

FRAGMENTABILITY OF SEQUENCES
OF SET-VALUED MAPPINGS WITH APPLICATIONS
TO VARIATIONAL PRINCIPLES

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ABSTRACT. We propose to study fragmentability of set-valued mappings not only for a given single mapping, but also for a sequence of mappings associated with the initial one. It turns out that this property underlies several variational principles, such as for example the Deville-Godefroy-Zizler variational principle and the Stegall variational principle, by providing a common path for proof.

1. INTRODUCTION AND PRELIMINARIES

Let $T : Z \rightrightarrows X$ be a set-valued mapping between a topological space Z and a metric space (X, d) . The mapping T is said to be *fragmented by d* (see [HJT]) if for any nonempty open set U of Z and every $\varepsilon > 0$ there is a nonempty open set $U' \subset U$ such that $d - \text{diam} T(U') < \varepsilon$. This notion, which has turned out to be very useful in the study of the points of single-valuedness of set-valued mappings, is closely related to (and also motivated by) two other notions: *fragmentability of topological spaces* and *minimal set-valued mappings*.

Recall that a topological space X is said to be *fragmented by a metric d in X* (see [JRo]) if for any nonempty set A of X and every $\varepsilon > 0$, there is a nonempty open (in the original topology) set V of X such that $d - \text{diam}(V \cap A) < \varepsilon$. Evidently, every metric space is fragmented by its own metric. But there are interesting cases of nonmetrizable topological spaces that are fragmentable by some metric. A typical situation is the case of certain real Banach spaces X with its weak topology (or dual Banach spaces with the weak star topology) which can be fragmented by the norm: for the study of fragmentability see e.g. [JNRo, Ri1, Ri2, KMo].

Further, a set-valued mapping $T : Z \rightrightarrows X$ between topological spaces Z and X is said to be *minimal* if for any open sets U of Z and V of X so that $T(U) \cap V \neq \emptyset$, there exists a nonempty open set $U' \subset U$ with the property $T(U') \subset V$. This definition is motivated by the fact that the latter property characterizes the minimal (in the sense of graph inclusion) mappings $T : Z \rightrightarrows X$ which are *usco in Z* (*upper semicontinuous with nonempty compact values*) ([ChrK]). However, there are important cases of mappings which are not necessarily usco but still possess the

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above minimality property, such as for example the solution mapping related to minimization of continuous bounded functions in a topological space X , or metric projections in certain Banach spaces (see [ČKR2]).

It is straightforward to check that if $T : Z \rightrightarrows X$ is minimal and X is fragmented by a metric d , then T is fragmented by d as well. Moreover, it is well known that a fragmentable mapping T between a Baire space Z and a space X (with fragmenting metric d) is *generically* no more than single-valued. The latter means that there is a dense G_δ -subset Z_1 of Z such that for any $z \in Z_1$ either $T(z) = \emptyset$ or T is single-valued at z . These facts have been used in proving, for instance, generic single-valuedness of maximal monotone mappings, in particular, generic differentiability of convex functions (cf. [Ph]).

The above approach is suitable for mappings that are nonempty-valued. But, often one is faced with a situation where a set-valued mapping to be used may have empty values. This is a typical situation in several well known variational principles, where the general scheme is the following one: one is given a lower semicontinuous extended real-valued and bounded from below function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ which is defined on a topological space X and which is *proper*. The latter means that the *effective domain* $\text{dom} f$ of f , i.e., the set $\text{dom} f := \{x \in X : f(x) < +\infty\}$ is nonempty. One possesses also a family Z of usually continuous and bounded from below on X functions $g : X \rightarrow \mathbb{R}$. The question is to investigate what is the set $Z_1 := \{g \in Z : f + g \text{ attains its minimum in } X \text{ (or stronger: the perturbed minimization problem } (X, f + g) \text{ is well-posed)}\}$.

Recall that a minimization problem (X, h) , with h proper in X , is well-posed if there is a unique minimum of h in X towards which converges any minimizing sequence for (X, h) , i.e., any sequence $\{x_n\}_{n \geq 1} \subset X$ with $h(x_n) \rightarrow \inf h := \inf_X h$. If (X, h) is well-posed, then every minimizing net for (X, h) also converges to the unique solution. The well-posedness of (X, h) is also termed as that h attains its *strong minimum* in X . The first aim in a typical variational principle is to prove that the set of good perturbations Z_1 is nonempty. If Z is equipped with a complete metric, a further aim is to show that Z_1 is *residual* in Z , which means that it contains a dense G_δ -subset of Z (equivalently, $Z \setminus Z_1$ is of the first Baire category in Z).

The above scheme has been the case in the Ekeland variational principle [Ek1, Ek2], Borwein-Preiss smooth variational principle [BPr], Stegall variational principle [St], the generic variational principle of Čoban, Kenderov and Revalski [ČK, ČKR1], Deville-Godefroy-Zizler variational principle [DGZ1, DGZ2] (see also a strengthening of the latter in [DR]) and Ioffe-Zaslavski principle [IZa]. With the exception of the first two principles, in the remaining ones the set of good perturbations contains a dense G_δ -subset of the space of all perturbations.

In [ČK, ČKR1, ČKR2], where the case $f \equiv 0$ and $Z = C_b(X) = \{g \in C_b(X) : g \text{ is bounded}\}$, equipped with the uniform norm $\|g\|_\infty := \sup\{|g(x)| : x \in X\}$, was considered, an approach was proposed via the solution mapping $M : C_b(X) \rightarrow X$ assigning to each $g \in C_b(X)$ the set of minimum points to g . This mapping is minimal and due to the fact that it is densely nonempty, the corresponding generic principle was deduced via the existence of generically defined selections of M ([ČKR2]). Unfortunately, the latter approach cannot be used directly in the case of the Stegall principle, neither in the Deville-Godefroy-Zizler nor in the Ioffe-Zaslavski principle, because in these principles one does not

know *a priori* (neither is it easily seen) whether the corresponding solution mapping is densely nonempty. In all these three principles the proofs have been based on a direct construction of a countable family of open dense subsets which gives as an intersection a dense G_δ -subset of good perturbations.

We show in this paper, however, that behind these principles there is something common, of topological nature, which allows us to avoid this repeated construction in each case. Namely, we propose below, instead of studying a single mapping T (corresponding to the solution mapping), to consider together with it a sequence of associated mappings $\{T_n\}_{n \geq 1}$ (which will correspond to the approximate solutions and) which give as an intersection the mapping T . We give a definition for the fragmentability of the sequence $\{T_n\}_{n \geq 1}$ and prove that in this case the initial mapping T is single-valued and possesses a continuity-like property (with respect to the fragmenting metric) at the points of a dense G_δ -subset of the domain space, provided the fragmenting metric is complete. The result is then applied to obtain by a common scheme the Deville-Godefroy-Zizler principle, the generic variational principle of Ćoban, Kenderov and Revalski and the Ioffe-Zaslavski principle. Stegall's variational principle is also shown to fit this general scheme.

Let us mention, finally, that a strengthening of the Borwein-Preiss variational principle in Banach spaces which could be used to derive the Deville-Godefroy-Zizler principle is provided by Loewen and Wang in [LW]. A one-perturbation variational principle (i.e. when Z consists of a single function) with applications to differentiability in Banach spaces is studied in the recent paper [BCFR].

In the sequel, given a metric space (X, d) , as usual $B_X(x, r)$ means the open ball in X around $x \in X$ with radius $r > 0$, and for a set $A \subset X$, $d - \text{diam } A$ is its diameter. All topological spaces in this article are supposed to be at least Hausdorff.

2. FRAGMENTABILITY OF SEQUENCES OF MAPPINGS

Let $\{T_n : Z \rightrightarrows X\}_{n \geq 1}$ be a decreasing sequence of nonempty-valued mappings between a topological space Z and a metric space (X, d) . In other words, the sequence $\{T_n\}_{n \geq 1}$ satisfies $\emptyset \neq T_{n+1}(z) \subset T_n(z)$ for every $z \in Z$ and $n \geq 1$. Given such a sequence, we will denote by $T(z) := \bigcap_{n=1}^{\infty} T_n(z)$, $z \in Z$, the limit (intersection) mapping generated by $\{T_n\}_{n \geq 1}$. Observe that the limit mapping T may have empty values.

Definition 2.1. We say that the sequence $\{T_n : Z \rightrightarrows (X, d)\}_{n \geq 1}$ is *fragmented by the metric d* if for every nonempty open set U in Z and $\varepsilon > 0$ there are $n' \geq 1$ and a nonempty open set $U' \subset U$ so that $d - \text{diam } T_{n'}(U') < \varepsilon$.

In this case we also say that d *fragments* the sequence $\{T_n\}_{n \geq 1}$. Evidently, if d fragments the sequence, it fragments also the limit mapping T .

The proof of the next theorem follows the general path of proving generic single-valuedness of a given (single) fragmentable mapping (see e.g. [ChrK, Ri1, Ph]). Recall that a *Baire space* is a topological space in which the intersection of a countable family of open and everywhere dense subsets is dense in the space.

Theorem 2.2. *Let $\{T_n\}_{n \geq 1}$ be a decreasing sequence of nonempty-valued mappings between a Baire topological space Z and a metric space (X, d) . Suppose that d fragments $\{T_n\}_{n \geq 1}$. Then there is a dense G_δ -subset Z_1 of Z such that for any*

point $z \in Z_1$ the limit mapping T satisfies:

- (a) either $T(z) = \emptyset$, or
- (b) $T(z)$ is a singleton, say x , and moreover, for any $\varepsilon > 0$ there exist an open set $U \ni z$ and $n' \geq 1$ with $T_{n'}(U) \subset B_X(x, \varepsilon)$.

If the metric d is complete and the values $T_n(z)$ are closed in X for any n and $z \in Z$, then each $z \in Z_1$ satisfies (b).

Proof. For each $k = 1, 2, \dots$, consider the sets:

$$W_k := \bigcup \left\{ U : U \neq \emptyset \text{ open in } Z \ \& \ \exists n \geq 1 : d - \text{diam } T_n(U) < 1/k \right\}.$$

Every W_k is open. We show that each of these sets is also dense in Z . Let $k \geq 1$ be fixed and take an arbitrary nonempty open set U in Z . By the fragmentability of $\{T_n\}_{n \geq 1}$ there are $n' \geq 1$ and a nonempty open $U' \subset U$ so that $d - \text{diam } T_{n'}(U') < 1/k$. Evidently, $U' \subset W_k$.

Let $Z_1 := \bigcap_{k=1}^{\infty} W_k$. This is a dense G_δ -subset of Z . Fix some $z \in Z_1$. Then there exist a sequence of open sets $\{U_k\}_{k \geq 1}$ and a strictly increasing sequence of integers $\{n_k\}_{k \geq 1}$, so that for any $k \geq 1$ we have $z \in U_k$ and $d - \text{diam } T_{n_k}(U_k) < 1/k$. Obviously, $T(z) = \bigcap_{k=1}^{\infty} T_{n_k}(z) \subset \bigcap_{k=1}^{\infty} T_{n_k}(U_k)$. The latter set is no more than a singleton. Suppose that $T(z) \neq \emptyset$. Then there is $x \in X$ so that

$$(1) \quad \{x\} = T(z) = \bigcap_{k=1}^{\infty} T_{n_k}(z) = \bigcap_{k=1}^{\infty} T_{n_k}(U_k).$$

Observe that if the metric d is complete and $T_n(z)$ are closed in X for any $n \in \mathbb{N}$, then by Cantor's theorem we always have (1) fulfilled for the chosen z .

Let $\varepsilon > 0$ be arbitrary. Take $k \geq 1$ so that $1/k < \varepsilon$ and put $U := U_k$ and $n' := n_k$. Since $\text{diam } T_{n_k}(U_k) < 1/k$, we conclude by (1) that $T_{n'}(U) \subset B_X(x, \varepsilon)$. The proof is completed. \square

Remark 2.3. Let us recall that a set-valued mapping $T : Z \rightrightarrows X$ between topological spaces Z and X is said to be *upper semicontinuous at* $z_0 \in Z$ if for any open set $V \subset X$ such that $T(z_0) \subset V$ there is an open set $U \subset Z$ with $z_0 \in U$ so that $T(U) \subset V$. Therefore, by (b) above, we see that T is single-valued and upper semicontinuous with respect to the metric d in X at any $z \in Z_1$ for which $T(z)$ is nonempty. In particular, if the metric d is complete and the values $T_n(z)$ are closed for any n and z , then T is single-valued and upper semicontinuous at any point of Z_1 . These two properties are well known for a single fragmentable mapping (for the latter property the mapping must be with nonempty values everywhere). What is new in Theorem 2.2 is, first the continuity property in (b) which, in general, is stronger than the upper semicontinuity of the limit mapping and which is important for some of the applications below. We may express this property as that the sequence $\{T_n\}_{n \geq 1}$ is *upper semicontinuous at* z . The second contribution of the latter theorem is that we obtain the generic single-valuedness of the limit mapping **without** supposing that this mapping has nonempty values. This again (see the next section) is important for the applications where, as a rule, the corresponding solution mapping may have empty values.

Call the sequence $\{T_n : Z \rightrightarrows X\}_{n \geq 1}$ of mappings between the topological spaces Z and X *minimal* if for any nonempty open sets U of Z and V of X such that

$T_n(U) \cap V \neq \emptyset$ for some $n \geq 1$, there are a nonempty open set $U' \subset U$ and $n' \in \mathbb{N}$ so that $T_{n'}(U') \subset V$.

Evidently, if $\{T_n\}_{n \geq 1}$ is minimal and the space X is fragmented by a metric d , then $\{T_n\}_{n \geq 1}$ is fragmented by d as well. Therefore, the following is an immediate corollary of Theorem 2.2.

Corollary 2.4. *Let $\{T_n\}_{n \geq 1}$ be a decreasing sequence of nonempty-valued mappings between a Baire topological space Z and a topological space X . Suppose that X is fragmented by a metric d . Then, if $\{T_n\}_{n \geq 1}$ is minimal, there exists a dense G_δ -subset Z_1 of Z such that for any point $z \in Z_1$ the limit mapping T satisfies:*

- (a) either $T(z) = \emptyset$, or
- (b) $T(z)$ is a singleton, say x , and moreover, for any $\varepsilon > 0$ there exist an open set $U \ni z$ and $n' \geq 1$ with $T_{n'}(U) \subset B_X(x, \varepsilon)$.

If the metric d is complete, and the values $T_n(z)$ are d -closed for any n and $z \in Z$, then each $z \in Z_1$ satisfies (b).

3. APPLICATIONS TO VARIATIONAL PRINCIPLES

In this section we show how the above general scheme applies to several well known variational principles. The framework is the following: we are given a completely regular topological space X , a subspace $X_1 \subset X$ whose topology inherited from X is metrizable by a complete metric d , and a Banach space $(Z, \|\cdot\|_Z)$ of real-valued continuous functions on X which are bounded on X_1 and satisfy:

(H1) There is $M > 0$ such that for all $g \in Z$, $M\|g\|_Z \geq \sup\{|g(x)| : x \in X_1\}$.

Observe that the functions from Z need not, in general, be bounded on X .

The space Z will serve as a space of perturbation functions. More precisely, the question we are concerned with reads: given a function $f : X_1 \rightarrow \mathbb{R} \cup \{+\infty\}$ which is proper, lower semicontinuous and bounded from below on X_1 , what are the descriptive properties of the set $\{g \in Z : \text{the perturbed minimization problem } (X_1, f + g) \text{ is well-posed}\}$.

To deal with this question, we shall follow the scheme developed in the previous section with the sequence of set-valued mappings $\{T_n : Z \rightrightarrows X_1\}_{n \geq 1}$ defined by

$$T_n(g) := \{x \in X_1 : f(x) + g(x) \leq \inf_{X_1}(f + g) + 1/n\}, \quad g \in Z, \quad n \geq 1,$$

which for any $g \in Z$ gives, as value of the limit mapping, the minimum points of $f + g$ in X_1 :

$$T(g) = \bigcap_{n=1}^{\infty} T_n(g) = \{x \in X_1 : f(x) + g(x) = \inf_{X_1}(f + g)\}, \quad g \in Z.$$

Given $A \subset X$, a proper function $h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ which is bounded from below on A and $\alpha > 0$, we denote by $S(A, h, \alpha)$ the sub-level set

$$S(A, h, \alpha) := \{x \in A : h(x) \leq \inf_A h + \alpha\}.$$

So, $T_n(g) = S(X_1, f + g, 1/n)$. Obviously, $\{T_n\}_{n \geq 1}$ is a decreasing sequence and all the sets $T_n(g)$ are nonempty and closed in X_1 . To apply Theorem 2.2 we need to find conditions assuring that this sequence is fragmented by the metric d . The following proposition gives such conditions.

Proposition 3.1. *In the above framework, sufficient conditions (in decreasing order of generality) in order for the sequence $\{T_n : Z \rightrightarrows X_1\}_{n \geq 1}$ to be fragmented by d are the following:*

- (F) *For each $h : X_1 \rightarrow \mathbb{R} \cup \{+\infty\}$ which is proper, lower semicontinuous and bounded from below in X_1 and every $\varepsilon > 0$ there are $g \in Z$ and $\alpha > 0$ so that $\|g\|_Z \leq \varepsilon$ and $d - \text{diam } S(X_1, h + g, \alpha) < \varepsilon$.*
- (S) *For every $\varepsilon > 0$ there is $\beta > 0$ such that for each $x_1 \in X_1$ there exists $g \in Z$ with $\|g\|_Z = 1$ and $g(x_1) + \beta \leq \inf \{g(x) : x \in X_1 \setminus B_{X_1}(x_1, \varepsilon)\}$.*

Proof. First we show that (F) implies that $\{T_n\}_{n \geq 1}$ is fragmented by d . Let U be a nonempty open subset of Z and $\varepsilon > 0$. Take $g_0 \in U$ and $\gamma > 0$ with $B_Z(g_0, \gamma) \subset U$. According to (F) there is $g \in Z$ such that $\|g\|_Z \leq \bar{\varepsilon} := \min\{\varepsilon, \gamma/2\}$ and $\alpha > 0$ with $d - \text{diam } S(X_1, f + g_0 + g, \alpha) < \bar{\varepsilon}$. On the other hand, because of (H1) it is well known (and straightforward to see) that there is a positive $\delta < \gamma/2$ so that $S(X_1, f + g', \alpha/2) \subset S(X_1, f + g_0 + g, \alpha)$ for any $g' \in B_Z(g_0 + g, \delta)$. Thus, $d - \text{diam } T_{n'}(U') < \varepsilon$ for $U' = B(g_0 + g, \delta) \subset U$ and n' such that $1/n' < \alpha/2$.

Next, we show that (S), which is a kind of separation of the points and closed sets of X_1 by functions from Z , implies (F). Let $h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be as in (F) and $\varepsilon > 0$. Take $\beta > 0$ as given by (S), put $\alpha := \varepsilon\beta/2$ and pick $x_1 \in X_1$ so that

$$h(x_1) < \inf_{X_1} h + \alpha.$$

Now let $g \in Z$ be the function provided by (S) for the point x_1 . Set $g_\varepsilon := \varepsilon g$ and let $x \in S(X_1, h + g_\varepsilon, \alpha)$. We have

$$(h + g_\varepsilon)(x) \leq (h + g_\varepsilon)(x_1) + \alpha < \inf_{X_1} h + g_\varepsilon(x_1) + 2\alpha;$$

hence $g(x) < g(x_1) + \beta$, so $x \in B_{X_1}(x_1, \varepsilon)$ according to (S). This shows that the set $S(X_1, f + g_\varepsilon, \alpha)$ is contained in $B_{X_1}(x_1, \varepsilon)$; hence its d -diameter is less than 2ε , while $g_\varepsilon \in Z$ with $\|g_\varepsilon\|_Z = \varepsilon$. The proof that (F) holds is complete. \square

As a result, we have the following variational principle:

Theorem 3.2. *Let X be a completely regular topological space, let $X_1 \subset X$ be a completely metrizable subspace by a metric d and let $(Z, \|\cdot\|_Z)$ be a Banach space satisfying (H1) and (S) (or, more generally, (H1) and (F)). Suppose that $f : X_1 \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper, lower semicontinuous and bounded from below on X_1 . Then, there is a dense G_δ -subset Z_1 of Z such that for any $g \in Z_1$ the perturbed minimization problem $(X_1, f + g)$ is well-posed.*

Proof. As observed, $\{T_n\}_{n \geq 1}$ is a decreasing sequence and the sets $T_n(g)$ are nonempty and closed in (X_1, d) . Moreover, it follows from Proposition 3.1 that this sequence is also fragmented by the metric d . Thus, by Theorem 2.2, there is a dense G_δ -subset Z_1 of Z so that for each $g \in Z_1$ we have the property (b) for the limit mapping (because the metric d is complete). The property (b) for g evidently implies the well-posedness of $(X_1, f + g)$. \square

The Deville-Godefroy-Zizler variational principle is a consequence of Theorem 3.2:

Theorem 3.3 ([DGZ1, DGZ2]). *Let $(X, \|\cdot\|)$ be a Banach space and let $(Z, \|\cdot\|_Z)$ be a Banach space of continuous bounded functions in X satisfying (H1) with $X_1 = X$*

and the following additional properties:

- (H2) For any $g \in Z$ and $x \in X$ the function $g_x : y \in X \mapsto g(x + y)$ is in Z and, moreover, $\|g_x\|_Z = \|g\|_Z$.
- (H3) For any $g \in Z$ and $t \in \mathbb{R}$ the function $x \in X \mapsto g(tx)$ is in Z .
- (H4) There exists a bump function in Z , i.e. a function $b : X \rightarrow \mathbb{R}$ with a bounded nonempty support.

Suppose that $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper, lower semicontinuous and bounded from below on X . Then, there is a dense G_δ -subset Z_1 of Z such that for any $g \in Z_1$ the perturbed minimization problem $(X, f + g)$ is well-posed.

Proof. The result will follow from Theorem 3.2 with $X_1 = X$ once we have checked that the properties (H2)–(H4) imply the separation property (S). First note that, because of (H2) and (H3), we may suppose that the support of the bump b given by (H4) lies in the (open) unit ball $B_X(0, 1)$ of X and that $b(0) = 1$. Let $\varepsilon > 0$ be arbitrary. Consider the function $b_\varepsilon(\cdot) = b(\cdot/\varepsilon)$ which is in Z and whose support lies in the open ball $B_X(0, \varepsilon)$. Put $\beta := 1/\|b_\varepsilon\|_Z$. Now let $x_1 \in X$ be arbitrary. The function $g(\cdot) := -\beta b_\varepsilon(\cdot - x_1)$ is in Z , $\|g\|_Z = \beta\|b_\varepsilon\|_Z = 1$, $g(x_1) + \beta = 0$ and g is zero outside the ball $B_X(x_1, \varepsilon)$. Hence, (S) is verified. \square

Let us mention that particular important cases when one has the conditions (H1)–(H4) fulfilled include the family of all bounded continuous functions in X , the family of all bounded Lipschitz functions in X and the family of all bounded Lipschitzian C^1 -smooth functions in a Banach space admitting a Lipschitzian C^1 -smooth bump with the corresponding norms. This allows many applications of the Deville-Godefroy-Zizler variational principle, among them the possibility to derive (the weak form) of the Ekeland variational principle and to prove existence of solutions to Hamilton-Jacobi equations (see [DGZ1, DGZ2] for more details).

Further, we show how (one part of) the generic variational principle of Čoban, Kenderov and Revalski ([ČK, ČKR1]) (in a slightly stronger form given in [DR]) follows also by Theorem 3.2.

Theorem 3.4 ([ČK, ČKR1, DR]). *Let X be a completely regular topological space which contains a dense completely metrizable subspace X_1 . Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and bounded from below function in X which is continuous at any point of $\text{dom} f$. Then the set $\{g \in C_b(X) : (X, f + g) \text{ is well-posed}\}$ contains a dense G_δ -subset of the space $(C_b(X), \|\cdot\|_\infty)$.*

Actually, in [ČK, ČKR1] it is proved (with $f \equiv 0$) that the property of X to contain a dense completely metrizable subspace characterizes the fact that the well-posed problems in $C_b(X)$ form a residual subset of $C_b(X)$. A strengthening of the above theorem (as well as of the Deville-Godefroy-Zizler principle) to get a stronger (than the first Baire category) property of smallness of the complement of the well-posed problems, the so-called σ -porosity, was done in [DR].

Proof. First, observe that because of the assumptions on f and X_1 the domain $\text{dom} f$ of f is open in X and moreover $\inf_X(f + g) = \inf_{X_1}(f + g)$ for any $g \in C_b(X)$. In particular, f is proper on X_1 too. The space $Z = C_b(X)$ supplied with the sup-norm $\|\cdot\|_\infty$ satisfies (H1), and because X is completely regular, the separation property (S) is satisfied too. Thus, by Theorem 3.2, there is a dense G_δ -subset Z_1 of $C_b(X)$ so that for each $g \in Z_1$ the problem $(X_1, f + g)$ is well-posed. We show

that for $g \in Z_1$ the minimization problem $(X, f + g)$ is well-posed too. Indeed, take $g \in Z_1$ and let $x_0 \in X_1$ be the unique minimum of the problem $(X_1, f + g)$. First, since every minimizing net for $(X_1, f + g)$ also converges to x_0 and X_1 is dense in X , the minimization problem $(X, f + g)$ cannot have other solutions than x_0 . Therefore, $\{x_0\} = \bigcap_{k \geq 1} O_k$, where $O_k := \{x \in X : f(x) + g(x) < \inf_X(f + g) + 1/k\}$. We show, in addition, that the family of open sets $\{O_k\}_{k \geq 1}$ is a (countable) local base for x_0 in X . From this the well-posedness of $(X, f + g)$ follows immediately.

Take an open set V of X so that $V \ni x_0$. Since X is completely regular there is an open set W in X with $W \ni x_0$ and the closure \overline{W} is contained in V . We claim that there is k_0 so that $O_{k_0} \subset \overline{W} \subset V$. Otherwise, for any $k \geq 1$ we have $O_k \setminus \overline{W} \neq \emptyset$ and since the latter set is open in X and X_1 is dense in X we obtain a sequence $x_k \in (O_k \setminus \overline{W}) \cap X_1$, $k \geq 1$. Evidently $\{x_k\}_{k \geq 1} \subset X_1$ is a minimizing sequence for $(X_1, f + g)$ which does not converge to x_0 . This contradiction completes the proof. \square

The above generic variational principle has been used for the study of the differentiability of the sup-norm in $C_b(X)$ (see [ČK, ČKR1]). It is also closely related to the existence of (special) winning strategies for one of the players in the Banach-Mazur game ([ČKR3, KR]).

Next, let us see that the well known Stegall variational principle also fits the above general scheme. Here again $(X, \|\cdot\|)$ denotes a real Banach space. Recall that when x^* is an element of the dual X^* of X the sets $S(A, x^*, \alpha)$ are called *slices* in $A \subset X$. A subset C of a Banach space X is said to satisfy the *Radon-Nikodým property* (in short RNP) if every bounded subset A of C is *dentable*. The set A is dentable if A admits slices of arbitrarily small (norm-) diameter.

Theorem 3.5 ([St]). *Let C be a nonempty bounded closed convex subset of a Banach space X . Suppose that C satisfies RNP and let $f : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous and bounded from below on C . Then the set $\{x^* \in X^* : (C, f + x^*) \text{ is well-posed}\}$ contains a dense G_δ -subset of X^* .*

Let us recall that, when the maximization problem (C, x^*) for some $x^* \in X^*$ is well-posed, then the unique maximum in C is said to be *strongly exposed (by x^*)* and the functional x^* is called a *strongly exposing functional for C* . An immediate consequence of the Stegall variational principle (with $f \equiv 0$) is that the set of strongly exposing functionals for a nonempty closed convex bounded subset $C \subset X$ with RNP is a residual subset of X^* . Another important consequence is that every nonempty closed bounded convex set with RNP in X is the closed convex hull of its strongly exposed points (see e.g. Phelps [Ph], Theorem 5.20).

One way to prove the above result passes by the following key fact (see e.g. Phelps [Ph], Proof of Theorem 5.15, page 90).

Proposition 3.6. *Let C be a nonempty bounded closed convex subset of a Banach space X . If C satisfies RNP, then for every $h : C \rightarrow \mathbb{R} \cup \{+\infty\}$ which is proper, lower semicontinuous and bounded from below on C and every $\varepsilon > 0$, there are $x^* \in X^*$ with $\|x^*\| < \varepsilon$ and $\alpha > 0$ such that $\text{diam } S(C, h + x^*, \alpha) < \varepsilon$.*

Proof of Theorem 3.5. We may apply Theorem 3.2 with $X_1 = C$ and $Z = X^*$ with its dual norm since (H1) is verified with $M = \sup\{\|x\| : x \in C\}$ and (F) holds according to Proposition 3.6. \square

Finally, as a further illustration of our approach, we show how the variational principle of Ioffe and Zaslavski follows readily from our general Theorem 2.2:

Theorem 3.7 ([IZa]). *Let (X, d) be a complete metric space and let Z be a Baire metric space such that for any $z \in Z$ there is associated a proper lower semicontinuous function $f_z : X \rightarrow [-\infty, +\infty]$ that satisfies the following hypothesis:*

(\mathcal{H}) *there is a dense subset Z_0 of Z so that for any $z \in Z_0$, any $\varepsilon > 0$ and any $\gamma > 0$, there exist a nonempty open set $U \subset Z$, $x \in X$, $\alpha > 0$ and $\eta > 0$ such that U belongs to the ball of radius ε around z and for any $u \in U$ one has:*

(i) $\inf f_u > -\infty$;

(ii) *if $y \in X$ is such that $f_u(y) \leq \inf f_u + \eta$, then $d(y, x) \leq \gamma$ and $|f_u(y) - \alpha| \leq \gamma$.*

Then, there exists a dense G_δ -subset Z_1 of Z so that for any $z \in Z_1$, the infimum of f_z is finite and attained at a unique point $x \in X$. Moreover, the following kind of strong well-posedness is true for f_z : if $z_n \rightarrow z$, then $\inf f_{z_n} > -\infty$ eventually, and if $x_n \in X$ satisfy $f_{z_n}(x_n) - \inf f_{z_n} \rightarrow 0$, then $x_n \rightarrow x$.

Proof. By the hypothesis (\mathcal{H}) there is an open and dense set Z' of Z , so that for any $z \in Z'$ one has $\inf f_z > -\infty$. Evidently Z' is a Baire subspace of Z . Define T, T_n in Z' as above: $T(z) := \{x \in X : f_z(x) = \inf f_z\}$ and $T_n(z) := \{x \in X : f_z(x) \leq \inf f_z + 1/n\}$, $n \geq 1$. Obviously $\{T_n\}_{n \geq 1}$ is decreasing and the sets $T_n(z)$ are nonempty and closed in X for any n and $z \in Z'$. Moreover, it is seen from (\mathcal{H}) that the sequence $\{T_n\}_{n \geq 1}$ is fragmented by the metric in X . By Theorem 2.2, there is a dense G_δ -subset Z_1 of Z' each point of which satisfies (b). Since Z' is open and dense in Z , then Z_1 is dense G_δ in Z as well. This completes the proof since the condition (b) implies also the required strong well-posedness. \square

The principle above has an additional conclusion for the continuity of the infimal-value function. This continuity is usually fulfilled in all the applications. Let us mention that the above principle is applicable also to situations where one considers perturbations with a distance between them weaker than the uniform one (for example the uniform convergence on bounded sets). Various such applications, concerning convex problems, quadratic problems, optimal control problems, etc., could be found in [IZa, ILR].

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