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CALDERÓN-ZYGMUND OPERATORS ON HARDY SPACES WITHOUT THE DOUBLING CONDITION

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ABSTRACT. Let μ be a non-negative Radon measure on $\mathbb{R}^d$ which only satisfies some growth condition. In this paper, the authors obtain the boundedness of Calderón-Zygmund operators in the Hardy space $H^1(\mu)$.

1. INTRODUCTION

In recent years, many papers focus on the analysis on $\mathbb{R}^d$ with non-doubling measure; see [2, 5, 6, 8, 3, 4, 1] and their references. Moreover, the analysis on such $\mathbb{R}^d$ was proved to play a striking role in solving the long open Painlevé's problem by Tolsa in [9]; see also [10] for more background of this. Throughout this paper, the Euclidean space $\mathbb{R}^d$ is endowed with a non-negative Radon measure μ which only satisfies the following growth condition that there exists $C_0 > 0$ such that

$$\mu(B(x,r)) \le C_0 r^n \tag{1.1}$$

for all $x \in \mathbb{R}^d$ and $r > 0$, where $B(x,r) = \{y \in \mathbb{R}^d : \ |y-x| < r\}$, n is a fixed number and $0 < n \le d$. Such a measure μ is not necessary to be doubling. We recall that μ is said to satisfy the doubling condition if there exists $C > 0$ such that $\mu(B(x,2r)) \le C\mu(B(x,r))$ for all $x \in \operatorname{supp}(\mu)$ and $r > 0$. It is well known that the doubling condition in the analysis on spaces of homogeneous type is a key assumption. However, some research has now indicated that the doubling condition is superfluous for most of the classical Calderón-Zygmund theory.

Let K be a function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,\, y) : x = y\}$ satisfying that for $x \ne y$,

$$|K(x,\, y)| \le C|x-y|^{-n}, \tag{1.2}$$

and for $|x-y| \ge 2|x-x'|$,

$$|K(x,\, y) - K(x',\, y)| + |K(y,\, x) - K(y,\, x')| \le C\frac{|x-x'|^\delta}{|x-y|^{n+\delta}}, \tag{1.3}$$

where $\delta \in (0,\, 1]$ and $C > 0$ is a constant. The Calderón-Zygmund operator associated to the above kernel K and the measure μ is formally defined by

$$Tf(x) = \int_{\mathbb{R}^d} K(x,\, y)f(y)d\mu(y).$$

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This integral may not be convergent for many functions. Thus we consider the truncated operators T_ε for $\varepsilon > 0$ defined by

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x, y) f(y) d\mu(y).$$

We say that T is bounded on $L^2(\mu)$ if the operators $\{T_\varepsilon\}_{\varepsilon>0}$ are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$. In this case, there is an operator $\widetilde{T}$ which is the weak limit as $\varepsilon \to 0$ of some subsequence of operators $\{T_\varepsilon\}_{\varepsilon>0}$; see [5]. It is easy to see that $\widetilde{T}$ is still bounded on $L^2(\mu)$; moreover, for $f \in L^2(\mu)$ with compact support and a. e. $x \in \mathbb{R}^d \setminus \operatorname{supp}(f)$,

$$\widetilde{T} f(x) = \int_{\mathbb{R}^d} K(x, y) f(y)\, d\mu(y) \tag{1.4}$$

with the same K as in (1.2) and (1.3). By the same argument of Tolsa as in [5, 7], we see that $\widetilde{T}$ is also bounded from $L^1(\mu)$ into weak-$L^1(\mu)$ and from $H^1(\mu)$ into $L^1(\mu)$.

In this paper, we will prove that $\widetilde{T}$ is bounded on the Hardy space $H^1(\mu)$ if $\widetilde{T}^*1 = 0$. Here, by $\widetilde{T}^*1 = 0$, we mean that for any bounded function b with compact support and $\int_{\mathbb{R}^d} b\, d\mu = 0$,

$$\int_{\mathbb{R}^d} \widetilde{T} b(x)\, d\mu(x) = 0. \tag{1.5}$$

We remark that for such a function b, $b \in H^1(\mu)$ and therefore, $\widetilde{T} b \in L^1(\mu)$. Also, if $\widetilde{T} b \in H^1(\mu)$, then $\widetilde{T} b$ should satisfy (1.5) by the definition of the Hardy space $H^1(\mu)$; see [5, 8] or Definition 2 below. Thus, in some sense, the condition (1.5) is also necessary.

If μ is the d-dimensional Lebesgue measure on $\mathbb{R}^d$, this result is well known and it was proved by verifying that $\widetilde{T}$ maps any atom of the Hardy space $H^1(\mathbb{R}^d)$ into some molecule. However, if μ only satisfies (1.1), it is still unknown if there is a molecular characterization for the Hardy space $H^1(\mu)$. We will prove that $\widetilde{T}$ is bounded on the Hardy space $H^1(\mu)$ via its "grand" maximal function characterization of Tolsa in [8] and its new atomic characterization of the authors in [1].

Definition 1. Given $f \in L^1_{\mathrm{loc}}(\mu)$, we set

$$M_\Phi f(x) = \sup_{\varphi \sim x} \left| \int_{\mathbb{R}^d} f \varphi\, d\mu \right|,$$

where the notation $\varphi \sim x$ means that $\varphi \in L^1(\mu) \cap C^1(\mathbb{R}^d)$ and satisfies

(i) $\|\varphi\|_{L^1(\mu)} \le 1$,
(ii) $0 \le \varphi(y) \le \frac{1}{|y-x|^n}$ for all $y \in \mathbb{R}^d$, and
(iii) $|\nabla \varphi(y)| \le \frac{1}{|y-x|^{n+1}}$ for all $y \in \mathbb{R}^d$, where $\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_d)$.

Based on Theorem 1.2 of Tolsa in [8], we define the Hardy space $H^1(\mu)$ as follows.

Definition 2. The Hardy space $H^1(\mu)$ is the set of all functions $f \in L^1(\mu)$ satisfying that $\int_{\mathbb{R}^d} f\, d\mu = 0$ and $M_\Phi f \in L^1(\mu)$. Moreover, we define the norm of $f \in H^1(\mu)$ by

$$\|f\|_{H^1(\mu)} = \|f\|_{L^1(\mu)} + \|M_\Phi f\|_{L^1(\mu)}.$$

Theorem 1. *Let K be the function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, y) : x = y\}$ satisfying* (1.2) *and* (1.3). *Suppose that the operator $\widetilde{T}$ in* (1.4) *is bounded on $L^2(\mu)$ and $\widetilde{T}^*1 = 0$ as in* (1.5). *Then $\widetilde{T}$ is bounded on $H^1(\mu)$.*

It is known that the dual space of $H^1(\mu)$ is the space $RBMO(\mu)$, which was introduced by Tolsa in [5]. From Theorem 1, the fact that $RBMO(\mu) = \left(H^1(\mu)\right)^*$ (see [5]) and a standard dual argument, it is easy to deduce the boundedness of the transpose operator of $\widetilde{T}$ in $RBMO(\mu)$ as below.

Corollary 1. *Let $\widetilde{T}$ be the same as in Theorem* 1. *Then $\widetilde{T}^*$, the transpose operator of $\widetilde{T}$, is bounded on $RBMO(\mu)$.*

Remark 1. Obviously, from different subsequences of operators $\{T_\varepsilon\}_{\varepsilon>0}$ which are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$, one may deduce different $\widetilde{T}$'s. However, they are all bounded on $L^2(\mu)$ and satisfy (1.4). But, the relation between these different $\widetilde{T}$'s is still open.

In what follows, C denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line.

2. Proof of Theorem 1

We begin with some necessary notation and definitions. Throughout this paper, we only consider the closed cubes with sides parallel to the coordinate axes. For any cube Q and any $\alpha > 0$, αQ denotes the cube with the same center as Q and $l(\alpha Q) = \alpha l(Q)$, where $l(Q)$ denotes the side length of the cube Q.

Given two cubes $Q \subset R$ in $\mathbb{R}^d$, set

$$K_{Q,\,R} = 1 + \sum_{k=1}^{N_{Q,\,R}} \frac{\mu(2^k Q)}{l(2^k Q)^n},$$

where $N_{Q,\,R}$ is the smallest positive integer k such that $l(2^k Q) \geq l(R)$; see [5] for some properties of $K_{Q,\,R}$.

To prove Theorem 1, we need to recall the atomic characterization of the Hardy space $H^1(\mu)$ as follows.

Definition 3. Let $\rho > 1$, $1 < p \leq \infty$ and $\gamma \in \mathbb{N}$. A function $b \in L^1_{\mathrm{loc}}(\mu)$ is called a (p, γ)-atomic block if

1. (1) there exists some cube R such that $\operatorname{supp}(b) \subset R$,
2. (2) $\int_{\mathbb{R}^d} b \, d\mu = 0$,
3. (3) for $j = 1, 2$, there are functions a_j supported on cube $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$, and

$$\|a_j\|_{L^p(\mu)} \leq [\mu(\rho Q_j)]^{1/p-1} \left[K_{Q_j,\,R}\right]^{-\gamma}.$$

Then we define

$$|b|_{H^{1,\,p}_{\mathrm{atb},\,\gamma}(\mu)} = |\lambda_1| + |\lambda_2|.$$

We say that $f \in H^{1,\,p}_{\mathrm{atb},\,\gamma}(\mu)$ if there are $(p,\,\gamma)$-atomic blocks $\{b_i\}_{i\in\mathbb{N}}$ such that

$$f = \sum_{i=1}^{\infty} b_i$$

with $\sum_{i=1}^{\infty} |b_i|_{H^{1,\,p}_{\mathrm{atb},\,\gamma}(\mu)} < \infty$. The $H^{1,\,p}_{\mathrm{atb},\,\gamma}(\mu)$ norm of f is defined by

$$\|f\|_{H^{1,\,p}_{\mathrm{atb},\,\gamma}(\mu)} = \inf\left\{\sum_i |b_i|_{H^{1,\,p}_{\mathrm{atb},\,\gamma}(\mu)}\right\},$$

where the infimum is taken over all the possible decompositions of f into (p,γ)-atomic blocks.

The above definition when $\gamma = 1$ was introduced by Tolsa in [5] and when $\gamma > 1$ by the authors in [1]. It was proved in [5, 1] that the definition of $H^{1,\,p}_{\mathrm{atb},\,\gamma}(\mu)$ is independent of the chosen constant $\rho > 1$, and for any integer $\gamma \geq 1$ and $1 < p \leq \infty$, all the atomic Hardy spaces $H^{1,\,p}_{\mathrm{atb},\,\gamma}(\mu)$ are just the Hardy space $H^1(\mu)$ with equivalent norms. We remark that in the proof of Theorem 1 below, we need to choose $\gamma > 1$, especially, $\gamma = 2$.

The following lemma will be used in the proof of Theorem 1.

Lemma 1. *Let M_Φ be as in Definition* 1 *and $1 < p < \infty$. Then M_Φ is bounded on $L^p(\mu)$.*

In fact, Tolsa proved that M_Φ is bounded from $H^1(\mu)$ into $L^1(\mu)$; see Lemma 3.1 in [8]. On the other hand, it is obvious that M_Φ is bounded on $L^\infty(\mu)$. By Theorem 7.2 in [5], we obtain that M_Φ is bounded on $L^p(\mu)$ for $1 < p < \infty$.

Now we turn to the proof of Theorem 1.

Proof of Theorem 1. By a standard argument, it suffices to verify that for any atomic block b as in Definition 3 with $\rho = 4$, $p = \infty$ and $\gamma = 2$, $\widetilde{T}b$ is in $H^1(\mu)$ with norm $C|b|_{H^{1,\,\infty}_{atb,\,2}(\mu)}$, where C is independent of b. Let all the notation be the same as in Definition 3. By our choices, a_j now satisfies the following size condition that

$$\|a_j\|_{L^\infty(\mu)} \leq \left[\mu(4Q_j)K^2_{Q_j,\,R}\right]^{-1}, \tag{2.1}$$

where $j = 1,\ 2$.

The assumption that $\widetilde{T}^*1 = 0$ tells us that $\int_{\mathbb{R}^d} \widetilde{T}b\,d\mu = 0$. Recalling that $\widetilde{T}$ is bounded from $H^1(\mu)$ into $L^1(\mu)$ (see [5]), we obtain

$$\|\widetilde{T}b\|_{L^1(\mu)} \leq C|b|_{H^{1,\,\infty}_{atb,\,2}(\mu)}.$$

By this and Definition 2, we deduce that the proof of Theorem 1 can be reduced to proving that

$$\|M_\Phi(\widetilde{T}b)\|_{L^1(\mu)} \leq C|b|_{H^{1,\,\infty}_{atb,\,2}(\mu)}. \tag{2.2}$$

Write

$$\|M_\Phi(\widetilde{T}b)\|_{L^1(\mu)} = \int_{4R} M_\Phi(\widetilde{T}b)(x)\,d\mu(x) + \int_{\mathbb{R}^d\setminus 4R} M_\Phi(\widetilde{T}b)(x)\,d\mu(x) = \mathrm{I} + \mathrm{II}.$$

Noting that M_Φ is sublinear, we can control I by

$$\mathrm{I} \le \int_{4R} M_\Phi \left[(\widetilde{T}b)\chi_{8R}\right](x)\,d\mu(x) + \int_{4R} M_\Phi \left[(\widetilde{T}b)\chi_{\mathbb{R}^d\setminus 8R}\right](x)\,d\mu(x) = \mathrm{I}_1 + \mathrm{I}_2.$$

From the fact that for $j = 1, 2$, $Q_j \subset R$, it follows that for any $z \in Q_j$ and any $y \in 2^{k+1}R \setminus 2^kR$, $k \ge 3$, $|y-z| \ge l(2^{k-2}R)$. By this fact, (ii) of Definition 1, (1.2) and (2.1), we obtain

$$\begin{aligned}
\mathrm{I}_2 &\le \int_{4R} \sup_{\varphi\sim x}\left[\int_{\mathbb{R}^d\setminus 8R} |\widetilde{T}b(y)|\varphi(y)\,d\mu(y)\right] d\mu(x)\\
&\le \sum_{j=1}^2 |\lambda_j| \int_{4R} \sum_{k=3}^\infty \int_{2^{k+1}R\setminus 2^kR} \left|\int_{Q_j} K(y,\,z)a_j(z)\,d\mu(z)\right| \frac{1}{|x-y|^n}\,d\mu(y)\,d\mu(x)\\
&\le C\sum_{j=1}^2 |\lambda_j| \sum_{k=3}^\infty \|a_j\|_{L^\infty(\mu)}\mu(Q_j)\frac{\mu(2^{k+1}R)}{l(2^{k-2}R)^n}\frac{\mu(4R)}{l(2^{k-2}R)^n}\\
&\le C\sum_{j=1}^2 |\lambda_j|.
\end{aligned}$$

To estimate I_1, we write

$$\begin{aligned}
\mathrm{I}_1 &\le \sum_{j=1}^2 |\lambda_j| \int_{4Q_j} M_\Phi\left[(\widetilde{T}a_j)\chi_{8R}\right](x)\,d\mu(x)\\
&\quad + \sum_{j=1}^2 |\lambda_j| \int_{4R\setminus 4Q_j} M_\Phi\left[(\widetilde{T}a_j)\chi_{2Q_j}\right](x)\,d\mu(x)\\
&\quad + \sum_{j=1}^2 |\lambda_j| \int_{4R\setminus 4Q_j} M_\Phi\left[(\widetilde{T}a_j)\chi_{8R\setminus 2Q_j}\right](x)\,d\mu(x)\\
&= \mathrm{I}_{11} + \mathrm{I}_{12} + \mathrm{I}_{13}.
\end{aligned}$$

The Hölder inequality, Lemma 1, the boundedness of $\widetilde{T}$ in $L^2(\mu)$ and (2.1) lead to

$$\begin{aligned}
\mathrm{I}_{11} &\le \sum_{j=1}^2 |\lambda_j|\mu(4Q_j)^{1/2}\left\|M_\Phi[(\widetilde{T}a_j)\chi_{8R}]\right\|_{L^2(\mu)}\\
&\le C\sum_{j=1}^2 |\lambda_j|\mu(4Q_j)^{1/2}\|\widetilde{T}a_j\|_{L^2(\mu)}\\
&\le C\sum_{j=1}^2 |\lambda_j|\mu(4Q_j)^{1/2}\|a_j\|_{L^2(\mu)}\\
&\le C\sum_{j=1}^2 |\lambda_j|\mu(4Q_j)\|a_j\|_{L^\infty(\mu)}\\
&\le C\sum_{j=1}^2 |\lambda_j|.
\end{aligned}$$

For $j = 1, 2$, denote $N_{Q_j,\,4R}$ simply by N_j. By (ii) of Definition 1, the Hölder inequality, the boundedness of $\widetilde{T}$ in $L^2(\mu)$ and (2.1), we have

$$\begin{aligned}
\mathrm{I}_{12} &\le \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \sup_{\varphi\sim x} \left| \int_{2Q_j} \widetilde{T}a_j(y)\varphi(y)\,d\mu(y) \right| d\mu(x)\\
&\le \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \frac{1}{l(2^{k-2}Q_j)^n}\,d\mu(x) \int_{2Q_j} \left|\widetilde{T}a_j(y)\right| d\mu(y)\\
&\le \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \frac{\mu(2^{k+1}Q_j)}{l(2^{k-2}Q_j)^n} \|\widetilde{T}a_j\|_{L^2(\mu)} \mu(2Q_j)^{1/2}\\
&\le C \sum_{j=1}^{2} |\lambda_j| K_{Q_j,\,R}\, \mu(2Q_j)^{1/2} \|a_j\|_{L^2(\mu)}\\
&\le C \sum_{j=1}^{2} |\lambda_j|,
\end{aligned}$$

where we have used the fact that

$$K_{Q_j,\,4R} \le C K_{Q_j,\,R}. \tag{2.3}$$

For I_{13}, we further decompose it into

$$\begin{aligned}
\mathrm{I}_{13} &= \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^k Q_j} M_\Phi \left[\left(\widetilde{T}a_j\right) \chi_{8R \setminus 2Q_j} \right](x)\,d\mu(x)\\
&\le \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^k Q_j} M_\Phi \left[|\widetilde{T}a_j| \chi_{2^{k+2}Q_j \setminus 2^{k-1}Q_j} \right](x)\,d\mu(x)\\
&\quad + \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^k Q_j} M_\Phi \left[|\widetilde{T}a_j| \chi_{\max\{2^{k+2}Q_j,\,8R\} \setminus 2^{k+2}Q_j} \right](x)\,d\mu(x)\\
&\quad + \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^k Q_j} M_\Phi \left[|\widetilde{T}a_j| \chi_{2^{k-1}Q_j \setminus 2Q_j} \right](x)\,d\mu(x)\\
&= \mathrm{E} + \mathrm{F} + \mathrm{G}.
\end{aligned}$$

Lemma 1, (1.2) and (2.1) tell us that

$$\begin{aligned}
\mathrm{E} &\le \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \mu(2^{k+1}Q_j)^{1/2} \left\| M_\Phi \left[|\widetilde{T}a_j| \chi_{2^{k+2}Q_j \setminus 2^{k-1}Q_j} \right] \right\|_{L^2(\mu)}\\
&\le C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \mu(2^{k+1}Q_j)^{1/2}\\
&\qquad \times \left\{ \int_{2^{k+2}Q_j \setminus 2^{k-1}Q_j} \left| \int_{Q_j} K(y,\,z) a_j(z)\,d\mu(z) \right|^2 d\mu(y) \right\}^{1/2}\\
&\le C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \frac{\mu(2^{k+2}Q_j)}{l(2^{k-3}Q_j)^n} \|a_j\|_{L^\infty(\mu)} \mu(Q_j)\\
&\le C \sum_{j=1}^{2} |\lambda_j|.
\end{aligned}$$

By (ii) of Definition 1, (1.2), (2.3) and (2.1), we easily see that

$$\begin{aligned}
\mathrm{G} &\leq \sum_{j=1}^{2}|\lambda_j|\sum_{k=2}^{N_j}\int_{2^{k+1}Q_j\setminus 2^kQ_j}\sup_{\varphi\sim x}\left[\int_{2^{k-1}Q_j\setminus 2Q_j}|\widetilde{T}a_j(y)|\varphi(y)\,d\mu(y)\right]d\mu(x)\\
&\leq \sum_{j=1}^{2}|\lambda_j|\sum_{k=2}^{N_j}\int_{2^{k+1}Q_j\setminus 2^kQ_j}\sum_{l=1}^{k-2}\int_{2^{l+1}Q_j\setminus 2^lQ_j}\left|\int_{Q_j}K(y,\,z)a_j(z)\,d\mu(z)\right|\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\frac{1}{|y-x|^n}\,d\mu(y)\,d\mu(x)\\
&\leq C\sum_{j=1}^{2}|\lambda_j|\sum_{k=2}^{N_j}\frac{\mu(2^{k+1}Q)}{l(2^{k+1}Q_j)^n}\sum_{l=1}^{k-2}\frac{\mu(2^{l+1}Q)}{l(2^{l+1}Q_j)^n}\|a_j\|_{L^\infty(\mu)}\mu(Q_j)\\
&\leq C\sum_{j=1}^{2}|\lambda_j|\left[K_{Q_j,\,R}\right]^2\|a_j\|_{L^\infty(\mu)}\mu(Q_j)\\
&\leq C\sum_{j=1}^{2}|\lambda_j|.
\end{aligned}$$

An argument similar to the estimate for G leads to

$$\mathrm{F}\leq C\sum_{j=1}^{2}|\lambda_j|.$$

The estimates for E, F and G give the desired estimate for I_{13}. Combining the estimates for I_{11}, I_{12}, I_{13} and I_2 yields

$$\mathrm{I}=\int_{4R}M_\Phi(\widetilde{T}b)(x)\,d\mu(x)\leq C\sum_{j=1}^{2}|\lambda_j|=C|b|_{H^{1,\,\infty}_{atb,\,2}(\mu)}. \tag{2.4}$$

Now we turn to the estimate for II. Let x_R be the center of the cube R. Invoking that $\widetilde{T}^*1=0$, we obtain

$$\begin{aligned}
\mathrm{II} &= \int_{\mathbb{R}^d\setminus 4R}\sup_{\varphi\sim x}\left|\int_{\mathbb{R}^d}\widetilde{T}b(y)[\varphi(y)-\varphi(x_R)]\,d\mu(y)\right|d\mu(x)\\
&\leq \int_{\mathbb{R}^d\setminus 4R}\sup_{\varphi\sim x}\left|\int_{2R}\widetilde{T}b(y)[\varphi(y)-\varphi(x_R)]\,d\mu(y)\right|d\mu(x)\\
&\quad+\int_{\mathbb{R}^d\setminus 4R}\sup_{\varphi\sim x}\left|\int_{\mathbb{R}^d\setminus 2R}\widetilde{T}b(y)\left[\varphi(y)-\varphi(x_R)\right]\,d\mu(y)\right|d\mu(x)\\
&= \mathrm{II}_1+\mathrm{II}_2.
\end{aligned}$$

Note that for any $z\in 2R$, $x\in 2^{k+1}R\setminus 2^kR$, and $k\geq 2$, we have $|x-z|\geq l(2^{k-2}R)$. This together with (iii) of Definition 1 and the mean value theorem leads to

$$|\varphi(y)-\varphi(x_R)|\leq C\frac{l(R)}{l(2^{k-2}R)^{n+1}} \tag{2.5}$$

for $y \in 2R$. By (2.5), (1.2), the Hölder inequality, the boundedness of $\widetilde{T}$ in $L^2(\mu)$ and (2.1), we have

$$\begin{aligned}
\mathrm{II}_1 &\le \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{\infty} \int_{2^{k+1}R\setminus 2^k R} \sup_{\varphi\sim x} \left[\int_{2R\setminus 2Q_j} |\widetilde{T}a_j(y)|\,|\varphi(y)-\varphi(x_R)|\, d\mu(y) \right] d\mu(x) \\
&\quad + \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{\infty} \int_{2^{k+1}R\setminus 2^k R} \sup_{\varphi\sim x} \left[\int_{2Q_j} |\widetilde{T}a_j(y)||\varphi(y)-\varphi(x_R)|\, d\mu(y) \right] d\mu(x) \\
&\le C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{\infty} \int_{2^{k+1}R\setminus 2^k R} \frac{l(R)}{l(2^{k-2}R)^{n+1}} \\
&\quad \times \sum_{l=1}^{N_j-1} \int_{2^{l+1}Q_j\setminus 2^l Q_j} \int_{Q_j} \frac{|a_j(z)|}{|y-z|^n}\, d\mu(z)\, d\mu(y)\, d\mu(x) \\
&\quad + C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{\infty} \int_{2^{k+1}R\setminus 2^k R} \frac{l(R)}{l(2^{k-2}R)^{n+1}} \left\| (\widetilde{T}a_j)\chi_{2Q_j} \right\|_{L^1(\mu)} d\mu(x) \\
&\le C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{\infty} 2^{-k} \sum_{l=1}^{N_j-1} \frac{\mu(2^{l+1}Q_j)}{l(2^{l+1}Q_j)^n} \|a_j\|_{L^\infty(\mu)} \mu(Q_j) \\
&\quad + C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{\infty} 2^{-k} \|(\widetilde{T}a_j)\chi_{2Q_j}\|_{L^2(\mu)} \mu(2Q_j)^{1/2} \\
&\le C \sum_{j=1}^{2} |\lambda_j| K_{Q_j,R} \|a_j\|_{L^\infty(\mu)} \mu(Q_j) + C \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^2(\mu)} \mu(2Q_j)^{1/2} \\
&\le C \sum_{j=1}^{2} |\lambda_j|.
\end{aligned}$$

We further estimate II_2 by

$$\begin{aligned}
\mathrm{II}_2 &= \sum_{k=2}^{\infty} \int_{2^{k+1}R\setminus 2^k R} \sup_{\varphi\sim x} \left| \int_{\mathbb{R}^d\setminus 2R} \widetilde{T}b(y)[\varphi(y)-\varphi(x_R)]\, d\mu(y) \right| d\mu(x) \\
&\le \sum_{k=2}^{\infty} \int_{2^{k+1}R\setminus 2^k R} M_\Phi \left[|\widetilde{T}b| \chi_{2^{k+2}R\setminus 2^{k-1}R} \right](x)\, d\mu(x) \\
&\quad + \sum_{k=2}^{\infty} \int_{2^{k+1}R\setminus 2^k R} \sup_{\varphi\sim x} \left[\int_{2^{k+2}R\setminus 2^{k-1}R} |\widetilde{T}b(y)| \varphi(x_R)\, d\mu(y) \right] d\mu(x) \\
&\quad + \sum_{k=2}^{\infty} \int_{2^{k+1}R\setminus 2^k R} \sup_{\varphi\sim x} \left[\int_{\mathbb{R}^d\setminus 2^{k+2}R} |\widetilde{T}b(y)| \{\varphi(y)+\varphi(x_R)\}\, d\mu(y) \right] d\mu(x) \\
&\quad + \sum_{k=2}^{\infty} \int_{2^{k+1}R\setminus 2^k R} \sup_{\varphi\sim x} \left[\int_{2^{k-1}R\setminus 2R} |\widetilde{T}b(y)| \{\varphi(y)+\varphi(x_R)\}\, d\mu(y) \right] d\mu(x) \\
&= \mathrm{II}_{21} + \mathrm{II}_{22} + \mathrm{II}_{23} + \mathrm{II}_{24}.
\end{aligned}$$

From Lemma 1, the fact that $\int_{\mathbb{R}^d} b\,d\mu = 0$ and (1.3), we can deduce that

$$\begin{aligned}
\mathrm{II}_{21} &\leq \sum_{k=2}^{\infty} \mu(2^{k+1}R)^{1/2} \left\| M_\Phi \left[|\widetilde{T}b| \chi_{2^{k+2}R\setminus 2^{k-1}R} \right] \right\|_{L^2(\mu)} \\
&\leq C \sum_{k=2}^{\infty} \mu(2^{k+1}R)^{1/2} \\
&\quad \times \left\{ \int_{2^{k+2}R\setminus 2^{k-1}R} \left| \int_R [K(y,\, z) - K(y,\, x_R)]\, b(z)\, d\mu(z) \right|^2 d\mu(y) \right\}^{1/2} \\
&\leq C \sum_{k=2}^{\infty} \mu(2^{k+1}R) \frac{l(R)^\delta}{l(2^k R)^{n+\delta}} \|b\|_{L^1(\mu)} \\
&\leq C \sum_{j=1}^{2} |\lambda_j|,
\end{aligned}$$

where we have used the fact that

$$\|b\|_{L^1(\mu)} \leq \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^1(\mu)} \leq C \sum_{j=1}^{2} |\lambda_j|.$$

An argument similar to the estimate for II_{21} tells us that

$$\mathrm{II}_{22} \leq C \sum_{j=1}^{2} |\lambda_j|.$$

Finally, we estimate II_{23}. By the fact that $\int_{\mathbb{R}^d} b\,d\mu = 0$, (ii) of Definition 1 and (1.3), we obtain

$$\begin{aligned}
\mathrm{II}_{23} &\leq \sum_{k=2}^{\infty} \int_{2^{k+1}R\setminus 2^k R} \sum_{l=k+2}^{\infty} \int_{2^{l+1}R\setminus 2^l R} \int_R |K(y,\, z) - K(y,\, x_R)||b(z)|\, d\mu(z) \\
&\quad \times \left[\frac{1}{|y-x|^n} + \frac{1}{|x_R - x|^n} \right] d\mu(y)\, d\mu(x) \\
&\leq C \sum_{k=2}^{\infty} \sum_{l=k+2}^{\infty} \frac{\mu(2^{k+1}R)}{l(2^{k+1}R)^n} \frac{\mu(2^{l+1}R) l(R)^\delta}{l(2^{l+1}R)^{n+\delta}} \|b\|_{L^1(\mu)} \\
&\leq C \sum_{j=1}^{2} |\lambda_j|.
\end{aligned}$$

An argument similar to the estimate for II_{23} indicates that

$$\mathrm{II}_{24} \leq C \sum_{j=1}^{2} |\lambda_j|.$$

Combining the estimates for II_{21}, II_{22}, II_{23} and II_{24}, we obtain the desired estimate for II_2. The estimates for II_1 and II_2 tell us that

$$\mathrm{II} = \int_{\mathbb{R}^d\setminus 4R} M_\Phi(\widetilde{T}b)(x)\, d\mu(x) \leq C|b|_{H^{1,\,\infty}_{atb,\,2}(\mu)}. \tag{2.6}$$

The estimates (2.4) and (2.6) lead to (2.2), and this completes the proof of our theorem.

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