PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 9, Pages 2533–2537
S 0002-9939(05)07846-9
Article electronically published on April 8, 2005

DERIVED CATEGORIES OF PROJECTIVE BUNDLES

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(Communicated by Michael Stillman)

Abstract. The goal of this short note is to prove that any projective bundle $\mathbb{P}(\mathcal{E}) \to X$ has a tilting bundle whose summands are line bundles whenever the same holds for X.

1. Introduction

Let X be a smooth projective variety defined over the complex numbers $\mathbb{C}$ and let $D^b(X) = D^b(\mathcal{O}_X\text{-}mod)$ be the derived category of bounded complexes of coherent sheaves of $\mathcal{O}_X$-modules. It is natural to ask when is $D^b(X)$ freely and finitely generated? In [2], Bondal pointed out that showing that $D^b(X)$ is freely and finitely generated by a coherent sheaf $T \in \mathcal{O}_X\text{-}mod$ (called a *tilting sheaf*) amounts to showing that the functors $\mathbf{R}\mathrm{Hom}_X(T, -) : D^b(X) \longrightarrow D^b(A)$ and $- \otimes^{\mathbf{L}}_A T : D^b(A) \longrightarrow D^b(X)$ define mutually inverse equivalences of the bounded derived categories of coherent sheaves on X and the finitely generated right modules over $A = \mathrm{Hom}_X(T, T)$, respectively.

The existence of tilting sheaves (see Definition 2.3) also plays an important role in the problem of characterizing the smooth projective varieties X determined by its bounded category of coherent sheaves $D^b(X)$ or, equivalently, in the problem of determining the set of smooth varieties Y for which there exists a Fourier-Mukai transform, i.e., an equivalence of categories $\phi : D^b(Y) \to D^b(X)$ preserving the triangles (for more information see [3] and [4]). Fourier-Mukai transforms are important tools for studying moduli spaces of sheaves ([5], [11] and [12]), and they provide the correct language for describing certain dualities suggested by string theory ([10]).

In this paper we will focus our attention on the existence of tilting sheaves on smooth projective varieties. The search for tilting sheaves on a smooth projective variety X splits naturally into two parts: First, we have to find the so-called *strongly exceptional collection* of coherent sheaves on X, $(F_0, F_1, \cdots, F_n)$ (See Definition 2.1); and second we have to show that the strongly exceptional collection $(F_0, F_1, \cdots, F_n)$ is full, i.e., F_0, F_1, $\cdots$, F_n generate the bounded derived category $D^b(X)$. Each full strongly exceptional collection defines a tilting sheaf $T = \bigoplus_{i=0}^n F_i$ because the endomorphism algebra of $T = \bigoplus_{i=0}^n F_i$ has global dimension at most n. Vice versa, each tilting bundle whose direct summands are line bundles gives

Received by the editors July 15, 2003 and, in revised form, May 17, 2004.
2000 *Mathematics Subject Classification.* Primary 14F05; Secondary 14M25.
The first author was partially supported by MTM2004-00666.
The second author was partially supported by MTM2004-00666.

rise to a full strongly exceptional collection. So we are led to pose the following question.

Question 1.1. Which smooth projective varieties have a tilting bundle whose summands are line bundles?

The goal of this paper is to prove that any projective bundle $\mathbb{P}(\mathcal{E})$ associated to a rank r vector bundle $\mathcal{E}$ on X has a tilting bundle whose summands are line bundles whenever the same holds for X.

2. Exceptional collections and tilting bundles

We start this section by recalling the notions of exceptional sheaves, exceptional collections of sheaves and strongly exceptional collections of sheaves. The notion of exceptional collections of sheaves was introduced by Gorodentsev and Rudakov in [7], and we will use strongly exceptional collections of locally free sheaves to construct tilting bundles.

Definition 2.1. Let X be a smooth projective variety.

(i) A coherent sheaf F on X is **exceptional** if $\mathrm{Hom}(F,F)=\mathbb{C}$ and $\mathrm{Ext}^i_X(F,F)=0$ for $i>0$.

(ii) An ordered collection $(F_0,F_1,\cdots,F_n)$ of coherent sheaves on X is an **exceptional collection** if each sheaf F_i is exceptional and $\mathrm{Ext}^i_X(F_k,F_j)=0$ for $j<k$ and $i\geq 0$.

(iii) An exceptional collection $(F_0,F_1,\cdots,F_n)$ is a **strongly exceptional collection** if in addition $\mathrm{Ext}^i_X(F_k,F_j)=0$ for $i\geq 1$ and $k\leq j$.

Let us illustrate the above definitions with precise examples:

Example 2.2. (1) $(\mathcal{O},\,\mathcal{O}(1),\,\mathcal{O}(2),\,\cdots,\,\mathcal{O}(r))$ is a strongly exceptional collection on a projective space $\mathbb{P}^r$.

(2) Let $X=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1})=\mathbb{P}^1\times\mathbb{P}^1\subset\mathbb{P}^3$ be a smooth quadric. It is well known that $Pic(X)\cong\mathbb{Z}^2=\mathbb{Z}l+\mathbb{Z}m$, with l and m lines in a different system. It is not difficult to see that $(\mathcal{O},\,\mathcal{O}(l),\,\mathcal{O}(m),\,\mathcal{O}(l+m))$ is a strongly exceptional collection on $\mathbb{P}^1\times\mathbb{P}^1$.

Since the fundamental paper of Beilinson [1], tilting theory has become a major tool in classifying vector bundles over smooth projective varieties. Following the terminology of representation theory we have

Definition 2.3. Let X be a smooth projective variety and let $T\in\mathcal{O}_X\text{-}mod$ be a coherent sheaf. T is called a **tilting sheaf** (or, when it is locally free, a **tilting bundle**) if

(i) it has no higher self-extensions, i.e. $\mathrm{Ext}^i_X(T,T)=0$ for all $i>0$,
(ii) the endomorphism algebra of T, $A=\mathrm{Hom}_X(T,T)$, has finite global homological dimension,
(iii) the direct summands of T generate the bounded derived category $D^b(\mathcal{O}_X\text{-}mod)$ of coherent sheaves of $\mathcal{O}_X$-modules.

Remark 2.4. Since there is no loss of generality in assuming that the indecomposable summands of a tilting sheaf T are pairwise non-isomorphic, we will make this assumption in the future.

Definition 2.5. Let X be a smooth projective variety. An ordered collection of coherent sheaves $(F_0, F_1, \cdots, F_n)$ on X is a **full (strongly) exceptional collection** if it is a (strongly) exceptional collection $(F_0, F_1, \cdots, F_n)$ and F_0, F_1, $\cdots$, F_n generate the bounded derived category $D^b(X)$.

Example 2.6. (i) The collection $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \cdots, \mathcal{O}(r))$ is a full strongly exceptional collection on a projective space $\mathbb{P}^r$.

(ii) The collection $(\mathcal{O}, \mathcal{O}(l), \mathcal{O}(m), \mathcal{O}(l+m))$ is a full strongly exceptional collection on $\mathbb{P}^1 \times \mathbb{P}^1$.

The importance of the existence of full strongly exceptional collections relies on the fact that each full strongly exceptional collection $(F_0, F_1, \cdots, F_n)$ of coherent sheaves on X defines a tilting sheaf $T = \bigoplus_{i=0}^n F_i$ because the endomorphism algebra of $T = \bigoplus_{i=0}^n F_i$ is a "triangular" algebra and it has global dimension at most n. (Recall that an algebra is said to be "triangular" if its indecomposable projective modules $P_1, \cdots, P_n$ all satisfy $Hom(P_i, P_i) = k$ and can be ordered in such a way that $Hom(P_j, P_i) = 0$ if $i < j$. It is easy to prove that any triangular algebra has a finite global dimension.) Vice versa, each tilting bundle whose direct summands are line bundles gives rise to a full strongly exceptional collection. So we are led to pose the following question.

Question 2.7. Which smooth projective variety have a tilting bundle whose summands are line bundles?

Examples of smooth projective varieties with a tilting bundle whose summands are line bundles can be found, for instance, in [6], [7], [9], [13] and [14].

3. Main Theorem

Let $\mathcal{E}$ be a rank r vector bundle on a smooth projective variety X. Denote by $\mathbb{P}(\mathcal{E})$ the corresponding projective bundle, $p : \mathbb{P}(\mathcal{E}) \to X$ the natural projection and $\mathcal{O}_{\mathcal{E}}(1)$ the tautological line bundle on $\mathbb{P}(\mathcal{E})$. The goal of this section is to prove the existence of a tilting bundle whose summands are line bundles on any projective bundle $\mathbb{P}(\mathcal{E})$ on X provided X has also a tilting bundle whose summands are line bundles. To this end, the following result on $\mathbb{P}^d$-bundles due to Orlov will be useful.

Proposition 3.1. *Let X be a smooth projective variety and let $\mathcal{E}$ be a rank r vector bundle on X. Denote by $\mathbb{P}(\mathcal{E})$ the corresponding projective bundle and let $p : \mathbb{P}(\mathcal{E}) \to X$ be the natural projection. If $(F_0, F_1, \cdots, F_n)$ is a full exceptional collection of coherent sheaves on X, then*

$$(p^*F_0 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), p^*F_1 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \cdots, p^*F_n \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \cdots, \\ p^*F_0, p^*F_1, \cdots, p^*F_n)$$

is a full exceptional collection of coherent sheaves on $\mathbb{P}(\mathcal{E})$.

Proof. See [13], Corollary 2.7. □

Notice that Orlov's result is not enough to construct tilting bundles whose summands are line bundles because for an arbitrary vector bundle $\mathcal{E}$, the collection constructed in Proposition 3.1 is a full exceptional collection but not necessarily

full strongly exceptional. In order to ensure that the collection is strongly exceptional we need some extra hypothesis on $\mathcal{E}$. In fact, we have:

Key Lemma. *Let $(F_0, F_1, \cdots, F_n)$ be a full exceptional collection of locally free sheaves on a smooth projective variety X and let $\mathcal{E}$ be a rank r vector bundle on X. Denote by $S^a\mathcal{E}$ the a-th symmetric power of $\mathcal{E}$ and assume that for any integer a, $0 \leq a \leq r-1$, and any l, m, $0 \leq l \leq m \leq n$,*

$$H^i(X, S^a\mathcal{E} \otimes F_m \otimes F_l^\vee) = 0, \quad i > 0.$$

Then,

$$(p^*F_0 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), p^*F_1 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \cdots, p^*F_n \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \cdots, \\ p^*F_0, p^*F_1, \cdots, p^*F_n)$$

is a full strongly exceptional collection of locally free sheaves on $\mathbb{P}(\mathcal{E})$.

Proof. By Proposition 3.1,

$$(p^*F_0 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), p^*F_1 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \cdots, p^*F_n \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \cdots, \\ p^*F_0, p^*F_1, \cdots, p^*F_n)$$

is a full exceptional collection on $\mathbb{P}(\mathcal{E})$. So, we only need to prove that for any k, j, l, m with $0 \leq k < j \leq r-1$ and $l \leq m$ or $0 \leq k = j \leq r-1$ and $l < m$, we have

$$\mathrm{Ext}^i(p^*F_l \otimes \mathcal{O}_{\mathcal{E}}(k-r+1), p^*F_m \otimes \mathcal{O}_{\mathcal{E}}(j-r+1)) = 0, \quad i > 0,$$

or equivalently

$$H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathcal{E}}(j-k) \otimes p^*(F_m \otimes F_l^\vee)) = 0, \quad i > 0.$$

By the Base Change Theorem ([8], III.12.9), $R^ip_*\mathcal{O}_{\mathcal{E}}(a) = 0$ for $0 < i < r-1$ and all $a \in \mathbb{Z}$, and $R^{r-1}p_*\mathcal{O}_{\mathcal{E}}(a) = 0$ for $a > -r$. On the other hand, it follows from the projection formula that for any line bundle $\mathcal{L}$ on X,

$$R^ip_*(\mathcal{O}_{\mathcal{E}}(a) \otimes p^*\mathcal{L}) \cong \mathcal{L} \otimes R^ip_*(\mathcal{O}_{\mathcal{E}}(a))$$

and thus, $R^ip_*(\mathcal{O}_{\mathcal{E}}(a) \otimes p^*\mathcal{L}) = 0$ if $i \geq 1$ and $a > -r$. Therefore, using the degeneration of the Leray spectral sequence, for $i \geq 0$ and $a > -r$, we obtain

$$H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathcal{E}}(a) \otimes p^*(\mathcal{L})) \cong H^i(X, p_*\mathcal{O}_{\mathcal{E}}(a) \otimes \mathcal{L}).$$

In particular, since $j-k \geq 0 > -r$ and $p_*\mathcal{O}_{\mathcal{E}}(a) \cong S^a(\mathcal{E})$, for any $a \geq 0$ we get

$$H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathcal{E}}(j-k) \otimes p^*(F_m \otimes F_l^\vee)) = H^i(X, S^{j-k}\mathcal{E} \otimes F_m \otimes F_l^\vee) = 0,$$

which proves what we want. □

Now, we can state and prove the main result of this paper.

Main Theorem. *Let $\mathcal{E}$ be a rank r vector bundle on a smooth projective variety X. Assume X has a tilting bundle whose summands are line bundles. Then $\mathbb{P}(\mathcal{E})$ has a tilting bundle whose summands are line bundles.*

Proof. By assumption, there exists $(F_0, F_1, \cdots, F_n)$, a full strongly exceptional collection of line bundles on X. By Serre's theorem, there exists a line bundle $\mathcal{L} = \mathcal{O}_X(t)$, $t >> 0$, on X such that for any integer a, $0 \leq a \leq r-1$, and any pair of integers l, m, $0 \leq l \leq m \leq n$,

$$H^i(X, S^a(\mathcal{E} \otimes \mathcal{L}) \otimes F_m \otimes F_l^\vee) = 0, \quad i > 0.$$

Hence, it follows from the Key Lemma that

$$(p^*F_0 \otimes \mathcal{O}_{\mathcal{E}\otimes\mathcal{L}}(-r+1), p^*F_1 \otimes \mathcal{O}_{\mathcal{E}\otimes\mathcal{L}}(-r+1), \cdots, p^*F_n \otimes \mathcal{O}_{\mathcal{E}\otimes\mathcal{L}}(-r+1), \cdots, p^*F_0, \cdots, p^*F_n)$$

is a full strongly exceptional collection of locally free sheaves on $\mathbb{P}(\mathcal{E}\otimes\mathcal{L})$. Finally, since $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}\otimes\mathcal{L})$ ([8], Exercise II.7.9), we conclude that $\mathbb{P}(\mathcal{E})$ also has a full strongly exceptional collection of locally free sheaves of rank one and hence, it has a tilting bundle whose summands are line bundles. □

Corollary 3.2. *Let $\mathcal{E}$ be a rank r vector bundle on $\mathbb{P}^n$. Then $\mathbb{P}(\mathcal{E})$ has a tilting bundle of rank $r(n+1)$ whose summands are line bundles.*

Proof. It follows from Example 2.6 and the Main Theorem. □

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