & t \in [\theta, 1]. \end{cases}$$

Theorem 4.2. *Suppose that $g_n, n \in Z$, defined by (4.3) with $g_0 = 0$ is a continuous spectral sequence of I . Let $G := \{g_n : n \in Z\}$. Then, $G = \{h_n : n \in Z\}$ where $h_n(t) = nt + b_n, t \in I, n \in Z$.*

Proof. According to Lemma 4.1, either $a_n = c_n$ or $\theta a_n \in Z, (1 - \theta)c_n \in Z$. If $a_n = c_n$, then due to the continuity of g_n the corresponding $g_n(t) = a_n t + b_n, t \in I$. Now, since $g_0 = 0$ and for $n \neq 0, \langle e^{2\pi i g_n}, e^{2\pi i g_0} \rangle = \int_0^1 e^{2\pi i g_n(t)} dt = 0$, we have that $a_n \in Z$. We denote by Z_1 the collection of all such a_n . Let $G_1 := \{h(t) = lt + e \in G : l \in Z_1\}$ and $G_2 = G \setminus G_1$. Then, $G = G_1 \cup G_2$. It suffices to show $G_2 = \emptyset$, since in this case the completeness of $\{e^{2\pi i h} : h \in G_1\}$ leads to $Z_1 = Z$ and furthermore to the desired result.

Suppose $g \in G_2$; we would have a contradiction. To see this, we let

$$g(t) = \begin{cases} at + b, & t \in [0, \theta), \\ ct + d, & t \in [\theta, 1]. \end{cases}$$

Then, by the definition of G_2 and the continuity of g , we see the properties that (i) $a \neq c$, (ii) $\theta a \in Z, (1 - \theta)c \in Z$, and (iii) $d = (a - c)\theta + b$.

We next consider two cases.

Case 1: $c \in Z_1$. In this case, we let $h(t) = ct + e, t \in I$. Then, $h \in G_1$. By orthogonality, we have that

$$(4.4) \quad \langle e^{2\pi i g}, e^{2\pi i h} \rangle = \int_0^\theta e^{2\pi i [(at+b)-(ct+e)]} dt + \int_\theta^1 e^{2\pi i [(ct+d)-(ct+e)]} dt.$$

By a direct computation, equation (4.4) becomes

$$e^{2\pi i (b-e)} \frac{e^{2\pi i (a-c)\theta} - 1}{2\pi i (a-c)} + (1 - \theta)e^{2\pi i (d-e)} = 0.$$

Note that $\theta a \in Z$ implies $e^{2\pi i a\theta} = 1$ and property (iii) implies

$$e^{2\pi i (d-e)} = e^{2\pi i [(a-c)\theta + (b-e)]}.$$

By using properties (ii) and (iii), the above identity reduces to

$$e^{2\pi i (b-e)} e^{-2\pi i c\theta} \frac{1}{2\pi i (a-c)} [1 - e^{2\pi i c\theta} + 2\pi i (1 - \theta)(a - lc)] = 0.$$

From this, we have arrived at $1 - e^{2\pi i c\theta} + 2\pi i (1 - \theta)(a - c) = 0$. However,

$$|1 - e^{2\pi i c\theta} + 2\pi i (1 - \theta)(a - c)| \geq |1 + 2\pi i (1 - \theta)(a - c)| - |e^{2\pi i c\theta}| > 0,$$

a contradiction.

Case 2. $c \notin Z_1$. We first show the claim that if $l \in Z_1$, then $l\theta \in Z$. If $l = a$, then $l\theta = \theta a \in Z$ due to property (ii). If $l \neq a$, define $h(t) = lt + e, t \in I$. Note that $c \notin Z_1$. Then

$$\int_0^1 e^{2\pi i [g(t) - h(t)]} dt = e^{2\pi i (b-e)} \frac{e^{2\pi i (a-l)\theta} - 1}{2\pi i (a-l)} + e^{2\pi i (d-e)} \frac{e^{2\pi i (c-l)} - e^{2\pi i (c-l)\theta}}{2\pi i (c-l)} = 0,$$

which by property (iii) reduces to

$$\frac{e^{2\pi i(a-l)\theta} - 1}{2\pi i(a-l)} + e^{2\pi i(a-c)\theta} \frac{e^{2\pi i(c-l)} - e^{2\pi i(c-l)\theta}}{2\pi i(c-l)} = 0.$$

Finally, it follows from property (ii) that $\frac{e^{-2\pi i l \theta} - 1}{(a-l)} + \frac{1 - e^{-2\pi i l \theta}}{(c-l)} = 0$. Since $a \neq c$, one receives $1 - e^{-2\pi i l \theta} = 0$. That is, $l\theta \in Z$. Our claim is proved.

Now we are in the position to receive a contradiction. Let $h_0 = \chi_{[0,\theta]}$. Then, for each $g \in G_2$, because of property (ii), we have that $\langle h_0, e^{2\pi i g} \rangle = \int_0^\theta e^{2\pi i(at+b)} dt = \theta \int_0^1 e^{2\pi i(a\theta t+b)} dt = 0$. Similarly, for each $h \in G_1$, by our claim, $\langle h_0, e^{2\pi i h} \rangle = \int_0^\theta e^{2\pi i(lt+b)} dt = \theta \int_0^1 e^{2\pi i(l\theta t+b)} dt = 0$. But the completeness of $f_n, n \in Z$, implies $h_0 = 0$, which is a contradiction. \square

5. A PIECEWISE CONSTANT SPECTRAL SEQUENCE AND THE WALSH SYSTEM

In this section, we will construct a piecewise constant spectral sequence of I . From this sequence, we recover the classical Walsh system (cf., [6, 7, 8]).

Let $J_n =: \{0, 1, \dots, 2^n - 1\}$. Then $J_n \subset J_{n+1}$ for $n \in \mathbb{N}_0$. We now describe the construction of the piecewise constant spectral sequence $g_n, n \in \mathbb{N}_0$. Beginning with $g_0 = 0$ on the interval I , we define for $j \in J_n$ recursively

$$g_{2^n+j}(t) = \begin{cases} g_j(t), & t \in [t_{n,2k}, t_{n,2k+1}), \\ g_j(t) + \frac{1}{2}, & t \in [t_{n,2k+1}, t_{n,2k+2}) \end{cases}$$

for $j \in J_n$, where $t_{n,k} := \frac{k}{2^{n+1}}, k = 0, 1, \dots, 2^{n+1}$. Clearly, g_j is well defined for $j \in J_{n+1}$ and each $g_n, n \in \mathbb{N}_0$, is a piecewise constant function with the values in $Z \cup (Z + \frac{1}{2})$.

Theorem 5.1. *Let $g_n, n \in \mathbb{N}_0$, be given as above. Then, $g_n, n \in \mathbb{N}_0$, is a spectral sequence of I and $f_n, n \in \mathbb{N}_0$, is the Walsh system on I .*

Proof. We use the induction on n to prove the orthogonality of system $f_n, n \in \mathbb{N}_0$, corresponding to $g_n, n \in \mathbb{N}_0$. That is, we will prove that $\langle e^{2\pi i g_j}, e^{2\pi i g_l} \rangle = 0$, for $j, l \in J_n$.

We first consider the case $n = 1$. Since $g_0 = 0$ and $g_1 = \frac{1}{2}\chi_{[\frac{1}{2},1]}$, it is easy to see $\langle e^{2\pi i g_0}, e^{2\pi i g_1} \rangle = 0$.

We assume that $\langle e^{2\pi i g_j}, e^{2\pi i g_l} \rangle = 0$, for $j, l \in J_n$ and show that it holds for $j, l \in J_{n+1}$. Suppose that both j and l are in $J_{n+1} \setminus J_n$. Hence, there exist $j_0, l_0 \in J_n$ such that $j = 2^n + j_0$ and $l = 2^n + l_0$. In this case, using the induction hypothesis, we have that $\langle e^{2\pi i g_j}, e^{2\pi i g_l} \rangle = \langle e^{2\pi i g_{j_0}}, e^{2\pi i g_{l_0}} \rangle = 0$. Suppose that j is in $J_{n+1} \setminus J_n$ and $l \in J_n$. Hence, there exists $j_0 \in J_n$ such that $j = 2^n + j_0$. Consequently, by the definition of $g_{2^n+j_0}$, we obtain that

$$\begin{aligned} \langle e^{2\pi i g_j}, e^{2\pi i g_l} \rangle &= \int_0^1 e^{2\pi i(g_{2^n+j_0} - g_l)} dt \\ &= \sum_{k=0}^{2^n-1} \left[\int_{t_{n,2k}}^{t_{n,2k+1}} e^{2\pi i(g_{j_0} - g_l)} dt + \int_{t_{n,2k+1}}^{t_{n,2k+2}} e^{2\pi i(g_{j_0} - g_l + \frac{1}{2})} dt \right]. \end{aligned}$$

Since on intervals $(t_{n,2k}, t_{n,2k+1})$ and $(t_{n,2k+1}, t_{n,2k+2})$, $c_{j_0,l} := g_{j_0}(t) - g_l(t)$ has the same constant value, we have that

$$\langle e^{2\pi i g_j}, e^{2\pi i g_l} \rangle = \sum_{k=0}^{2^n-1} \frac{1}{2^{n+1}} (e^{2\pi i c_{j_0,l}} - e^{2\pi i c_{j_0,l}}) = 0.$$

This advances the induction hypothesis and proves the orthogonality.

To show the completeness, we denote by X_n the space of the piecewise constants with the knots at dyadic points $\frac{k}{2^n}$, $k = 1, 2, \dots, 2^n - 1$. Note that the dimension of X_n is 2^n . Also, $\{e^{2\pi i g_j} : j \in J_n\} \subseteq X_n$, $\#J_n = 2^n$ and $e^{2\pi i g_j}$, $j \in J_n$, are orthonormal. We then conclude that $\{e^{2\pi i g_j} : j \in J_n\}$ spans X_n . Since $\{X_n : n \in \mathbb{N}_0\}$ is dense in $L^2(I)$, the completeness follows.

The identification of f_n , $n \in \mathbb{N}_0$, with the classical Walsh system on I can be verified by an induction on n . \square

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