CALDERÓN-ZYGMUND OPERATORS ON HARDY SPACES
WITHOUT THE DOUBLING CONDITION

WENGU CHEN, YAN MENG, AND DACHUN YANG

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Abstract. Let \( \mu \) be a non-negative Radon measure on \( \mathbb{R}^d \) which only satisfies some growth condition. In this paper, the authors obtain the boundedness of Calderón-Zygmund operators in the Hardy space \( H^1(\mu) \).

1. Introduction

In recent years, many papers focus on the analysis on \( \mathbb{R}^d \) with non-doubling measure; see [2, 5, 6, 8, 3, 4, 1] and their references. Moreover, the analysis on such \( \mathbb{R}^d \) was proved to play a striking role in solving the long open Painlevé’s problem by Tolsa in [9]; see also [10] for more background of this. Throughout this paper, the Euclidean space \( \mathbb{R}^d \) is endowed with a non-negative Radon measure \( \mu \) which only satisfies the following growth condition that there exists \( C_0 > 0 \) such that

\[
\mu(B(x, r)) \leq C_0 r^n
\]

for all \( x \in \mathbb{R}^d \) and \( r > 0 \), where \( B(x, r) = \{ y \in \mathbb{R}^d : |y - x| < r \} \), \( n \) is a fixed number and \( 0 < n \leq d \). Such a measure \( \mu \) is not necessary to be doubling. We recall that \( \mu \) is said to satisfy the doubling condition if there exists \( C > 0 \) such that \( \mu(B(x, 2r)) \leq C \mu(B(x, r)) \) for all \( x \in \text{supp} (\mu) \) and \( r > 0 \). It is well known that the doubling condition in the analysis on spaces of homogeneous type is a key assumption. However, some research has now indicated that the doubling condition is superfluous for most of the classical Calderón-Zygmund theory.

Let \( K \) be a function on \( \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, y) : x = y\} \) satisfying that for \( x \neq y \),

\[
|K(x, y)| \leq C|x - y|^{-n},
\]

and for \( |x - y| \geq 2|x - x'| \),

\[
|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C \frac{|x - x'|^\delta}{|x - y|^{n+\delta}},
\]

where \( \delta \in (0, 1] \) and \( C > 0 \) is a constant. The Calderón-Zygmund operator associated to the above kernel \( K \) and the measure \( \mu \) is formally defined by

\[
Tf(x) = \int_{\mathbb{R}^d} K(x, y)f(y)d\mu(y).
\]
This integral may not be convergent for many functions. Thus we consider the
truncated operators $T_\varepsilon$ for $\varepsilon > 0$ defined by

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x, y) f(y) d\mu(y).$$

We say that $T$ is bounded on $L^2(\mu)$ if the operators $\{T_\varepsilon\}_{\varepsilon>0}$ are bounded on $L^2(\mu)$
uniformly on $\varepsilon > 0$. In this case, there is an operator $\overline{T}$ which is the weak limit as
$\varepsilon \to 0$ of some subsequence of operators $\{T_\varepsilon\}_{\varepsilon>0}$; see [5]. It is easy to see that $\overline{T}$
is also bounded on $L^2(\mu)$; moreover, for $f \in L^2(\mu)$ with compact support and a. e.
$x \in \mathbb{R}^d \setminus \text{supp}(f)$,

$$\overline{T} f(x) = \int_{\mathbb{R}^d} K(x, y) f(y) \, d\mu(y)$$

with the same $K$ as in (1.2) and (1.3). By the same argument of Tolsa as in [5, 7],
we see that $\overline{T}$ is also bounded from $L^1(\mu)$ into weak-$L^1(\mu)$ and from $H^1(\mu)$ into
$L^1(\mu)$.

In this paper, we will prove that $\overline{T}$ is bounded on the Hardy space $H^1(\mu)$ if
$\overline{T}^* 1 = 0$. Here, by $\overline{T}^* 1 = 0$, we mean that for any bounded function $b$
with compact support and $\int_{\mathbb{R}^d} b \, d\mu = 0$,

$$\int_{\mathbb{R}^d} \overline{T} b(x) \, d\mu(x) = 0.$$  

We remark that for such a function $b$, $b \in H^1(\mu)$ and therefore, $\overline{T} b \in L^1(\mu)$.
Also, if $\overline{T} b \in H^1(\mu)$, then $\overline{T} b$ should satisfy (1.5) by the definition of the Hardy space
$H^1(\mu)$; see [5, 8] or Definition 2 below. Thus, in some sense, the condition (1.5) is
also necessary.

If $\mu$ is the $d$-dimensional Lebesgue measure on $\mathbb{R}^d$, this result is well known and it
was proved by verifying that $\overline{T}$ maps any atom of the Hardy space $H^1(\mathbb{R}^d)$ into some
molecule. However, if $\mu$ only satisfies (1.1), it is still unknown if there is a molecular
characterization for the Hardy space $H^1(\mu)$. We will prove that $\overline{T}$ is bounded on
the Hardy space $H^1(\mu)$ via its “grand” maximal function characterization of Tolsa
in [8] and its new atomic characterization of the authors in [1].

**Definition 1.** Given $f \in L^1_{\text{loc}}(\mu)$, we set

$$M_\phi f(x) = \sup_{\varphi \sim x} \left| \int_{\mathbb{R}^d} f \varphi \, d\mu \right|,$$

where the notation $\varphi \sim x$ means that $\varphi \in L^1(\mu) \cap C^1(\mathbb{R}^d)$ and satisfies

(i) $\|\varphi\|_{L^1(\mu)} \leq 1$,

(ii) $0 \leq \varphi(y) \leq \frac{1}{|y-x|^a}$ for all $y \in \mathbb{R}^d$, and

(iii) $|\nabla \varphi(y)| \leq \frac{1}{|y-x|^{a+\varepsilon}}$ for all $y \in \mathbb{R}^d$, where $\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_d)$.

Based on Theorem 1.2 of Tolsa in [8], we define the Hardy space $H^1(\mu)$ as follows.

**Definition 2.** The Hardy space $H^1(\mu)$ is the set of all functions $f \in L^1(\mu)$
satisfying that $\int_{\mathbb{R}^d} f \, d\mu = 0$ and $M_\phi f \in L^1(\mu)$. Moreover, we define the norm of
$f \in H^1(\mu)$ by

$$\|f\|_{H^1(\mu)} = \|f\|_{L^1(\mu)} + \|M_\phi f\|_{L^1(\mu)}.$$
Theorem 1. Let $K$ be the function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, y) : x = y\}$ satisfying (1.2) and (1.3). Suppose that the operator $\tilde{T}$ in (1.4) is bounded on $L^2(\mu)$ and $\tilde{T}^*1 = 0$ as in (1.5). Then $\tilde{T}$ is bounded on $H^1(\mu)$.

It is known that the dual space of $H^1(\mu)$ is the space $RBMO(\mu)$, which was introduced by Tolsa in [5]. From Theorem 1, the fact that $RBMO(\mu) = (H^1(\mu))^*$ (see [5]) and a standard dual argument, it is easy to deduce the boundedness of the transpose operator of $\tilde{T}$ in $RBMO(\mu)$ as below.

Corollary 1. Let $\tilde{T}$ be the same as in Theorem 1. Then $\tilde{T}^*$, the transpose operator of $\tilde{T}$, is bounded on $RBMO(\mu)$.

Remark 1. Obviously, from different subsequences of operators $\{T_\varepsilon\}_{\varepsilon > 0}$ which are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$, one may deduce different $\tilde{T}$’s. However, they are all bounded on $L^2(\mu)$ and satisfy (1.4). But, the relation between these different $\tilde{T}$’s is still open.

In what follows, $C$ denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line.

2. Proof of Theorem 1

We begin with some necessary notation and definitions. Throughout this paper, we only consider the closed cubes with sides parallel to the coordinate axes. For any cube $Q$ and any $\alpha > 0$, $\alpha Q$ denotes the cube with the same center as $Q$ and $l(\alpha Q) = \alpha l(Q)$, where $l(Q)$ denotes the side length of the cube $Q$.

Given two cubes $Q \subset R$ in $\mathbb{R}^d$, set

$$K_{Q, R} = 1 + \sum_{k=1}^{N_{Q, R}} \frac{\mu(2^k Q)}{l(2^k Q)^n},$$

where $N_{Q, R}$ is the smallest positive integer $k$ such that $l(2^k Q) \geq l(R)$; see [5] for some properties of $K_{Q, R}$.

To prove Theorem 1, we need to recall the atomic characterization of the Hardy space $H^1(\mu)$ as follows.

Definition 3. Let $\rho > 1$, $1 < p \leq \infty$ and $\gamma \in \mathbb{N}$. A function $b \in L^1_{\text{loc}}(\mu)$ is called a $(\rho, \gamma)$-atomic block if

1. there exists some cube $R$ such that $\text{supp}(b) \subset R$,
2. $\int_{\mathbb{R}^d} b d\mu = 0$,
3. for $j = 1, 2$, there are functions $a_j$ supported on cube $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$, and

$$\|a_j\|_{L^p(\mu)} \leq \left[ \mu(\rho Q_j) \right]^{1/p - 1} \left[ K_{Q_j, R} \right]^{-\gamma}.$$

Then we define

$$|b|_{H^{1, p}_{\text{at}}, \gamma}(\mu) = |\lambda_1| + |\lambda_2|.$$
We say that \( f \in H^{1,p}_{\text{atb},\gamma}(\mu) \) if there are \((p, \gamma)\)-atomic blocks \( \{b_i\}_{i \in \mathbb{N}} \) such that

\[
f = \sum_{i=1}^{\infty} b_i
\]

with \( \sum_{i=1}^{\infty} |b_i|_{H^{1,p}_{\text{atb},\gamma}(\mu)} < \infty \). The \( H^{1,p}_{\text{atb},\gamma}(\mu) \) norm of \( f \) is defined by

\[
\|f\|_{H^{1,p}_{\text{atb},\gamma}(\mu)} = \inf \left\{ \sum_{i} |b_i|_{H^{1,p}_{\text{atb},\gamma}(\mu)} \right\},
\]

where the infimum is taken over all the possible decompositions of \( f \) into \((p, \gamma)\)-atomic blocks.

The above definition when \( \gamma = 1 \) was introduced by Tolsa in \([5]\) and when \( \gamma > 1 \) by the authors in \([1]\). It was proved in \([5, 1]\) that the definition of \( H^{1,p}_{\text{atb},\gamma}(\mu) \) is independent of the chosen constant \( \rho > 1 \), and for any integer \( \gamma \geq 1 \) and \( 1 < p \leq \infty \), all the atomic Hardy spaces \( H^{1,p}_{\text{atb},\gamma}(\mu) \) are just the Hardy space \( H^1(\mu) \) with equivalent norms. We remark that in the proof of Theorem 1 below, we need to choose \( \gamma > 1 \), especially, \( \gamma = 2 \).

The following lemma will be used in the proof of Theorem 1.

**Lemma 1.** Let \( M_\Phi \) be as in Definition 1 and \( 1 < p < \infty \). Then \( M_\Phi \) is bounded on \( L^p(\mu) \).

In fact, Tolsa proved that \( M_\Phi \) is bounded from \( H^1(\mu) \) into \( L^1(\mu) \); see Lemma 3.1 in \([8]\). On the other hand, it is obvious that \( M_\Phi \) is bounded on \( L^\infty(\mu) \). By Theorem 7.2 in \([5]\), we obtain that \( M_\Phi \) is bounded on \( L^p(\mu) \) for \( 1 < p < \infty \).

Now we turn to the proof of Theorem 1.

**Proof of Theorem 1.** By a standard argument, it suffices to verify that for any atomic block \( b \) as in Definition 3 with \( \rho = 4, p = \infty \) and \( \gamma = 2 \), \( \tilde{T}b \) is in \( H^1(\mu) \) with norm \( C|b|_{H^{1,\infty}_{\text{atb},2}(\mu)} \), where \( C \) is independent of \( b \). Let all the notation be the same as in Definition 3. By our choices, \( a_j \) now satisfies the following size condition that

\[
\|a_j\|_{L^\infty(\mu)} \leq \left[ \mu(4Q_j)K_{Q_j,j}^2 \right]^{-1},
\]

where \( j = 1, 2 \).

The assumption that \( \tilde{T}^*1 = 0 \) tells us that \( \int_{\mathbb{R}^d} \tilde{T}b \, d\mu = 0 \). Recalling that \( \tilde{T} \) is bounded from \( H^1(\mu) \) into \( L^1(\mu) \) (see \([5]\)), we obtain

\[
\|\tilde{T}b\|_{L^1(\mu)} \leq C|b|_{H^{1,\infty}_{\text{atb},2}(\mu)}.
\]

By this and Definition 2, we deduce that the proof of Theorem 1 can be reduced to proving that

\[
\|M_\Phi(\tilde{T}b)\|_{L^1(\mu)} \leq C|b|_{H^{1,\infty}_{\text{atb},2}(\mu)}.
\]

Write

\[
\|M_\Phi(\tilde{T}b)\|_{L^1(\mu)} = \int_{4\mathbb{R}} M_\Phi(\tilde{T}b)(x) \, d\mu(x) + \int_{\mathbb{R}^d \setminus 4\mathbb{R}} M_\Phi(\tilde{T}b)(x) \, d\mu(x) = I + II.
\]
Noting that $M_\Phi$ is sublinear, we can control $I$ by

$$I \leq \int_{4R} M_\Phi \left[ (\tilde{T}b)\chi_{8R} \right](x) \, d\mu(x) + \int_{4R} M_\Phi \left[ (\tilde{T}b)\chi_{8R\setminus 8R} \right](x) \, d\mu(x) = I_1 + I_2.$$  

From the fact that for $j = 1, 2, Q_j \subset R$, it follows that for any $z \in Q_j$ and any $y \in 2^{k+1}R \setminus 2^k R$, $k \geq 3$, $|y - z| \geq l(2^{k-2}R)$. By this fact, (ii) of Definition 1, (1.2) and (2.1), we obtain

$$I_2 \leq \int_{4R} \sup_{\varphi \sim x} \left[ \int_{\mathbb{R}^n \setminus 8R} |\tilde{T}b(y)| \varphi(y) \, d\mu(y) \right] \, d\mu(x)$$

$$\leq \sum_{j=1}^2 |\lambda_j| \int_{4R} \sum_{k=1}^\infty \int_{2^{k+1}R \setminus 2^k R} \left| \int_{Q_j} K(y, z) a_j(z) \, d\mu(z) \right| \frac{1}{|x - y|^n} \, d\mu(y) \, d\mu(x)$$

$$\leq C \sum_{j=1}^2 |\lambda_j| \sum_{k=3}^\infty \|a_j\|_{L^\infty(\mu)} \mu(Q_j) \frac{\mu(2^{k+1}R)}{l(2^{k-2}R)^n} \frac{\mu(4R)}{l(2^{k-2}R)^n}$$

$$\leq C \sum_{j=1}^2 |\lambda_j|.$$  

To estimate $I_1$, we write

$$I_1 \leq \sum_{j=1}^2 |\lambda_j| \int_{4Q_j} M_\Phi \left[ (\tilde{T}a_j)\chi_{8R} \right](x) \, d\mu(x)$$

$$+ \sum_{j=1}^2 |\lambda_j| \int_{4R \setminus 4Q_j} M_\Phi \left[ (\tilde{T}a_j)\chi_{2Q_j} \right](x) \, d\mu(x)$$

$$+ \sum_{j=1}^2 |\lambda_j| \int_{4R \setminus 4Q_j} M_\Phi \left[ (\tilde{T}a_j)\chi_{8R \setminus 2Q_j} \right](x) \, d\mu(x)$$

$$= I_{11} + I_{12} + I_{13}.$$  

The Hölder inequality, Lemma 1, the boundedness of $\tilde{T}$ in $L^2(\mu)$ and (2.1) lead to

$$I_{11} \leq \sum_{j=1}^2 |\lambda_j| |\mu(4Q_j)|^{1/2} \left\| M_\Phi [(\tilde{T}a_j)\chi_{8R}] \right\|_{L^2(\mu)}$$

$$\leq C \sum_{j=1}^2 |\lambda_j| |\mu(4Q_j)|^{1/2} \|\tilde{T}a_j\|_{L^2(\mu)}$$

$$\leq C \sum_{j=1}^2 |\lambda_j| |\mu(4Q_j)|^{1/2} \|a_j\|_{L^2(\mu)}$$

$$\leq C \sum_{j=1}^2 |\lambda_j| |\mu(4Q_j)| \|a_j\|_{L^\infty(\mu)}$$

$$\leq C \sum_{j=1}^2 |\lambda_j|.$$
For \( j = 1, 2 \), denote \( N_{Q_j, 4R} \) simply by \( N_j \). By (ii) of Definition 1, the Hölder inequality, the boundedness of \( \tilde{T} \) in \( L^2(\mu) \) and (2.1), we have

\[
I_{12} \leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \sup_{\varphi \sim x} \left| \int_{2^k Q_j} \tilde{T} a_j(y) \varphi(y) \, d\mu(y) \right| \, d\mu(x)
\]

\[
\leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \frac{1}{(2^{k+2}Q_j)^n} \, d\mu(x) \left| \int_{2^k Q_j} \tilde{T} a_j(y) \, d\mu(y) \right|
\]

\[
\leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \mu(2^{k+1}Q_j) \| \tilde{T} a_j \|_{L^2(\mu)}(2Q_j)^{1/2}
\]

\[
\leq C \sum_{j=1}^{2} |\lambda_j| K_{Q_j, R} \mu(2Q_j)^{1/2} \| a_j \|_{L^2(\mu)}
\]

where we have used the fact that

\[
(2.3) \quad K_{Q_j, 4R} \leq CK_{Q_j, R}.
\]

For \( I_{13} \), we further decompose it into

\[
I_{13} = \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^k Q_j} M_{\Phi} \left( \tilde{T} a_j \chi_{8R \setminus 2Q_j} \right)(x) \, d\mu(x)
\]

\[
\leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^k Q_j} M_{\Phi} \left( |\tilde{T} a_j| \chi_{2^{k+2}Q_j \setminus 2^k Q_j} \right)(x) \, d\mu(x)
\]

\[
+ \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^k Q_j} M_{\Phi} \left( |\tilde{T} a_j| \chi_{\max\{2^{k+2}Q_j, 8R \setminus 2^{k+1}Q_j \}} \right)(x) \, d\mu(x)
\]

\[
+ \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^k Q_j} M_{\Phi} \left( |\tilde{T} a_j| \chi_{2^{k-1}Q_j \setminus 2Q_j} \right)(x) \, d\mu(x)
\]

\[= E + F + G.\]

Lemma 1, (1.2) and (2.1) tell us that

\[
E \leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \mu(2^{k+1}Q_j)^{1/2} \left\| M_{\Phi} \left( |\tilde{T} a_j| \chi_{2^{k+2}Q_j \setminus 2^k Q_j} \right) \right\|_{L^2(\mu)}
\]

\[
\leq C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \mu(2^{k+1}Q_j)^{1/2}
\]

\[
\times \left\{ \int_{2^{k+2}Q_j \setminus 2^k Q_j} \left| \int_{Q_j} K(y, z) a_j(z) \, d\mu(z) \right|^2 \, d\mu(y) \right\}^{1/2}
\]

\[
\leq C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \mu(2^{k+2}Q_j) \| a_j \|_{L^\infty(\mu)} \mu(Q_j)
\]

\[
\leq C \sum_{j=1}^{2} |\lambda_j|.
\]
By (ii) of Definition 1, (1.2), (2.3) and (2.1), we easily see that

\[ G \leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^{k}Q_j}^\infty \left[ \int_{2^{k-1}Q_j} |\tilde{T}a_j(y)| \varphi(y) \, d\mu(y) \right] \, d\mu(x) \]

\[ \leq C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \frac{\mu(2^{k+1}Q_j)}{(2^{k+1}Q_j)^n} \sum_{l=1}^{k-2} \frac{\mu(2^{l+1}Q_j)}{(2^{l+1}Q_j)^n} \|a_j\|_{L^\infty(\mu)} \mu(Q_j) \]

\[ \leq C \sum_{j=1}^{2} |\lambda_j| \left[ K_{Q_j, n} \right]^2 \|a_j\|_{L^\infty(\mu)} \mu(Q_j) \]

\[ \leq C \sum_{j=1}^{2} |\lambda_j|. \]

An argument similar to the estimate for G leads to

\[ F \leq C \sum_{j=1}^{2} |\lambda_j|. \]

The estimates for E, F and G give the desired estimate for I_{13}. Combining the estimates for I_{11}, I_{12}, I_{13} and I_2 yields

\[ (2.4) \quad I = \int_{4R} M_\Phi(\tilde{T}b)(x) \, d\mu(x) \leq C \sum_{j=1}^{2} |\lambda_j| = C |b|_{H^{1, \infty}_{\text{atb}, 2}}. \]

Now we turn to the estimate for II. Let x_R be the center of the cube R. Invoking that \( T^*1 = 0 \), we obtain

\[ II = \int_{\mathbb{R}^n \setminus 4R}^\infty \sup_{\varphi \sim x} \left[ \int_{2R} \tilde{T}b(y) [\varphi(y) - \varphi(x_R)] \, d\mu(y) \right] \, d\mu(x) \]

\[ \leq \int_{\mathbb{R}^n \setminus 4R}^\infty \sup_{\varphi \sim x} \left[ \int_{2R} \tilde{T}b(y) [\varphi(y) - \varphi(x_R)] \, d\mu(y) \right] \, d\mu(x) \]

\[ + \int_{\mathbb{R}^n \setminus 2R}^\infty \sup_{\varphi \sim x} \int_{2R} \tilde{T}b(y) [\varphi(y) - \varphi(x_R)] \, d\mu(y) \, d\mu(x) \]

\[ = II_1 + II_2. \]

Note that for any z \in 2R, x \in 2^{k+1}R \setminus 2^kR, and k \geq 2, we have |x - z| \geq l(2^{k-2}R). This together with (iii) of Definition 1 and the mean value theorem leads to

\[ (2.5) \quad |\varphi(y) - \varphi(x_R)| \leq C \frac{l(R)}{l(2^{k-2}R)^{n+1}} \]
for \( y \in 2R \). By (2.5), (1.2), the Hölder inequality, the boundedness of \( \tilde{T} \) in \( L^2(\mu) \) and (2.1), we have

\[
\Pi_1 \leq \sum_{j=1}^{2} |\lambda_j| \int_{2^{k+1}R \backslash 2^k R} \sup_{x} \left[ \int_{2^k R \backslash 2^{k-1} R} |\tilde{T}a_j(y)||\varphi(y) - \varphi(xR)| \, d\mu(y) \right] \, d\mu(x) \\
+ \sum_{j=1}^{2} |\lambda_j| \int_{2^{k+1}R \backslash 2^k R} \sup_{x} \left[ \int_{2^k R \backslash 2^{k-1} R} |\tilde{T}a_j(y)||\varphi(y) - \varphi(xR)| \, d\mu(y) \right] \, d\mu(x) \\
\leq C \sum_{j=1}^{2} |\lambda_j| \int_{2^{k+1}R \backslash 2^k R} \frac{l(R)}{l(2^{k-1}R)^n} \cdot \sum_{j=1}^{N_j-1} \left[ \int_{2^k R \backslash 2^{k-1} R} |a_j(z)| \, d\mu(z) \right] \, d\mu(x) \\
+ C \sum_{j=1}^{2} |\lambda_j| \int_{2^{k+1}R \backslash 2^k R} \frac{l(R)}{l(2^{k-1}R)^n} \left\| (\tilde{T}a_j) \chi_{2Q_j} \right\|_{L^2(\mu)} \, d\mu(x) \\
\leq C \sum_{j=1}^{2} |\lambda_j| K_{Q_j, R} \|a_j\|_{L^\infty(\mu)} \mu(Q_j) + C \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^2(\mu)} \mu(2Q_j)^{1/2} \\
\leq C \sum_{j=1}^{2} |\lambda_j|.
\]

We further estimate \( \Pi_2 \) by

\[
\Pi_2 = \sum_{k=2}^{\infty} \int_{2^{k+1}R \backslash 2^k R} M_{\Phi} \left[ \tilde{T}b(y) \{ \varphi(y) \} \right] (x) \, d\mu(x) \\
\leq \sum_{k=2}^{\infty} \int_{2^{k+1}R \backslash 2^k R} M_{\Phi} \left[ \tilde{T}b(y) \chi_{2^{k+2}R \backslash 2^{k-1} R} \right] (x) \, d\mu(x) \\
+ \sum_{k=2}^{\infty} \int_{2^{k+1}R \backslash 2^k R} \sup_{x} \left[ \int_{2^{k-2}R \backslash 2^{k-1} R} |\tilde{T}b(y)||\varphi(y) - \varphi(xR)| \, d\mu(y) \right] \, d\mu(x) \\
+ \sum_{k=2}^{\infty} \int_{2^{k+1}R \backslash 2^k R} \sup_{x} \left[ \int_{2^{k-2}R \backslash 2^{k-1} R} |\tilde{T}b(y)||\varphi(y) - \varphi(xR)| \, d\mu(y) \right] \, d\mu(x) \\
+ \sum_{k=2}^{\infty} \int_{2^{k+1}R \backslash 2^k R} \sup_{x} \left[ \int_{2^{k-2}R \backslash 2^{k-1} R} |\tilde{T}b(y)||\varphi(y) - \varphi(xR)| \, d\mu(y) \right] \, d\mu(x) \\
= \Pi_{21} + \Pi_{22} + \Pi_{23} + \Pi_{24}.
\]
From Lemma 1, the fact that \( \int_{\mathbb{R}^d} b \, d\mu = 0 \) and \( (1.3) \), we can deduce that

\[
\Pi_{21} \leq \sum_{k=2}^{\infty} \mu(2^{k+1}R)^{1/2} \left\| M_{\Phi} \left[ \tilde{T}b|_{2^{k+2}R}, 2^{k-1}R \right] \right\|_{L^2(\mu)}
\]

\[
\leq C \sum_{k=2}^{\infty} \mu(2^{k+1}R)^{1/2}
\]

\[
\times \left\{ \int_{2^{k+2}R}^{2^{k+1}R} \left| \int_{2^{k+1}R}^{2^{i+1}R} \left| K(y, z) - K(y, x_R) \right| b(z) \, d\mu(z) \right| \, d\mu(y) \right\}^{1/2}
\]

\[
\leq C \sum_{k=2}^{\infty} \mu(2^{k+1}R) \frac{l(R)\delta}{l(2^kR)^{n+\delta}} \|b\|_{L^1(\mu)}
\]

\[
\leq C \sum_{j=1}^{2} |\lambda_j|,
\]

where we have used the fact that

\[
\|b\|_{L^1(\mu)} \leq \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^1(\mu)} \leq C \sum_{j=1}^{2} |\lambda_j|.
\]

An argument similar to the estimate for \( \Pi_{21} \) tells us that

\[
\Pi_{22} \leq C \sum_{j=1}^{2} |\lambda_j|.
\]

Finally, we estimate \( \Pi_{23} \). By the fact that \( \int_{\mathbb{R}^d} b \, d\mu = 0 \), (ii) of Definition 1 and (1.3), we obtain

\[
\Pi_{23} \leq \sum_{k=2}^{\infty} \int_{2^{k+1}R}^{2^{k+2}R} \sum_{i=k+2}^{\infty} \int_{2^{i+1}R}^{2^{i+2}R} \int_{2^iR}^{2^{i+1}R} \left| K(y, z) - K(y, x_R) \right| b(z) \, d\mu(z)
\]

\[
\times \left\{ \frac{1}{|y-x|^n} + \frac{1}{|x_R-x|^n} \right\} \, d\mu(y) \, d\mu(x)
\]

\[
\leq C \sum_{k=2}^{\infty} \sum_{i=k+2}^{\infty} \mu(2^{k+1}R) \mu(2^{i+1}R) \frac{l(R)\delta}{l(2^iR)^{n+\delta}} \|b\|_{L^1(\mu)}
\]

\[
\leq C \sum_{j=1}^{2} |\lambda_j|.
\]

An argument similar to the estimate for \( \Pi_{23} \) indicates that

\[
\Pi_{24} \leq C \sum_{j=1}^{2} |\lambda_j|.
\]

Combining the estimates for \( \Pi_{21}, \Pi_{22}, \Pi_{23} \) and \( \Pi_{24} \), we obtain the desired estimate for \( \Pi_2 \). The estimates for \( \Pi_1 \) and \( \Pi_2 \) tell us that

\[
(2.6) \quad \Pi = \int_{\mathbb{R}^d \setminus 4R} M_{\Phi}(\tilde{T}b)(y) \, d\mu(y) \leq C |b|_{H^{1, \infty}_{\mu ba, 2}(\mu)}.
\]

The estimates (2.4) and (2.6) lead to (2.2), and this completes the proof of our theorem.
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References


Institute of Applied Physics and Computational Mathematics, P.O. 8009, Beijing, 100088, People’s Republic of China
E-mail address: chenwg@mail.iapcm.ac.cn

School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, People’s Republic of China
E-mail address: mengyan@mail.bnu.edu.cn

School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, People’s Republic of China
E-mail address: dcyang@bnu.edu.cn