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ERGODIC ISOSPECTRAL THEORY OF THE LAX PAIRS OF EULER EQUATIONS WITH HARMONIC ANALYSIS FLAVOR

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Abstract. Isospectral theory of the Lax pairs of both 3D and 2D Euler equations of inviscid fluids is developed. Eigenfunctions are represented through an ergodic integral. The Koopman group and mean ergodic theorem are utilized. Further harmonic analysis results on the ergodic integral are introduced. The ergodic integral is a limit of the oscillatory integral of the first kind.

1. Introduction

Lax pairs for both 3D and 2D Euler equations of inviscid fluids were developed in [1], [2]. The greatest significance of these Lax pairs is their aid in understanding the global well-posedness of 3D Euler equations through isospectral theory. The isospectral theory of Lax pairs provides infinitely many conserved quantities for the underlying evolution equations. In this article, an isospectral theory of the Lax pairs of both 3D and 2D Euler equations is developed. By defining a Koopman group [3], [4], eigenfunctions can be represented through an ergodic integral. In fact, the mean ergodic theorem implies that a nonzero ergodic integral starting from an arbitrary function always produces an eigenfunction [4]. Whether or not the ergodic integral is zero poses an interesting question in harmonic analysis. In fact, the ergodic integral is the limit of an oscillatory integral of the first kind [5]. In summary, the present isospectral theory is built upon an interplay of classical isospectral theory, ergodic theory, and harmonic analysis.

2. Preliminary isospectral theory

The 3D Euler equation can be written in the vorticity form

$$\partial_t \Omega + (u \cdot \nabla)\Omega - (\Omega \cdot \nabla)u = 0 \ , \tag{2.1}$$

where $u = (u_1, u_2, u_3)$ is the velocity, $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the vorticity, $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$, $\Omega = \nabla \times u$, and $\nabla \cdot u = 0$. Here u can be represented by Ω through the Biot-Savart law or Fourier series depending on the boundary condition. We shall consider (2.1) as a Cauchy problem in the Sobolev space $H^k(S)$, $k = 0, 1, 2, \cdots$,

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$S = \mathbb{R}^3$ or $\mathbb{T}^3$. In the latter case, we are considering periodic boundary conditions. In this setting, the system is locally well-posed. For instance, the 3D Euler equation is locally well-posed in $C^0([0,T]; H^k(\mathbb{R}^3))$, $k > 5/2$, in the velocity variables, where T depends on the norm of the initial data [6].

A Lax pair for the 3D Euler equation can be written as [2],

$$(\Omega \cdot \nabla)\varphi = i\lambda\varphi \ , \tag{2.2}$$

$$\partial_t \varphi + (u \cdot \nabla)\varphi = 0 \ , \tag{2.3}$$

where λ is a complex spectral parameter, and φ is a complex scalar-valued function. The temporal part of the Lax pair (2.3) is a passive scalar equation, and the spatial part of the Lax pair (2.2) can be regarded as a stationary passive scalar equation. The 3D Euler equation (2.1) is a compatibility condition for the Lax pair (2.2)-(2.3). If Ω evolves in time according to the 3D Euler equation, the spectrum of (2.2) is frozen in time. This fact leads to the terminology "isospectral theory". As a result, any $\lambda \in \mathbb{C}$ as a functional of Ω is a conserved quantity for the 3D Euler equation. In the periodic case $S = \mathbb{T}^3$, of particular importance are the so-called periodic points, that is, such λ's which correspond to periodic eigenfunctions. Isospectral theory focuses upon the spatial part of the Lax pair (2.2). Here the temporal part of the Lax pair (2.3) can be solved using the characteristic curve method. Denote by $x = \xi(t, x_0)$ the solution to

$$\frac{dx}{dt} = u(t,x) \ , \qquad x(0) = x_0 \ .$$

Denote by $x_0 = \eta(t,x)$ the inverse function of the solution. Then

$$\varphi(t,x) = \varphi_0(\eta(t,x))$$

solves (2.3) with the initial condition $\varphi(0,x) = \varphi_0(x)$ since $\eta(0,x) = x$.

3. Ergodic theory

Now we turn our focus to the spatial part of the Lax pair (2.2). We will investigate its eigenfunctions in $H^k(S)$, $k = 0, 1, 2, \cdots$, $S = \mathbb{R}^3$ or $\mathbb{T}^3$. In the latter case, we are investigating the so-called periodic eigenfunctions, and the corresponding λ's are called periodic points. We assume that Ω is smooth enough, say $\Omega \in H^\ell(S)$ for a large ℓ. The fact that $\nabla \cdot \Omega = 0$ and (2.2) clearly imply that

$$(\lambda - \bar{\lambda}) \int |\varphi|^2 dx = 0 \ .$$

Thus λ is real. In fact, $i(\Omega \cdot \nabla)$ is a self-adjoint operator on $H^0(S)$ under the Hermitian inner product. Denote by $x(\tau)$ the solution to

$$\frac{dx}{d\tau} = \Omega(x) \ , \qquad x(0) = x \ .$$

One can define a Koopman group U_τ which is slightly different from the original Koopman group [3], [4]:

$$(U_\tau f)(x) = e^{-i\lambda\tau} f(x(\tau)) \ , \qquad x(0) = x \ . \tag{3.1}$$

Lemma 3.1 (Koopman's Lemma)**.** *U_τ is a one-parameter group of unitary operators on $H^0(S)$, where $S = \mathbb{R}^3$ or $\mathbb{T}^3$, under the Hermitian inner product.*

Proof. The crucial step is to show that the Jacobian of the transform $x \mapsto x(\tau)$ is 1. In fact, the equality

$$\frac{d}{d\tau} \det \left(\frac{\partial x(\tau)}{\partial x} \right) = (\nabla \cdot \Omega) \det \left(\frac{\partial x(\tau)}{\partial x} \right)$$

implies that the Jacobian is indeed 1 because $\nabla \cdot \Omega = 0$. The rest of the proof is routine. □

Theorem 3.2 (Stone's Theorem)**.** *U_τ is generated by iA, for a self-adjoint operator A.*

In fact,

$$\left(\frac{\partial}{\partial \tau} U_\tau f \right)(x) \bigg|_{\tau=0} = iAf(x) \ , \quad Af = -i[(\Omega \cdot \nabla) f - i\lambda f] \ .$$

Let $E_{\tilde{\lambda}}$ be the spectral resolution of the identity for A. Then

$$\langle U_\tau f, g \rangle = \int_{-\infty}^{+\infty} e^{i\tilde{\lambda}\tau} d\langle E_{\tilde{\lambda}} f, g \rangle \ , \quad \langle Af, g \rangle = \int_{-\infty}^{+\infty} \tilde{\lambda} d\langle E_{\tilde{\lambda}} f, g \rangle \ .$$

A fixed point f of the Koopman group U_τ is defined by

$$U_\tau f = f \qquad \text{for all } \tau \ . \tag{3.2}$$

The set F of all such fixed points is a closed linear subspace of $H^0(S)$. The corresponding orthogonal projection onto F is denoted by $\mathbf{P}$.

Theorem 3.3 (von Neumann's Mean Ergodic Theorem)**.** *For any $f \in H^0(S)$, where $S = \mathbb{R}^3$ or $\mathbb{T}^3$,*

$$\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} U_\tau f d\tau = \mathbf{P} f \ .$$

For E. Hopf's elegant proof of this theorem, see [4].

Theorem 3.4. *The function φ is an eigenfunction of* (2.2) *in $H^0(S)$, $S = \mathbb{R}^3$ or $\mathbb{T}^3$ if and only if $\varphi \in H^1(S)$ is a fixed point of the Koopman group U_τ.*

Proof. The identity obtained by differentiating (3.2) implies that φ is a fixed point of U_τ if and only if $A\varphi = 0$, i.e. φ is an eigenfunction of (2.2). □

Theorems 3.4 and 3.3 indicate an elegant ergodic integral representation for the eigenfunctions of (2.2):

$$\begin{aligned} \varphi(x) &= \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} (U_\tau f)(x) d\tau \\ &= \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda\tau} f(x(\tau)) d\tau \ , \quad x(0) = x \ , \end{aligned} \tag{3.3}$$

where f is an arbitrary function in $H^k(S)$, $k = 1, 2, \cdots$, $S = \mathbb{R}^3$ or $\mathbb{T}^3$. Substituting (3.3) into (2.2) shows that $\varphi(x)$ indeed solves (2.2):

$$\frac{\partial}{\partial(-\tau)} f(x(\tau)) + (\Omega \cdot \nabla)(x) f(x(\tau)) = 0 \ .$$

Thus

$$\begin{aligned}(\Omega\cdot\nabla)(x)\varphi(x) &= \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}e^{-i\lambda\tau}\frac{\partial}{\partial\tau}f(x(\tau))d\tau\\ &= i\lambda\varphi(x)+\lim_{T\to+\infty}\frac{1}{2T}e^{-i\lambda\tau}f(x(\tau))\Big|_{-T}^{T}\ ,\end{aligned}$$

and the last term vanishes if f is bounded, which will be the case if k is large enough.

The advantage of the representation (3.3) is that one can seek an eigenfunction by looking for an $f\in H^k(S)$ such that the limit (3.3) is not identically zero.

4. Harmonic analysis

To answer the question whether or not (3.3) is identically zero, one can use the Fourier transform. When $S=\mathbb{T}^3$,

$$f(x(\tau))=\sum_{n\in\mathbb{Z}^3}\tilde{f}_n e^{in\cdot x(\tau)}\ .$$

Then

$$\varphi(x)=\sum_{n\in\mathbb{Z}^3}\tilde{f}_n\varphi_n(x)\ ,$$

where

$$\varphi_n(x)=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}e^{i[n\cdot x(\tau)-\lambda\tau]}d\tau\ . \tag{4.1}$$

When $S=\mathbb{R}^3$,

$$f(x(\tau))=\int_{\mathbb{R}^3}\tilde{f}(\zeta)e^{i\zeta\cdot x(\tau)}d\zeta\ .$$

Then

$$\varphi(x)=\int_{\mathbb{R}^3}\tilde{f}(\zeta)\varphi_\zeta(x)d\zeta\ ,$$

where

$$\varphi_\zeta(x)=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}e^{i[\zeta\cdot x(\tau)-\lambda\tau]}d\tau\ . \tag{4.2}$$

Representations (4.1) and (4.2) are limits of oscillatory integrals of the first kind [5]. Let

$$\theta(\tau)=n\cdot x(\tau)-\lambda\tau\quad\text{or}\quad\zeta\cdot x(\tau)-\lambda\tau\ .$$

Theorem 4.1 (Stein [5]). *If $|\theta^{(k)}(\tau)|\geq 1$ for all τ, then for any T,*

$$\left|\int_{-T}^{T}e^{i\theta(\tau)}d\tau\right|\leq 5\cdot 2^{k-1}-2$$

holds when

(1) *$k\geq 2$, or*
(2) *$k=1$ and $\theta'(\tau)$is monotonic.*

Of course, if the condition in Theorem 4.1 holds, then all the Fourier coefficients $\varphi_n(x)$ or $\varphi_\zeta(x)$ will be zero. On the other hand, the condition rarely holds for all ζ. Often one can find a solution ζ to the equation

$$\theta'(\tau) = \zeta \cdot x'(\tau) - \lambda = 0 \quad \text{or} \quad \theta^{(k)}(\tau) = \zeta \cdot x^{(k)}(\tau) = 0, \quad k \geq 2 \ .$$

In the case $\theta(\tau) = n \cdot x(\tau) - \lambda\tau$, $|\theta^{(k)}(\tau)|$ can often get very small as n changes, say $|\theta^{(k)}(\tau)| \geq \epsilon$ for all τ. Let $\theta(\tau) = \epsilon\vartheta(\tau)$. Then [5]

$$\left| \int_{-T}^{T} e^{i\epsilon\vartheta(\tau)} d\tau \right| \leq (5 \cdot 2^{k-1} - 2)\epsilon^{-1/k} \ .$$

Here the Diophantine theory is relevant. Of course, in general, the principal contribution to the integral comes from the neighborhoods of the critical points of $\theta(\tau)$ — a principle of the method of stationary phase. The critical points of $\theta(\tau)$ are given by

$$n \cdot \Omega(x(\tau)) - \lambda = 0 \quad \text{or} \quad \zeta \cdot \Omega(x(\tau)) - \lambda = 0 \ .$$

Asymptotics of the integral (3.3) in λ can be developed through integrations by parts. In the case that f is smooth and bounded in τ,

$$\left| \int_{-T}^{T} e^{-i\lambda\tau} f d\tau \right| \leq \frac{1}{|\lambda|} \left[|f(x(T))| + |f(x(-T))| + \int_{-T}^{T} |f'| d\tau \right] \ .$$

In fact, the following is true [5]:

$$\left| \int_{-T}^{T} e^{-i\lambda\tau} f d\tau \right| \leq \frac{2}{|\lambda|} \left[|f(x(T))| + \int_{-T}^{T} |f'| d\tau \right] \ .$$

Thus the ergodic integral (3.3) has the asymptotics

$$|\varphi(x)| \leq \frac{1}{|\lambda|} \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |f'| d\tau \ .$$

5. Examples

Example 1. Let $S = \mathbb{T}^3$, $\Omega(x) = (1, 1, 1)$. Then

$$x(\tau) = \tau(1, 1, 1) + x \ ,$$

$$\varphi_n(x) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} e^{i[n_1 + n_2 + n_3 - \lambda]\tau} e^{in \cdot x} d\tau \ .$$

If $\lambda = n_1 + n_2 + n_3$, then

$$\varphi_n(x) = e^{in \cdot x}$$

is a fixed point of the Koopman group U_τ. Thus the periodic points are given by

$$\lambda_p = n_1 + n_2 + n_3 \ , \quad \forall n \in \mathbb{Z}^3 \ .$$

At each periodic point λ_p, the number of eigenfunctions is determined by all the n's that satisfy the above relation.

Example 2. Let $S = \mathbb{T}^3$. Assume that $n \in \mathbb{Z}^3$ is such that $n \cdot \Omega = c$, a constant. Then $\alpha n \cdot \Omega = \alpha c$, $\forall \alpha \in \mathbb{Z}$. Thus, $\alpha n \cdot x(\tau) = \alpha c\tau + \alpha n \cdot x$,

$$\begin{aligned} \varphi_{\alpha n}(x) &= \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} e^{i[\alpha n \cdot x(\tau) - \lambda\tau]} d\tau \\ &= \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} e^{i[\alpha c - \lambda]\tau} e^{i\alpha n \cdot x} d\tau \ . \end{aligned}$$

If $\lambda = \alpha c$, then

$$\varphi_{\alpha n}(x) = e^{i\alpha n \cdot x},$$

which is a fixed point of the Koopman group U_τ. Thus the following are the periodic points:

$$\lambda_p = \alpha c \ , \quad \forall \alpha \in \mathbb{Z} \ .$$

The drawback of the above two examples is that the corresponding velocity u cannot be periodic.

6. 2D Euler equation

The 2D Euler equation can be written in the vorticity form

$$\partial_t \Omega + \{\Psi, \Omega\} = 0 \ , \tag{6.1}$$

where Ω is a real scalar-valued function, and the bracket $\{\ ,\ \}$ is defined as

$$\{f, g\} = (\partial_{x_1} f)(\partial_{x_2} g) - (\partial_{x_2} f)(\partial_{x_1} g) \ ,$$

where Ψ is the stream function given by

$$u_1 = -\partial_{x_2}\Psi \ , \quad u_2 = \partial_{x_1}\Psi \ ,$$

and the relation between the vorticity Ω and the stream function Ψ is

$$\Omega = \partial_{x_1} u_2 - \partial_{x_2} u_1 = \Delta\Psi \ .$$

The Lax pair of the 2D Euler equation can be written as [1], [2],

$$\{\Omega, \varphi\} = i\lambda\varphi \ , \tag{6.2}$$

$$\partial_t \varphi + \{\Psi, \varphi\} = 0 \ , \tag{6.3}$$

where λ is a complex spectral parameter and φ is a complex scalar-valued function. The theory developed in the previous sections can be easily applied to the 2D Euler case by simply replacing the $\Omega(x)$ in the previous sections by $(-\partial_{x_2}\Omega, \partial_{x_1}\Omega)$ here.

Results on the essential spectrum of (6.2) have been obtained. As an example [7], for $\Omega = \cos(x_1 + x_2)$, the entire $H^0(\mathbb{T}^2)$ spectrum is the continuous spectrum, which is the imaginary axis (in $i\lambda$). The point and residue spectra are empty. In general, if the vector field $(-\partial_{x_2}\Omega, \partial_{x_1}\Omega)$ has a saddle, and Λ is the largest eigenvalue of all its saddles, then the essential spectrum of (6.2) in $H^k(\mathbb{T}^2)$ $(k = 1, 2, \cdots)$ is a vertical band of width $2k\Lambda$, symmetric with respect to the imaginary axis [8]. Notice that here there can be a point spectrum embedded in the essential spectrum. Also notice that the width of the band increases with k.

The same result holds for the linear 2D Euler operator [8], where the increase of the band width with k corresponds to the well-known phenomenon of cascade and inverse cascade. For the linear 2D Euler operator, the point spectrum can be calculated through continued fractions [7]. The eigenfunctions belong to the Schwartz class. Thus the point spectrum is independent of k. For $\Omega = \cos(x_1 + x_2)$, the velocity field has no saddle [7]; thus $\Lambda = 0$. In this case, the essential spectrum in $H^k(\mathbb{T}^2)$ $(k = 1, 2, \cdots)$ is the imaginary axis [8]. The invariant manifold problem here is still open due to no spectral resolution and non-Lipschitz nature of the nonlinearity.

Returning to (6.2), if the vector field $(-\partial_{x_2}\Omega, \partial_{x_1}\Omega)$ has at least two fixed points, then the essential spectrum of (6.2) in $H^0(\mathbb{T}^2)$ is the imaginary axis [8]. Again there can be a point spectrum embedded in the essential spectrum.

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