ALMOST-DISJOINT CODING
AND STRONGLY SATURATED IDEALS

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Abstract. We show that Martin’s Axiom plus $\mathfrak{c} = \aleph_2$ implies that there is no $(\aleph_2, \aleph_2, \aleph_0)$-saturated $\sigma$-ideal on $\omega_1$.

Given cardinals $\lambda$, $\kappa$ and $\gamma$, a $\sigma$-ideal $I$ on a set $X$ is said to be $(\lambda, \kappa, \gamma)$-saturated if for every set $\{A_\alpha: \alpha < \lambda\} \subseteq \mathcal{P}(X) \setminus I$ there exists a set $Y \in [\lambda]^\gamma$ such that for all $Z \subseteq [Y]^\kappa$, $\bigcap\{A_\alpha: \alpha \in Z\} \not\in I$. Laver [6] was the first to show the consistency of an $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal on $\omega_1$, using a huge cardinal. Shelah [10] later showed that the nonstationary ideal on $\omega_1$ restricted to a given stationary set can be $(\aleph_2, \aleph_2, \aleph_0)$-saturated, using a supercompact cardinal.

The cardinal characteristic $\text{ap}$ is defined to be the least $\kappa$ such that there exist an almost disjoint family $\{e_\alpha: \alpha < \kappa\}$ (i.e., each $e_\alpha$ is an infinite subset of $\omega$, and for each distinct pair $\alpha, \beta < \kappa$, $e_\alpha \cap e_\beta$ is finite) and a set $A \subseteq \kappa$ such that for no $x \subseteq \omega$ does it hold for all $\alpha < \kappa$ that $\alpha \in A$ if and only if $e_\alpha \cap x$ is infinite (in [5] we called this $q$, but [1] shows that we should not have, as consistently every set of reals of cardinality $\text{ap}$ is a Q-set). We let $\mathfrak{c}$ denote the cardinality of the continuum. It follows easily that $2^\gamma = \mathfrak{c}$ for every infinite $\gamma < \text{ap}$.

Given a cardinal $\gamma$, $\text{MA}_\gamma$ is the variant of Martin’s Axiom that says that if $P$ is a c.c.c. partial order and $D_\alpha (\alpha < \gamma)$ are dense subsets of $P$, then there is a filter $G \subseteq P$ such that $G \cap D_\alpha$ is nonempty for each $\alpha < \gamma$. It is a standard fact that $\text{MA}_\gamma$ implies that $\text{ap} > \gamma$. [4].

In this note, we show that the statement $\text{ap} = \mathfrak{c} = \aleph_2$ implies that there is no countably complete $(\aleph_2, \aleph_2, \aleph_0)$-saturated $\sigma$-ideal on $\omega_1$. This contradicts statements in [2, 5] to the effect that the axiom PFA (see [10]) had been shown to be consistent with the existence of a stationary subset of $\omega_1$ such that the nonstationary ideal restricted to this set is $(\aleph_2, \aleph_2, \aleph_0)$-saturated. This situation is addressed by Nyikos in [9] and in another corrigendum to appear.

For a fixed cardinal $\kappa$, an ideal on a set $X$ is $\kappa$-dense if there is a subset $A$ of $\mathcal{P}(X) \setminus I$ of cardinality $\kappa$ such that every $I$-positive subset of $X$ contains a member of $A$ modulo $I$. It follows easily that every $\aleph_1$-dense $\sigma$-ideal is $(\aleph_2, \aleph_2, \aleph_0)$-saturated.

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It is a classical fact due to Ulam that there is no $\aleph_0$-dense $\sigma$-ideal on $\omega_1$ (see, for instance, Lemma 10.13 of [3]). Taylor [11] showed that under MA$_{\aleph_1}$ there is no $\aleph_1$-dense $\sigma$-ideal on $\omega_1$. The proof of Theorem 18 in [2] shows that $\mathfrak{ap} > \aleph_1$ suffices, i.e., that the following holds (a version of the argument appears also in [5]).

**Fact 0.1.** If $\mathfrak{ap} > \aleph_1$, then there is no $\aleph_1$-dense $\sigma$-ideal on $\omega_1$.

For the rest of this note, we fix an almost disjoint family $\{e_\alpha : \alpha < \omega_1\}$. For each $n \in \omega$ and each (possibly finite) $\sigma \subset \omega$, we let $E^m_\sigma = \{ \alpha < \omega_1 \mid |e_\alpha \cap \sigma| \geq n \}$, and we let $F^m_\sigma = \omega_1 \setminus E^m_\sigma$.

**Lemma 0.2.** Assume that $\mathfrak{ap} > \aleph_1$. Let $I$ be a $\sigma$-ideal on $\omega_1$ and let $\beta$ be a cardinal such that $\mathcal{P}(\omega_1)/I$ is not $\beta$-dense. Let $\{A_\alpha : \alpha < \beta\}$ be a subset of $\mathcal{P}(\omega_1) \setminus I$. Then there exist an $x \subset \omega$ and an $n \in \omega$ such that

- $F^m_x \notin I$,
- for each $\alpha < \beta$ there exists an $m \in \omega$ such that $E^m_{x \cap m} \cap A_\alpha \notin I$.

**Proof.** Since $\mathcal{P}(\omega_1)/I$ is not $\beta$-dense, we may fix $\{B_\alpha : \alpha < \beta\} \subset \mathcal{P}(\omega_1) \setminus I$ and $D \subset \mathcal{P}(\omega_1) \setminus I$ such that each $B_\alpha \subset A_\alpha$ and each $B_\alpha \cap D = \emptyset$. Since $\mathfrak{ap} > \aleph_1$, there exists an $x \subset \omega$ such that $e_x \cap x$ is infinite for each $\gamma \in \bigcup\{B_\alpha : \alpha < \beta\}$ and $e_x \cap x$ is finite for each $\gamma \in D$. Since $D \subset \bigcup\{F^m_x : n < \omega\}$, we may fix an $n \in \omega$ such that $F^m_x \notin I$. Similarly, for each $\alpha < \beta$, since $B_\alpha \subset \bigcup\{E^m_{x \cap m} : m < \omega\}$, there is an $m \in \omega$ such that $E^m_{x \cap m} \cap A_\alpha \notin I$.

The following theorem shows that $\mathfrak{ap} > \aleph_1$ implies that there is no $\sigma$-ideal I on $\omega_1$ which is $(\gamma, \gamma, \aleph_0)$-saturated, where $\gamma$ is the least cardinality of a dense subset of $\mathcal{P}(\omega_1)/I$. In particular, if $\mathfrak{ap} = \mathfrak{c} = \aleph_2$, then there is no $(\aleph_2, \aleph_2, \aleph_0)$-saturated $\sigma$-ideal on $\omega_1$.

**Theorem 0.3.** Assume that $\mathfrak{ap} > \aleph_1$, and let $I$ be a $\sigma$-ideal on $\omega_1$. Let $\gamma$ be the least cardinal such that there exists a dense (modulo $I$) subset of $\mathcal{P}(\omega_1) \setminus I$ of cardinality $\gamma$. Then there is a sequence $\langle D_\alpha : \alpha < \gamma \rangle$ of members of $\mathcal{P}(\omega_1) \setminus I$ such that for every cofinal $X \subset \gamma$ there exists a countable $y \subset X$ such that $\bigcap\{D_\alpha : \alpha \in y\} \notin I$.

**Proof.** Let $\{A_\alpha : \alpha < \gamma\}$ enumerate a dense subset of $\mathcal{P}(\omega_1) \setminus I$ modulo $I$. For each $\beta < \gamma$, apply Lemma [12] to $\{A_\alpha : \alpha < \beta\}$, obtaining $x_\beta, n_\beta$ and $D_\beta = F^{n_\beta}_{x_\beta}$ such that $D_\beta \notin I$ and such that for each $\alpha < \beta$ there exists an $m \in \omega$ such that $A_\alpha \cap E^{n_\beta}_{x_\beta \cap m} \notin I$.

Now let $X \subset \gamma$ be cofinal. Let $Z$ be the set of pairs $(n, \sigma)$ ($n \in \omega, \sigma \subset \omega$ finite) such that there exists a $\beta \in X$ with $E^n_\sigma \cap D_\beta \notin I$. We claim that $\{E^n_{(n, \sigma)} : (n, \sigma) \in Z\}$ is dense in $\mathcal{P}(\omega_1) \setminus I$, i.e., that for every $\alpha < \gamma$ there exist a $\beta \in X$, an $n \in \omega$ and a finite $\sigma \subset \omega$ such that $E^n_\sigma \cap D_\beta \notin I$ and $E^n_\sigma \cap A_\alpha \notin I$. To verify this, fix $\alpha < \gamma$ and let $\beta$ be any member of $X$ greater than $\alpha$. Then $D_\beta = F^{n_\beta}_{x_\beta}$, and there exists an $m$ such that $E^{n_\beta}_{x_\beta \cap m} \cap A_\alpha \notin I$, so $\beta, n_\beta$ and $x_\beta \cap m$ suffice for $\alpha$.

Now, for each $(n, \sigma) \in Z$, choose $\beta_{(n, \sigma)} \in X$ such that $E^n_\sigma \cap D_{\beta_{(n, \sigma)}} \in I$. Then since $\{E^n_{(n, \sigma)} : (n, \sigma) \in Z\}$ is predense in $\mathcal{P}(\omega_1) \setminus I$, $\bigcap\{D_{\beta_{(n, \sigma)}} : (n, \sigma) \in Z\} \in I$. We do not know whether some forcing axiom implies that the nonstationary ideal on $\omega_1$ is not $(\aleph_2, \aleph_2, \aleph_0)$-saturated. On the other hand, for all we know, some forcing axiom implies that the nonstationary ideal on $\omega_1$ is $(\aleph_2, \aleph_2, \aleph_0)$-saturated.
References


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