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UNIQUENESS OF POSITIVE SOLUTIONS FOR SINGULAR PROBLEMS INVOLVING THE p-LAPLACIAN

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Abstract. We study existence and uniqueness of positive eigenfunctions for the singular eigenvalue problem: $-\Delta_p u - \lambda\eta(x)\frac{u^{p-1}}{|x|^p} = \mu\frac{u^{p-1}}{|x|^p}$ on a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with zero boundary condition. We also characterize all positive solutions of $-\Delta_p u = |\frac{N-p}{p}|^p \frac{u^{p-1}}{|x|^p}$ in $\mathbb{R}^N \setminus \{0\}$.

1. Introduction

This note is concerned with solutions of the following nonlinear eigenvalue problem:

$$(1.1) \qquad \begin{cases} -\Delta_p u - \lambda\eta(x)\frac{|u|^{p-2}u}{|x|^p} &= \mu\frac{|u|^{p-2}u}{|x|^p} \quad \text{in } \Omega\setminus\{0\},\\ u &> 0 \quad \text{in } \Omega\setminus\{0\},\\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ of class C^2 with $0 \in \Omega$, $p \in (1,\infty)\setminus\{N\}$, $\eta \in C^\alpha(\overline{\Omega})$ $(\alpha \in (0,1))$ such that $\eta \geq 0, \eta \not\equiv 0$ in $\overline{\Omega}$ and $\eta(0) = 0$, $\mu, \lambda \in \mathbb{R}$ are two parameters and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. By the regularity results of Tolksdorf [11] and Di Benedetto [4] any weak solution $u \in \bigcap_{r>0} W^{1,p}(\Omega \setminus B_r(0))$ to (1.1) actually belongs to $C^1(\overline{\Omega}\setminus\{0\})$.

Naturally related to (1.1) is the following variational problem:

$$(1.2) \qquad I[\lambda] := \inf_{0\neq v\in W_0^{1,p}(\Omega\setminus\{0\})} \frac{\int_\Omega |\nabla v|^p - \lambda\int_\Omega \frac{\eta|v|^p}{|x|^p}}{\int_\Omega (|v|/|x|)^p}\,.$$

Recall that in the case $\lambda = 0$ it follows from the well-known Hardy inequality,

$$(1.3) \qquad \int_\Omega |\nabla u|^p \geq \left|\frac{N-p}{p}\right|^p \int_\Omega (|u|/|x|)^p\,, \quad \forall u \in W_0^{1,p}(\Omega\setminus\{0\}),$$

that $I[0] = c^*_{p,N} := |\frac{N-p}{p}|^p$. Moreover, by a simple construction of test functions approximating $|x|^{1-N/p}$ (see [8]), it can be shown that we always have $I[\lambda] \leq c^*_{p,N}$. It was also proved in [8] (in the spirit of [3]) that there exists $\lambda^* > 0$ such that $I[\lambda] = c^*_{p,N}$ for $\lambda \leq \lambda^*$ and $I[\lambda] < c^*_{p,N}$ for $\lambda > \lambda^*$. If $u \in W_0^{1,p}(\Omega\setminus\{0\})$ is a minimizer for (1.2), then u is clearly a solution to the equation in (1.1) with

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$\mu = I[\lambda]$. Since $|u|$ is also a minimizer, hence a solution to the same equation, it follows from Serrin's Harnack principle [9] that $|u| > 0$ in $\Omega \setminus \{0\}$. Therefore, either u or $-u$ is a solution to (1.1). It turns out that solutions of (1.1) can be obtained as minimizers for (1.2) if and only if $I[\lambda] < c^*_{p,N}$. In this direction we have:

Theorem 1.1. *A solution u to* (1.1) *with $\mu = I[\lambda]$ which belongs to $W_0^{1,p}(\Omega \setminus \{0\})$ exists if and only if $I[\lambda] < c^*_{p,N}$. In this case u is a minimizer for* (1.2) *and it is the unique solution to* (1.1) *up to a multiplicative factor.*

The critical role played by the value $c^*_{p,N}$ for the existence of a minimizer is analogous to a phenomenon that was observed first by Marcus, Mizel and Pinchover [6] in the context of another version of Hardy's inequality. The existence of a minimizer in the subcritical case $I[\lambda] < c^*_{p,N}$ is proved in the appendix, using a "concentration-compactness principle" [5], similar to the one used in [6, 7]. The uniqueness part in the subcritical case is then essentially known ([2, 1, 7]); see the appendix for details.

The next natural question is what happens in the critical case $I[\lambda] = c^*_{p,N}$. In that case the existence of a solution $u \in \bigcap_{q\in[1,p)} W_0^{1,q}(\Omega \setminus \{0\}) \setminus W_0^{1,p}(\Omega \setminus \{0\})$ was proved in [8]. We are then left with the problem of uniqueness. This is the subject of the main theorem of this note:

Theorem 1.2. *In the case $I[\lambda] = c^*_{p,N}$ the solution to* (1.1) *with $\mu = c^*_{p,N}$ is unique up to a multiplicative factor.*

Uniqueness in the linear case $p = 2$ is known and easy (for any value of μ, see Remark 3.1), but we could not find an easy proof for general p. Our argument, presented in Section 3, is based on Harnack's inequality and on the following Liouville type theorem, which may be of independent interest.

Theorem 1.3. *Let u be a solution of*

$$\begin{cases} -\Delta_p u &= c^*_{p,N} \frac{u^{p-1}}{|x|^p} \quad \text{in } \mathbb{R}^N \setminus \{0\}, \\ u &> 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \end{cases} \tag{1.4}$$

Then $u(x) = C|x|^{1-N/p}$ for some positive constant C.

2. Characterization of global solutions

In the proof of Theorem 1.3 we will make use of an extension of Picone's identity for the p-Laplacian, due to Allegretto and Huang [1], that we now recall. Let Ω be a subdomain of $\mathbb{R}^N$. For u, v in $C^1(\Omega)$ such that $u \geq 0$ and $v > 0$ in Ω, denote

$$\begin{aligned} L(u,v) &= |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u|\nabla v|^{p-2}\nabla v, \\ R(u,v) &= |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v. \end{aligned} \tag{2.1}$$

Then $L(u,v) = R(u,v) \geq 0$ in Ω and $L(u,v) = R(u,v) = 0$ in Ω if and only if $u = Cv$ for some constant C; see [1].

Proof of Theorem 1.3. By the regularity results of Tolksdorf [11] and Di Benedetto [4] any weak solution to (1.4) belongs to $C^1(\mathbb{R}^N \setminus \{0\})$. Clearly $v(x) = |x|^{1-N/p}$ is a solution of (1.4). We must show that u is a multiple of v. For each $R > 0$ we define a cut-off function ψ_R as follows:

$$\psi_R(x) = \begin{cases} 0, & 0 \le |x| \le \frac{1}{R^2}, \\ 2 + \frac{\log|x|}{\log R}, & \frac{1}{R^2} \le |x| \le \frac{1}{R}, \\ 1, & \frac{1}{R} \le |x| \le R, \\ 2 - \frac{\log|x|}{\log R}, & R \le |x| \le R^2, \\ 0, & |x| \ge R^2. \end{cases} \tag{2.2}$$

Clearly $\psi_R(x)$ is a Lipschitz function with compact support in $\mathbb{R}^N$. By Picone's identity (2.1) and (1.4),

(2.3)
$$\begin{aligned} 0 \le \int_{\mathbb{R}^N} L(\psi_R v, u) &= \int_{\mathbb{R}^N} R(\psi_R v, u) = \int_{\mathbb{R}^N} |\nabla(\psi_R v)|^p - \int_{\mathbb{R}^N} \nabla\Big(\frac{\psi_R^p v^p}{u^{p-1}}\Big)|\nabla u|^{p-2}\nabla u \\ &= \int_{\mathbb{R}^N} \Big(|\nabla(\psi_R v)|^p - \frac{c^*_{p,N}}{|x|^p}\psi_R^p v^p\Big). \end{aligned}$$

On the other hand, testing (1.4) for v against $\psi_R^p v$ yields

$$\int_{\mathbb{R}^N} \frac{c^*_{p,N}}{|x|^p}\psi_R^p v^p = \int_{\mathbb{R}^N} \psi_R^p|\nabla v|^p + p\psi_R^{p-1} v|\nabla v|^{p-2}\nabla v\nabla\psi_R. \tag{2.4}$$

Combining (2.3) with (2.4) we obtain,

$$\begin{aligned} 0 \le I(R) &:= \int_{R^N} L(\psi_R v, u) \\ &= \int_{\mathbb{R}^N} \Big(|\psi_R\nabla v + v\nabla\psi_R|^p - \psi_R^p|\nabla v|^p - p\psi_R^{p-1} v|\nabla v|^{p-2}\nabla v\nabla\psi_R\Big). \end{aligned} \tag{2.5}$$

Next we shall prove that $\lim_{R\to\infty} I(R) = 0$. We recall the following elementary inequality which holds for all $z_1, z_2 \in \mathbb{R}^N$ (see [10, Lemma A.4]):

$$|z_1+z_2|^p - |z_1|^p - p|z_1|^{p-2}z_1\cdot z_2 \le \begin{cases} \frac{p(p-1)}{2}(|z_1|+|z_2|)^{p-2}|z_2|^2, & \text{for } p>2, \\ \gamma_p|z_2|^p & \text{for } p \le 2, \end{cases} \tag{2.6}$$

for some constant $\gamma_p > 0$. Assume first that $p \le 2$. Then from (2.2), (2.5) and (2.6) we get

$$\begin{aligned} I(R) &\le C\Big(\int_{\frac{1}{R^2}}^{\frac{1}{R}} r^{p-N}\big(\frac{1}{r\log R}\big)^p r^{N-1}dr + \int_R^{R^2} r^{p-N}\big(\frac{1}{r\log R}\big)^p r^{N-1}dr\Big) \\ &= \frac{2C}{(\log R)^{p-1}}\,, \end{aligned} \tag{2.7}$$

hence $\lim_{r\to\infty} I(R) = 0$ in this case. Now consider the case $p > 2$. Thanks to (2.6) and (2.7) we need only show that

$$\int_{A_R} |\nabla v|^{p-2}v^2|\nabla\psi_R|^2 \to 0 \text{ as } R\to\infty, \tag{2.8}$$

where $A(R) = \{\frac{1}{R^2} < |x| < \frac{1}{R}\} \cup \{R < |x| < R^2\}$. Indeed, (2.8) follows from

$$\int_{A_R} |\nabla v|^{p-2} v^2 |\nabla \psi_R|^2 \le C \int_{\frac{1}{R^2}}^{\frac{1}{R}} r^{-\frac{N}{p}(p-2)} r^{2(1-\frac{N}{p})} \frac{r^{N-1}dr}{r^2(\log R)^2} + C \int_R^{R^2} r^{-\frac{N}{p}(p-2)} r^{2(1-\frac{N}{p})} \frac{r^{N-1}dr}{r^2(\log R)^2} = \frac{2C}{\log R} \to 0 \text{ as } R \to \infty.$$

Therefore, in all cases we have $\lim_{R\to\infty} \int_{R^N} L(\psi_R v, u) = 0$. By Fatou's Lemma we infer that $L(v,u) \in L^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} L(v,u) = 0$. Hence, $L(v,u) = 0$ in $\mathbb{R}^N$ which implies that for some constant $C > 0$ we have $u = Cv = C|x|^{1-N/p}$. □

From Theorem 1.3 we next deduce the asymptotic radial symmetry at the origin of solutions on a bounded domain.

Corollary 2.1. *Let $u \in W^{1,p}_{loc}(B_R(0) \setminus \{0\})$ be a solution of*

$$\begin{cases} -\Delta_p u &= \frac{u^{p-1}}{|x|^p}\big(c^*_{p,N} + \eta(x)\big) \text{ on } B_R(0) \setminus \{0\}, \\ u &> 0 \text{ on } B_R(0) \setminus \{0\}, \end{cases} \tag{2.9}$$

where $\eta(x) \in C(B_R(0))$ satisfies $\eta(0) = 0$. Then,

$$\lim_{r\to 0^+} \frac{\overline{u}(r)}{\underline{u}(r)} = 1, \tag{2.10}$$

where for each $r \in (0,R)$ we denote

$$\overline{u}(r) = \max_{\{|x|=r\}} u(x)\ ,\ \underline{u}(r) = \min_{\{|x|=r\}} u(x). \tag{2.11}$$

Proof. Fix $n \in \mathbb{R}^N$ with $|n| = 1$ and for any $r \in (0, R/2)$ set

$$v_r(y) = \frac{u(ry)}{u(rn)}\,, \quad \text{for } y \in B_{R/r}(0) \setminus \{0\}. \tag{2.12}$$

Then v_r satisfies

$$-\Delta_p v_r = \frac{{v_r}^{p-1}}{|y|^p}(c^*_{p,N} + \eta(ry)) \quad \text{on } B_{R/r}(0) \setminus \{0\}\,, \tag{2.13}$$

and $v_r(n) = 1$. For any $D > 1$ denote $\mathcal{A}_D = \{1/D < |y| < D\}$. Then, by the Harnack inequality of Serrin [9] we infer that for each $D > 1$ there exist $K_D > 1$, such that for any $r < R/(2D)$ we have

$$\frac{1}{K_D} \le v_r(y) \le K_D \quad \text{on } \mathcal{A}_D. \tag{2.14}$$

Then, from the regularity results of [11, 4] it follows that there exist $L_D > 0$ and $\alpha_D \in (0,1)$, such that for each $r < R/(2D)$ we have

$$\|v_r(y)\|_{C^{1,\alpha_D}(\mathcal{A}_{D/2})} \le L_D, \tag{2.15}$$

where we denoted $\mathcal{A}_{D/2} = \{2/D < |y| < D/2\}$. Set $Q := \limsup_{r\to 0^+} \frac{\overline{u}(r)}{\underline{u}(r)}$ and choose a sequence $r_n \to 0$ such that $\lim_{n\to\infty} \frac{\overline{u}(r_n)}{\underline{u}(r_n)} = Q$. Using (2.15) we deduce from (2.13) that for a subsequence, still denoted by $\{r_n\}$, we have $v_{r_n} \to w_D$ in $C^1(\mathcal{A}_{D/2})$, where w_D is a positive solution of

$$-\Delta_p w_D = c^*_{p,N} \frac{w_D^{p-1}}{|y|^p} \quad \text{on } \mathcal{A}_{D/2}.$$

Choosing a sequence $D_m \nearrow \infty$ and passing to a diagonal subsequence, we obtain a subsequence $r_{n_k} \to 0$ such that $v_{r_{n_k}} \to v_0$ in $C^1_{loc}(\mathbb{R}^N \setminus \{0\})$, where v_0 is a positive solution of (1.4). By Theorem 1.3, $v_0(x) = C|x|^{1-N/p}$, hence $Q = 1$. □

3. Uniqueness of the eigenfunction in the critical case

This section is devoted to the proof of Theorem 1.2. We shall need the following comparison lemma.

Lemma 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2 and let $K \subset\subset \Omega$ be a compact subset. Suppose that $u_1, u_2 \in C(\overline{\Omega \setminus K}) \cap W^{1,p}(\Omega \setminus K)$ are such that $u_2 > 0$ on $(\Omega \setminus K) \cup \partial K$ and $u_1 = 0$ on $\partial\Omega$. Suppose further that*

$$-\Delta_p u_1 - a(x)|u_1|^{p-2}u_1 \leq 0 \quad \text{in } \Omega \setminus K,$$
$$-\Delta_p u_2 - a(x)|u_2|^{p-2}u_2 \geq 0 \quad \text{in } \Omega \setminus K,$$

for some $a(x) \in L^\infty_{loc}(\Omega \setminus K)$, and that

$$u_2 \geq u_1 \quad \text{on } \partial K.$$

Then,

$$u_2(x) \geq u_1(x) \quad \text{on } \Omega \setminus K.$$

Proof. The result follows directly from the argument of Proposition A.1 and Corollary A.1 in [10]. The results in [10] deal with the case where $K = \{x \in \Omega, \operatorname{dist}(x, \partial\Omega) \geq \beta\}$ for some $\beta > 0$, but it is clear that the argument there works for general K. □

Proof of Theorem 1.2. Let u_1, u_2 be two solutions to (1.1). Again by regularity theory we have $u_1, u_2 \in C^1(\overline{\Omega} \setminus \{0\})$. For all sufficiently small $r > 0$ set

$$f(r) = \min_{\{|x|=r\}} \frac{u_2(x)}{u_1(x)} \quad , \quad F(r) = \max_{\{|x|=r\}} \frac{u_2(x)}{u_1(x)}. \tag{3.1}$$

Thanks to Corollary 2.1 we have $\lim_{r\to 0^+} F(r)/f(r) = 1$, and we can define

$$m := \liminf_{r\to 0^+} f(r) = \liminf_{r\to 0^+} F(r) \in [0, \infty],$$
$$M := \limsup_{r\to 0^+} f(r) = \limsup_{r\to 0^+} F(r) \in [0, \infty].$$

We can assume without loss of generality that $m < \infty$ (otherwise we interchange u_1 and u_2). For every $\varepsilon > 0$ there exists a sequence $r_n \to 0$ such that

$$u_2(x) \leq (m + \varepsilon)u_1(x) \quad \text{on } \{|x| = r_n\}.$$

By Lemma 3.1 it follows that

$$u_2(x) \leq (m + \varepsilon)u_1(x) \quad \text{on } \Omega \setminus B_{r_n}(0). \tag{3.2}$$

Letting n tend to ∞ in (3.2) and then sending ε to 0, we obtain that

$$u_2(x) \leq m u_1(x) \quad \text{on } \Omega \setminus \{0\}.$$

It follows that $M = m > 0$. Interchanging u_1 and u_2 and repeating the above argument we also get that $u_2(x) \geq m u_1(x)$ and we conclude that $u_2 = m u_1$. □

Remark 3.1. In the *linear case* $p = 2$ the uniqueness (up to a multiplicative factor) of a solution to (1.1) is easy for any μ and λ. Indeed, if u and v are two solutions, we have by (4.12) below that $v \geq c_1 u$ in Ω for some $c_1 > 0$. Defining $\bar{c}_1 := \sup\{c_1 > 0;\ v \geq c_1 u\}$ we have $v - \bar{c}_1 u \geq 0$ in Ω. If $v - \bar{c}_1 u = 0$, we are done. Otherwise, since $v - \bar{c}_1 u$ is also a solution of the equation in (1.1) (recall that we are in the linear case), we have by the strong maximum principle $v - \bar{c}_1 u > 0$ in Ω. Hence by the above argument there exists $c_2 > 0$ such that $v - \bar{c}_1 u > c_2 u$ in Ω. This is a contradiction to the maximality of $\bar{c}_1$.

4. Appendix: Existence and uniqueness in the subcritical case

This section is devoted to the subcritical case $I[\lambda] < c^*_{p,N}$. Existence of a minimizer in (1.2) is established in the next proposition.

Proposition 4.1. *Assume that $I[\lambda] < c^*_{p,N}$. Let $\{u_n\} \subset W^{1,p}_0(\Omega)$ be a sequence of nonnegative functions satisfying*

$$\int\limits_\Omega \frac{u_n^p}{|x|^p} = 1,\ \forall n, \tag{4.1}$$

and

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^p - \lambda \int_\Omega \frac{\eta |u_n|^p}{|x|^p} = I[\lambda]. \tag{4.2}$$

Then $u_n \to u$ strongly in $W^{1,p}_0(\Omega \setminus \{0\})$, where u is a minimizer for (1.2).

Proof. We shall first prove the convergence of a subsequence. It is clear from (4.1)–(4.2) that $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega \setminus \{0\})$, and passing to a subsequence we may assume that

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}_0(\Omega \setminus \{0\}) \quad \text{and} \quad u_n \to u \text{ in } L^p(\Omega) \tag{4.3}$$

for some nonnegative $u \in W^{1,p}_0(\Omega \setminus \{0\})$. We denote $v_n = u_n - u$. Fix any $r > 0$ such that $B_{2r}(0) \subset \Omega$. Then

$$\Big| \int_{B_{2r}(0)} |\nabla u_n|^p - |\nabla v_n|^p \Big| \leq C \|\nabla (u_n - v_n)\|_{L^p(B_{2r}(0))} = C \|\nabla u\|_{L^p(B_{2r}(0))} := \varepsilon_r, \tag{4.4}$$

with $\varepsilon_r \to 0$ as $r \to 0$. Let $\varphi_r \in C^\infty_c(\mathbb{R}^N)$ denote a cut-off function satisfying:

(i) $\varphi_r(x) = 1$ for $|x| \leq r$,
(ii) $0 \leq \varphi_r(x) \leq 1$ for $r < |x| < 2r$,
(iii) $\varphi_r(x) = 0$ for $|x| \geq 2r$,
(iv) $\|\nabla \varphi_r\|_{L^\infty} \leq 4/r$.

By Hölder's inequality,

$$\begin{aligned}\int_{B_{2r}(0)} |\nabla(\varphi_r v_n)|^p &\leq \int_{B_{2r}(0)} |\varphi_r|^p |\nabla v_n|^p + p \int_{B_{2r}(0)} |v_n \nabla \varphi_r| \big(|v_n \nabla \varphi_r| + |\varphi_r \nabla v_n|\big)^{p-1} \\ &\leq \int_{B_{2r}(0)} |\nabla v_n|^p + \frac{4p}{r} \Big(\int_{B_{2r}(0)} \big(|v_n \nabla \varphi_r| + |\varphi_r \nabla v_n|\big)^p \Big)^{\frac{p-1}{p}} \\ &\quad \cdot \Big(\int_{B_{2r}(0)} |v_n|^p \Big)^{\frac{1}{p}}.\end{aligned}$$

Therefore, using (4.3) we infer that

$$\int_{B_{2r}(0)} |\nabla(\varphi_r v_n)|^p \le \int_{B_{2r}(0)} |\nabla v_n|^p + D_r\alpha_n, \tag{4.5}$$

with $D_r > 0$ and $\alpha_n \to 0$. By Hardy's inequality (1.3) we get

$$\int_{B_{2r}(0)} |\nabla(\varphi_r v_n)|^p \ge c^*_{p,N}\int_{B_{2r}(0)} \frac{|\varphi_r|^p|v_n|^p}{|x|^p} \ge c^*_{p,N}\int_{B_r(0)} \frac{|v_n|^p}{|x|^p},$$

which together with (4.4) and (4.5) yields

$$\int_{B_{2r(0)}} |\nabla u_n|^p \ge c^*_{p,N}\int_{B_r(0)} \frac{|v_n|^p}{|x|^p} - D_r\alpha_n - \varepsilon_r. \tag{4.6}$$

On the other hand, as in (4.4) we get

$$\Big|\int_{B_r(0)} \frac{u_n^p}{|x|^p} - \int_{B_r(0)} \frac{|v_n|^p}{|x|^p}\Big| \le C\Big(\int_{B_r(0)} (\frac{u}{|x|})^p\Big)^{1/p} := \delta_r\,, \tag{4.7}$$

with $\delta_r \to 0$ as $r \to 0$ (by (1.3)). Moreover, by (4.3) and (1.3) we easily deduce that

$$\lim_{n\to\infty}\int_\Omega \frac{\eta u_n^p}{|x|^p} - \int_\Omega \frac{\eta u^p}{|x|^p} = 0\,. \tag{4.8}$$

By (4.1), (4.6) and (4.7) we obtain

$$\begin{aligned}\int_\Omega |\nabla u_n|^p - \lambda\int_\Omega \frac{\eta u_n^p}{|x|^p} \ge \int_{\Omega\setminus B_{2r}(0)} |\nabla u_n|^p + c^*_{p,N}\Big(1 - \int_{\Omega\setminus B_r(0)} \frac{u_n^p}{|x|^p}\Big) - \lambda\int_\Omega \frac{\eta u_n^p}{|x|^p}\\ - c^*_{p,N}\delta_r - D_r\alpha_n - \varepsilon_r\,.\end{aligned} \tag{4.9}$$

Passing to the limit $n \to \infty$ in (4.9) and using (4.2), (4.3) and (4.8), yields

$$I[\lambda] \ge \int_{\Omega\setminus B_{2r}(0)} |\nabla u|^p + c^*_{p,N}\Big(1 - \int_{\Omega\setminus B_r(0)} \frac{u^p}{|x|^p}\Big) - \lambda\int_\Omega \frac{\eta u^p}{|x|^p} - c^*_{p,N}\delta_r - \varepsilon_r. \tag{4.10}$$

Passing to the limit $r \to 0$ in (4.10) gives, using the definition of $I[\lambda]$,

$$\begin{aligned} I[\lambda] &\ge \int_\Omega |\nabla u|^p + c^*_{p,N}\Big(1 - \int_\Omega \frac{u^p}{|x|^p}\Big) - \lambda\int_\Omega \frac{\eta u^p}{|x|^p}\\ &\ge c^*_{p,N}\Big(1 - \int_\Omega \frac{u^p}{|x|^p}\Big) + I[\lambda]\int_\Omega \frac{u^p}{|x|^p}.\end{aligned} \tag{4.11}$$

By (4.11) and our assumption $I[\lambda] < c^*_{p,N}$ it follows that $\int_\Omega \frac{u^p}{|x|^p} = 1$, and then by (4.11) we infer that u is a minimizer. Since $\int_\Omega \frac{u_n^p}{|x|^p} \to \int_\Omega \frac{u^p}{|x|^p} = 1$ we also get that $\int_\Omega |\nabla u_n|^p \to \int_\Omega |\nabla u|^p$, and the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega \setminus \{0\})$ follows. So far we proved the convergence of a subsequence of $\{u_n\}$. But since it is known that a solution to (1.1) in $W_0^{1,p}(\Omega \setminus \{0\})$ is unique up to a multiplicative factor (see [1, Theorem 2.1] and [7, Proposition 3.2]) the full convergence $u_n \to u$ follows as well. □

Proof of Theorem 1.1. Existence of a positive minimizer u for (1.2), which is then a solution of (1.1), follows directly from Proposition 4.1. Next we turn to the proof of the uniqueness part. Let v be another solution to (1.1) with $\mu = I[\lambda]$. We claim that there exists $c_1 > 1$ such that

$$\frac{u}{c_1} \le v \le c_1 u \quad \text{on } \Omega \setminus \{0\}. \tag{4.12}$$

Indeed, to see for example why u/v is bounded, assume by negation that $\sup_\Omega u/v = \infty$. By Hopf's boundary lemma [12] we must have $\lim_{r\to 0} \sup_{B_r(0)\setminus\{0\}} u/v = \infty$. Using Harnack's inequality [9] we obtain for some sequence $r_n \to 0$ that $u \ge nv$ on $\partial B_{r_n}(0)$. But then we get by Lemma 3.1 that $u \ge nv$ on $\Omega \setminus B_{r_n}(0)$. This is clearly a contradiction for n large enough, and (4.12) follows. On the other hand, by a simple rescaling argument, Harnack inequality ([9]) and C^1 regularity estimates ([11, 4]), it follows (see [10, Lemma A.3]) that, for some $c_2 > 0$, we have

$$\Big|\big(\frac{\nabla v}{v}\big)(x)\Big| \le \frac{c_2}{|x|} \qquad \text{on } B_d(0)\setminus\{0\},\ \text{with } d := \frac{1}{2}\min\{|y|;\, y \in \partial\Omega\}\,. \tag{4.13}$$

Combining (4.12) and (4.13) we get that

$$|\nabla v(x)| \le c_1 c_2 \frac{u(x)}{|x|} \quad \text{on } B_d(0)\setminus\{0\}. \tag{4.14}$$

By (4.14) and (1.3) we obtain that $v \in W_0^{1,p}(\Omega\setminus\{0\})$ and therefore v is a minimizer for (1.2). The fact that v is a multiple of u follows from the argument cited at the end of the proof of Proposition 4.1. To complete the proof we remark that in [8] it was proved that there is no solution in $W_0^{1,p}(\Omega\setminus\{0\})$ to (1.1) with $\mu = c^*_{p,N}$. $\square$

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