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ON MORDELL-TORNHEIM ZETA VALUES

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ABSTRACT. We prove that the Mordell-Tornheim zeta value of depth r can be expressed as a rational linear combination of products of the Mordell-Tornheim zeta values of lower depth than r when r and its weight are of different parity.

1. INTRODUCTION

For complex numbers $s_1, s_2, \ldots, s_r, s$, Matsumoto defined the Mordell-Tornheim r-fold zeta function $\zeta_{MT,r}(s_1, s_2, \ldots, s_r; s)$ by

$$\sum_{m_1,m_2,\ldots,m_r=1}^{\infty} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^s}, \tag{1.1}$$

where the sum is over r-tuples of positive integers (see [4, 5]). He showed that this function can be continued meromorphically to the whole r-dimensional complex space. The origin of this function goes back to Tornheim and Mordell. Tornheim investigated the properties of $\zeta_{MT,2}(k_1, k_2; k)$ for positive integers k_1, k_2, k and discovered some relations (see [8]). Later Mordell independently considered $\zeta_{MT,2}(k, k; k)$ for any even positive integer k and also studied the values

$$\sum_{m_1,m_2,\ldots,m_r=1}^{\infty} \frac{1}{m_1 m_2 \cdots m_r (m_1 + \cdots + m_r + a)}$$

with $a > r$ (see [6], see also [7]). By using this result, Hoffman gave some evaluation formulas for $\zeta_{MT,r}(1, \ldots, 1; k)$ for any positive integer k (see [1]).

In the present paper, we aim to consider $\zeta_{MT,r}(k_1, \ldots, k_r; k)$ for positive integers $k_1, \ldots, k_r, k$ with $k \geq 2$. We call it the Mordell-Tornheim zeta value of depth r and of weight $\sum_{j=1}^{r} k_j + k$ as well as the multiple zeta values defined by

$$\sum_{1 \leq m_1 < m_2 < \cdots < m_r} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}} \tag{1.2}$$

for positive integers $k_1,\ k_2,\ \ldots,\ k_r$ with $k_r \geq 2$ (see, for example, [1]).

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The main aim of this paper is to prove

Theorem 1.1. *The Mordell-Tornheim zeta value of depth r with $r \geq 2$ and of weight w can be expressed as a rational linear combination of products of the Mordell-Tornheim zeta values of lower depth than r, when its depth r and its weight w are of different parity.*

Note that this theorem is an analogue of the result on multiple zeta values, which was proved by Zagier (see [3]), and has recently been proved in a different method by the author (see [9]).

The case $r = 2$ of Theorem 1.1 was proved in [8] and the explicit formulas were given in [2]. The case $r = 3$ was proved in [10].

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2. Preliminaries

We use the same notation as in [9]. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, and $\mathbb{R}$ the field of real numbers. Throughout this paper we fix $\delta \in \mathbb{R}$ with $\delta > 0$. For $u \in \mathbb{R}$ with $1 \leq u \leq 1 + \delta$, we define

$$\phi(s;u) = \sum_{m=1}^{\infty} \frac{(-u)^{-m}}{m^s} \qquad (s \in \mathbb{R}). \tag{2.1}$$

If $u > 1$, then $\phi(s;u)$ is convergent for any $s \in \mathbb{R}$. In the case when $u = 1$, let $\phi(s) = \phi(s;1) = (2^{1-s} - 1)\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function. Corresponding to $\phi(s;u)$, we define a set of numbers $\{\varepsilon_m(u)\}$ by

$$F(x;u) = \frac{(1+u)e^x}{e^x + u} = \sum_{m=0}^{\infty} \varepsilon_m(u) \, \frac{x^m}{m!} \quad (|x| < \pi). \tag{2.2}$$

Note that

$$\varepsilon_{2j}(1) = 0 \quad (j \in \mathbb{N}). \tag{2.3}$$

It follows from (2.2) that if $u \in [1, 1+\delta]$, then

$$\liminf_{m \to \infty} \left(\frac{|\varepsilon_m(u)|}{m!} \right)^{-1/m} \geq \pi \quad \text{and} \quad \frac{|\varepsilon_n(u)|}{n!} \leq \frac{M}{\gamma^n} \quad (n \in \mathbb{N}_0) \tag{2.4}$$

for any γ with $0 < \gamma < \pi$, where M is the constant independent of n and u.

$$\phi(-k;u) = -\frac{1}{1+u}\varepsilon_k(u) \tag{2.5}$$

for $k \in \mathbb{N}_0$ and $u \in (1, 1+\delta]$ (see [9], Lemma 1). For simplicity we let

$$\mathcal{R}_n(\theta; k_1, \ldots, k_n; u) = \sum_{m_1, \ldots, m_n = 1}^{\infty} \frac{(-u)^{-\sum_{\nu=1}^{n} m_\nu} e^{i\left(\sum_{\nu=1}^{n} m_\nu\right)\theta}}{{m_1}^{k_1} \cdots {m_n}^{k_n}} \tag{2.6}$$

for $k_1, \ldots, k_n \in \mathbb{N}$, $u \in [1, 1+\delta]$ and $\theta \in \mathbb{R}$, where $i = \sqrt{-1}$. We define $\lambda_j = (1 + (-1)^j)/2$ for $j \in \mathbb{Z}$ and

$$H(\theta; k; u) = \mathcal{R}_1(\theta; k; u) - \sum_{\nu=0}^{k} \phi(k-\nu; u)\lambda_{k-\nu} \frac{(i\theta)^\nu}{\nu!} \tag{2.7}$$

for $k \in \mathbb{N}$, $\theta \in \mathbb{R}$ and $u \in [1, 1+\delta]$. It follows from (2.5) and (2.6) that

$$H(\theta;k;u) = \sum_{\nu=0}^{k} \phi(k-\nu;u)\lambda_{k-\nu+1}\frac{(i\theta)^\nu}{\nu!} - \frac{1}{1+u}\sum_{N=1}^{\infty}\varepsilon_N(u)\frac{(i\theta)^{N+k}}{(N+k)!}, \tag{2.8}$$

when $u \in (1, 1+\delta]$. From (2.4) we can see that each side of (2.8) is uniformly convergent with respect to $u \in (1, 1+\delta]$ if $\theta \in (-\pi, \pi)$ and $k \in \mathbb{N}$. So (2.8) holds for $u = 1$. Hence it follows from (2.3) that if k is even (resp. odd), then $H(\theta; k; 1)$ is odd (resp. even) function. Namely

$$H(-\theta;k;1) = (-1)^{k+1}H(\theta;k;1). \tag{2.9}$$

For $k_1, \dots, k_r \in \mathbb{N}$, $\theta \in \mathbb{R}$ and $u \in [1, 1+\delta]$, we define

$$G(\theta;k_1,\dots,k_r;u) = \prod_{j=1}^{r} H(\theta;k_j;u). \tag{2.10}$$

By (2.9), we have

$$G(-\theta;k_1,\dots,k_r;1) = (-1)^{\sum_{j=1}^{r} k_j + r}G(\theta;k_1,\dots,k_r;1). \tag{2.11}$$

Now we prepare some notation. Let

$$\Delta_s = \{J = \{j_1,\dots,j_s\} \mid J \subset \{1,2,\dots,r\}\} \text{ and } \Delta = \bigcup_{s=1}^{r-1}\Delta_s.$$

For $J = \{j_1, \dots, j_s\} \in \Delta$, let $\overline{J} = \{1, \dots, r\} \smallsetminus J$. Let $\Omega(u)$ be the $\mathbb{Q}$-algebra generated by $\{\phi(2j; u) \mid j \in \mathbb{N}_0\}$. Note that $\Omega(1) = \mathbb{Q}[\pi^2]$ because $\phi(s; 1) = \left(2^{1-s} - 1\right)\zeta(s)$.

By (2.7) and (2.10), we have the expansion

$$\begin{aligned} G(\theta;&k_1,\dots,k_r;u) \\ &= \mathcal{R}_r(\theta;k_1,\dots,k_r;u) \\ &\quad + \sum_{\substack{J=\{j_1,\dots,j_s\} \\ \in\Delta}} \sum_{\mu}{}^{*} B(J;\mu;u)\frac{(i\theta)^\mu}{\mu!}\mathcal{R}_s(\theta;k_{j_1},\dots,k_{j_s};u) \\ &\quad + \sum_{\nu}{}^{*} C(\nu;u)\frac{(i\theta)^\nu}{\nu!}, \end{aligned} \tag{2.12}$$

where the sums $\sum_\mu{}^*$ and $\sum_\nu{}^*$ are taken over all $\mu, \nu \in \mathbb{N}_0$ with

$$0 \le \mu \le \sum_{j\in\overline{J}} k_j,\ 0 \le \nu \le \sum_{l=1}^{r} k_l,$$

respectively, and $B(J; \mu; u), C(\nu; u) \in \Omega(u)$. Note that

$$B(J;\mu;1), C(\nu;1) \in \Omega(1) = \mathbb{Q}[\pi^2]. \tag{2.13}$$

3. Some lemmas

We define the multiple series

$$\rho_n(k_1,\ldots,k_n;s;u)=\sum_{m_1,m_2,\ldots,m_n=1}^{\infty}\frac{(-u)^{-\sum_{\nu=1}^{n}m_\nu}}{m_1^{k_1}\cdots m_n^{k_n}(m_1+\cdots+m_n)^s} \tag{3.1}$$

for $k_1,\ldots,k_n\in\mathbb{N}$, $s\in\mathbb{R}$ and $u\in[1,1+\delta]$. In particular when $n=1$, we have $\rho_1(k_1;s;u)=\phi(k_1+s;u)$. If $u\in(1,1+\delta]$, then

$$\mathcal{R}_n(\theta;k_1,\ldots,k_n;u)=\sum_{N=0}^{\infty}\rho_n(k_1,\ldots,k_n;-N;u)\frac{(i\theta)^N}{N!}. \tag{3.2}$$

For $N\in\mathbb{Z}$ and $u\in(1,1+\delta]$ we define

$$\begin{aligned}\mathcal{A}_N(k_1,\ldots,k_r;u)&\\ &=\rho_r(k_1,\ldots,k_r;-N;u)\\ &\quad+\sum_{\substack{J=\{j_1,\ldots,j_s\}\\ \in\Delta}}\sideset{}{^*}\sum_{\mu}B(J;\mu;u)\binom{N}{\mu}\rho_s(k_{j_1},\ldots,k_{j_s};\mu-N;u).\end{aligned} \tag{3.3}$$

Furthermore we define

$$\begin{aligned}\widetilde{\mathcal{A}}_N(k_1,\ldots,k_r;u)&\\ &=\begin{cases}\mathcal{A}_N(k_1,\ldots,k_r;u)+C(N;u) & (\text{if } 0\le N\le\sum_{j=1}^{r}k_j),\\ \mathcal{A}_N(k_1,\ldots,k_r;u) & (\text{otherwise}).\end{cases}\end{aligned} \tag{3.4}$$

Note that if $N\le -1$, then we can define

$$\widetilde{\mathcal{A}}_N(k_1,\ldots,k_r;1)=\lim_{u\to 1}\widetilde{\mathcal{A}}_N(k_1,\ldots,k_r;u).$$

By combining (2.12), (3.2) and (3.4), we have

Lemma 3.1. *With the above notation,*

$$G(\theta;k_1,\ldots,k_r;u)=\sum_{N=0}^{\infty}\widetilde{\mathcal{A}}_N(k_1,\ldots,k_r;u)\frac{(i\theta)^N}{N!}. \tag{3.5}$$

By (2.2), (2.8), (2.10) and (3.5), we have

Lemma 3.2. *With the above notation and for* $u\in(1,1+\delta]$,

$$\liminf_{N\to\infty}\left(\frac{|\widetilde{\mathcal{A}}_N(k_1,\ldots,k_r;u)|}{N!}\right)^{-1/N}\ge\pi\quad\textit{and}\quad\frac{|\widetilde{\mathcal{A}}_n(k_1,\ldots,k_r;u)|}{n!}\le\frac{\widetilde{M}}{\gamma^n} \tag{3.6}$$

for any $\gamma\in(0,\pi)$ *and* $n\in\mathbb{N}_0$, *where* $\widetilde{M}(>0)$ *is independent of* n *and* u. *Further*

$$\lim_{u\to 1}\widetilde{\mathcal{A}}_N(k_1,\ldots,k_r;u)\lambda_{N+1+r+\sum_{j=1}^{r}k_j}=0 \tag{3.7}$$

for $N\in\mathbb{N}_0$.

For simplicity we let

$$\mathcal{S}_n^p(\theta;k_1,\ldots,k_n;s;u)=\sum_{m_1,\ldots,m_n=1}^{\infty}\frac{(-u)^{-\sum_{\nu=1}^{n}m_\nu}\sin^{(p)}\left(\left(\sum_{\nu=1}^{n}m_\nu\right)\theta\right)}{{m_1}^{k_1}\cdots{m_n}^{k_n}(m_1+\cdots+m_n)^s} \tag{3.8}$$

for $p\in\mathbb{N}_0$, $s\in\mathbb{R}$, $k_1,\ldots,k_n\in\mathbb{N}$ and $u\in[1,1+\delta]$, where we denote the lth derivative of a function $f(\theta)$ by $f^{(l)}(\theta)$. Then we have the following lemma.

Lemma 3.3. *With the above notation and for $l \in \mathbb{N}$ with $l \geq 2$,*

$$\begin{aligned}
(3.9)\qquad & \mathcal{S}_r^p(\theta; k_1, \ldots, k_r; l+p; u) \\
&+ \sum_{\substack{J=\{j_1,\ldots,j_s\}\\ \in\Delta}} {\sum_{\mu}}^{*} B(J;\mu;u)(-1)^{\mu} \sum_{\sigma=0}^{\mu} \binom{l+p+\mu-1-\sigma}{\mu-\sigma} \\
&\qquad \times \mathcal{S}_s^{(\sigma+p)}(\theta; k_{j_1}, \ldots, k_{j_s}; l+p+\mu-\sigma; u) \frac{(-\theta)^{\sigma}}{\sigma!} \\
&+ i^{p-1} {\sum_{\nu}}^{*} C(\nu;u)\lambda_{\nu+l+1} \frac{(i\theta)^{\nu+l+p}}{(\nu+l+p)!} \\
&= i^{p-1} \sum_{n=-l-p}^{\infty} \widetilde{\mathcal{A}}_n(k_1, \ldots, k_r; u)\lambda_{n+l+1} \frac{(i\theta)^{n+l+p}}{(n+l+p)!}.
\end{aligned}$$

Proof. It is known that

$$\begin{aligned}
(3.10)\qquad & \sum_{\nu=0}^{b} \binom{a-1+b-\nu}{b-\nu} \frac{(-\theta)^{\nu}}{\nu!} \frac{\sin^{(\nu+p)}(\theta x)}{x^{a+b-\nu}} \\
&= i^{p-1} \sum_{N\geq 0} \binom{a-1+b-N}{b} \frac{(i\theta)^N}{N!} \lambda_{p+1+N}\ x^{-a-b+N}
\end{aligned}$$

for $a, b \in \mathbb{N}_0$ (see [9], (2.16)). We assume that $u \in (1, 1+\delta]$ and apply (3.10) with $a = l+p$ and $b = 0$. Then we obtain

$$\begin{aligned}
(3.11)\qquad & \mathcal{S}_r^p(\theta; k_1, \ldots, k_r; l+p; u) \\
&= i^{p-1} \sum_{N=0}^{\infty} \rho_r(k_1, \ldots, k_r; l+p-N; u)\lambda_{p+1+N} \frac{(i\theta)^N}{N!}.
\end{aligned}$$

Applying (3.10) with $a = l+p$ and $b = \mu$, we have

$$\begin{aligned}
(3.12)\qquad & \sum_{\substack{J=\{j_1,\ldots,j_s\}\\ \in\Delta}} {\sum_{\mu}}^{*} B(J;\mu;u) \sum_{\sigma=0}^{\mu} \binom{l+p+\mu-1-\sigma}{\mu-\sigma} \\
&\qquad \times \mathcal{S}_s^{\sigma+p}(\theta; k_{j_1}, \ldots, k_{j_s}; l+p+\mu-\sigma; u) \frac{(-\theta)^{\sigma}}{\sigma!} \\
&= i^{p-1} \sum_{N=0}^{\infty} \sum_{\substack{J=\{j_1,\ldots,j_s\}\\ \in\Delta}} {\sum_{\mu}}^{*} B(J;\mu;u) \binom{l+p+\mu-1-N}{\mu} \\
&\qquad \times \rho_s(k_{j_1}, \ldots, k_{j_s}; l+p+\mu-N; u)\lambda_{p+1+N} \frac{(i\theta)^N}{N!}.
\end{aligned}$$

Let $n = N-l-p$. By (3.3), (3.4), (3.11), (3.12) and using the well-known relation

$$\binom{-X}{j} = (-1)^j \binom{X+j-1}{j},$$

we obtain (3.9). □

4. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. Let $\mathcal{MT}_n$ be the $\mathbb{Q}$-algebra generated by the Mordell-Tornheim zeta values of depth lower than or equal to n. Assume $r \geq 2$. Then we aim to prove that the Mordell-Tornheim zeta values of depth r and of weight w belongs to $\mathcal{MT}_{r-1}$ if r and w are of different parity.

By (3.6), we can see that (3.9) is uniformly convergent with respect to $u \in (1, 1+\delta]$ if $l+p \geq 2$ and $\theta \in [-\pi, \pi]$. So we let $\theta = \pi$ and $u \to 1$ in both sides of (3.9). For simplicity we let

$$\begin{aligned}(4.1)\quad \mathcal{F}^p(k_1,\ldots,k_r;l) = &\sum_{\substack{J=\{j_1,\ldots,j_s\}\\ \in\Delta}} \sum_{\mu}{}^{*} (-1)^{\mu} B(J;\mu;1) \sum_{\sigma=0}^{\mu} \binom{l+p+\mu-1-\sigma}{\mu-\sigma} \\ &\times \mathcal{S}_s^{\sigma+p}(\pi; k_{j_1},\ldots,k_{j_s}; l+p+\mu-\sigma; 1) \frac{(-\pi)^{\sigma}}{\sigma!} \\ &+ i^{p-1} \sum_{\nu}{}^{*} C(\nu;1)\lambda_{\nu+l+1} \frac{(i\pi)^{\nu+l+p}}{(\nu+l+p)!}.\end{aligned}$$

By the facts $\mathcal{S}_s^{2m}(\pi; k_{j_1},\ldots,k_{j_s};k;1)=0$, $\mathcal{S}_s^{2m+1}(\pi; k_{j_1},\ldots,k_{j_s};k;1) \in \mathcal{MT}_s$ and (2.13), we have

$$(4.2)\qquad \pi^{p-1}\mathcal{F}^p(k_1,\ldots,k_r;l) \in \mathcal{MT}_{r-1} \quad (p=0,1),$$

when $l+p \geq 2$, because $\pi^2 \subset \mathcal{MT}_1 \subset \mathcal{MT}_{r-1}$ for $r \geq 2$. Suppose $l \equiv \sum_{\eta=1}^{r} k_\eta + r \pmod 2$ and $l+p \geq 2$. It follows from (3.7) that

$$\begin{aligned}(4.3)\qquad &\mathcal{S}_r^p(\pi; k_1,\ldots,k_r; l+p; 1) + \mathcal{F}^p(k_1,\ldots,k_r;l) \\ &= i^{p-1} \sum_{\mu=0}^{l+p-1} \widetilde{\mathcal{A}}_{\mu-l-p}(k_1,\ldots,k_r;1)\lambda_{\mu+p+1} \frac{(i\pi)^{\mu}}{\mu!}.\end{aligned}$$

Let $p=0$ and $l \geq 2$. Since $\mathcal{S}_r^0(\pi;k_1,\ldots,k_r;l;1)=0$, we have

$$(4.4)\qquad \mathcal{F}^0(k_1,\ldots,k_r;l) = i^{-1} \sum_{\nu=0}^{[(l-2)/2]} \widetilde{\mathcal{A}}_{2\nu+1-l}(k_1,\ldots,k_r;1) \frac{(i\pi)^{2\nu+1}}{(2\nu+1)!}.$$

Now we recall the following lemma.

Lemma 4.1 ([9], Lemma 7). *Suppose $\{P_m\}$ and $\{Q_m\}$ are sequences which satisfy the relation*

$$\sum_{j=0}^{[m/2]} P_{m-2j} \frac{(i\pi)^{2j}}{(2j+1)!} = Q_m,$$

for any $m \in \mathbb{N}_0$. Then the relation

$$P_m = -2 \sum_{\nu=0}^{m} \phi(m-\nu)\lambda_{m-\nu} Q_\nu$$

holds for any $m \in \mathbb{N}_0$.

Now we give the proof of Theorem 1.1. Applying Lemma 4.1 with

$$\begin{aligned}P_m &= \widetilde{\mathcal{A}}_{-m-1}(k_1,\ldots,k_r;1)\lambda_{\sum_{\eta=1}^{r} k_\eta + r + m}, \\ Q_m &= \frac{1}{\pi}\mathcal{F}^0(k_1,\ldots,k_r;m+2)\lambda_{\sum_{\eta=1}^{r} k_\eta + r + m},\end{aligned}$$

it follows from (4.2) and (4.4) that

$$\begin{aligned}(4.5)\qquad &\widetilde{\mathcal{A}}_{-m-1}(k_1,\ldots,k_r;1)\\ &= -2\sum_{\nu=0}^{m}\phi(m-\nu)\lambda_{m+\nu}\frac{1}{\pi}\mathcal{F}^0(k_1,\ldots,k_r;m-\nu+2)\in\mathcal{MT}_{r-1}\end{aligned}$$

for any $m\in\mathbb{N}_0$ with $m\equiv\sum_{\eta=1}^{r}k_\eta+r \pmod 2$. Furthermore let $k\in\mathbb{N}$ with $k\geq 2$ and $k\not\equiv\sum_{\eta=1}^{r}k_\eta+r \pmod 2$. Namely $\sum_{\eta=1}^{r}k_\eta+k$ and r are of different parity. Applying (4.3) with $l=k-1$ and $p=1$, we have

$$\begin{aligned}(4.6)\qquad &\zeta_{MT,r}(k_1,\ldots,k_r;k)+\mathcal{F}^1(k_1,\ldots,k_r;k-1)\\ &= \sum_{\nu=0}^{[(k-2)/2]}\widetilde{\mathcal{A}}_{2\nu-k}(k_1,\ldots,k_r;1)\frac{(i\pi)^{2\nu}}{(2\nu)!},\end{aligned}$$

because $\mathcal{S}_r^1(\pi;k_1,\ldots,k_r;k;1)=\zeta_{MT,r}(k_1,\ldots,k_r;k)$. By combining (4.2), (4.5) and (4.6), we obtain the proof of Theorem 1.1.

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