ON NATURAL HOMOMORPHISMS OF WITT RINGS

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ABSTRACT. We prove that the kernel of the ring homomorphism between the Witt ring of any order of a global field and the Witt ring of the field is a nilideal.

1. Introduction

For a commutative ring $A$, let $W(A)$ denote the Witt ring of nondegenerate symmetric bilinear forms on finitely generated projective modules (bilinear spaces) over $A$, as defined by Knebusch in [5]. For an integral domain $A$ and its field of fractions $K$ we consider the ring homomorphism

$$ \varphi : W(A) \rightarrow W(K) $$

induced by the inclusion $A \hookrightarrow K$. If $A$ is a Dedekind domain it is known that $\ker \varphi$ is zero. This was first proved by Knebusch ([5 Satz 11.1.1]). A more general case was considered by Craven, Rosenberg and Ware in [3]. They proved that when $A$ is a regular noetherian domain of an arbitrary Krull dimension, then $\ker \varphi$ is a nilideal, that is, every element belonging to the kernel is a nilpotent element of the Witt ring $W(A)$. They also produced examples emphasizing the importance of the regularity condition. However, it turns out that the class of domains $A$ with $\ker \varphi$ a nilideal does not only include regular domains. We prove that if $A$ is an order in a global field $K$, then the kernel of $\varphi$ is a nilideal. Since nonmaximal orders are not regular rings, this result shows that regularity of the domain $A$ is not a necessary condition for the kernel of $\varphi$ to be a nilideal. When the field $K$ is not formally real we find a necessary and sufficient condition for $\ker \varphi$ to be a nilideal (Theorem 2.3) immediately applicable to global fields. When $K$ is a formally real number field we use the theory of signatures on Witt rings.

For a commutative ring $A$ we write $A^*$ for the group of invertible elements in $A$. If $p$ is a prime ideal in $A$ we write $A_p$ for the localization of $A$ with respect to $p$. The symbol $\langle E \rangle$ denotes the element of the Witt ring $W(A)$ determined by the bilinear space $E$ over $A$.
If $B$ is a commutative ring and $A$ is a subring of $B$, then by the natural homomorphism induced by the inclusion of $A$ into $B$ we mean the map

$$\varphi : W(A) \to W(B), \quad \varphi(E) = \langle E \otimes_A B \rangle.$$ 

We also use the symbol $\langle E \rangle_B$ for $\varphi(E)$.

$W(A)_t$ denotes the torsion part of the Witt ring $W(A)$. If $A$ is a local ring, then each element of $W(A)$ is represented by a free bilinear module. Then $W(A)_0$ denotes the ideal of classes of even ranks. We say that a ring $A$ has finite level if $-1 \in A$ can be represented as a sum of squares of elements of $A$. Otherwise $A$ has infinite level. A field with infinite level is said to be formally real, and nonreal otherwise.

A global field $K$ is a finite extension of the rational number field $\mathbb{Q}$ or a finite extension of a rational function field in one variable over a finite field. The ring of integers of a number field $K$ is denoted by $R$. In the function field case $R$ denotes the integral closure in $K$ of the polynomial ring of the underlying rational function field. In any case $R$ is a Dedekind domain. We adopt the following definition (see [4]).

**Definition 1.1.** An order $O$ in $K$ is a subring of $R$ such that $R/O$ is a finitely generated torsion $O-$module.

The ring $R$ is the maximal order in $K$. When $K$ is a number field, an order $O$ in $K$ is a subring of $R$ containing an integral basis of length $[K : \mathbb{Q}]$. Every order is a one-dimensional noetherian domain ([8, p. 73]).

We recall that a nonmaximal order $O$ is not a regular ring, i.e., there exists a prime ideal $p$ such that the localization $O_p$ is not a regular local ring and this is equivalent to the fact that $O_p$ is not a discrete valuation ring. This follows from [8, Theorem 12.10] where it is shown that $O_p$ is regular if and only if $f \notin p$, where $f$ is the conductor of the ring extension $R \supset O$. Since for a nonmaximal order $O$ we have $f \neq (0)$ and $f \neq O$, there is a nonzero prime ideal containing $f$ and so $O$ is not a regular ring.

## 2. $K$ nonreal

When the field $K$ is nonreal, there is no need to restrict the consideration to global fields. Theorem 2.3 below gives a convenient necessary and sufficient condition for the kernel of the natural ring homomorphism $W(A) \to W(K)$ to be a nilideal. This applies to nonreal global fields. We begin the discussion with the case of local domains.

**Theorem 2.1.** Let $A$ be a local domain. If $A$ has finite level, then the kernel of the natural ring homomorphism $W(A) \to W(K)$ is a nilideal.

**Proof.** If $A$ has finite level, then $W(A)_t = W(A)$ (see [1] A.4, p. 178). On the other hand $\ker(W(A) \to W(K)) \subseteq W(A)_0$, since any element in the kernel becomes metabolic over $K$, hence of even dimension. Thus

$$\ker(W(A) \to W(K)) \subseteq W(A)_0 = W(A)_0 \cap W(A)_t = \text{Nil} W(A),$$

the latter by [1] Theorem 8.9, p. 158].

**Remark 2.2.** Craven, Rosenberg and Ware proved that if $A$ is a regular noetherian local domain, then $\ker(W(A) \to W(K))$ is a nilideal ([3 Thm. 2.4]). Theorem 2.1
shows that if the local domain $A$ has finite level, then the conclusion holds even if it is not regular or noetherian.

For a ring $A$ and a prime ideal $p$ of $A$ we write $A(p)$ for the field of fractions of $A/p$. In particular, if $A$ is a domain, $A(0) = K$. We shall use below the following Dress’ local-global principle (see [3 Cor. 2.7]):

For a commutative ring $A$ and any bilinear $A$-space $E$, the class $\langle E \rangle \in W(A)$ is nilpotent if and only if $\langle E \rangle_{A_p} \in W(A_p)$ is nilpotent for all maximal ideals $p$ in $A$.

**Theorem 2.3.** Let $A$ be an integral domain. Assume that the field of fractions $K$ is not formally real. Then the following are equivalent:

(a) $\ker(W(A) \to W(K))$ is a nilideal.

(b) For each prime ideal $p$ in $A$ the field $A(p)$ is not formally real.

**Proof.** (a) $\Rightarrow$ (b) The proof in [3 Prop. 3.2] applies with a slight complication. Here are the details. Suppose there is a prime ideal $p$ of $A$ such that $A(p)$ is formally real. Then $r \cdot \langle 1 \rangle \neq 0$ in $W(A(p))$ for all positive integers $r$. On the other hand $2^n \langle 1 \rangle = 0$ in $W(K)$ for some $n \geq 0$ since $K$ is a nonreal field. Hence $2^n \langle 1 \rangle \in \ker(W(A) \to W(K))$. It remains to show that $2^n \langle 1 \rangle$ is not a nilpotent element in $W(A)$. If it were, then $2^n \langle 1 \rangle = 0$ in $W(A)$ for some $n$. Then the composition of natural homomorphisms

$$W(A) \to W(A/p) \to W(A(p))$$

sends $2^n \langle 1 \rangle$ to zero in $W(A(p))$, a contradiction.

(b) $\Rightarrow$ (a) By [2 Theorem 1], (b) implies that the level of the ring $A$ is finite. Hence the level of the localization $A_p$ is also finite for each nonzero prime ideal in $A$, and so $\ker(W(A_p) \to W(K))$ is a nilideal by Theorem 2.1. Now let $\langle E \rangle \in \ker(W(A) \to W(K))$. Then

$$\langle E \otimes_A A_p \rangle \in \ker(W(A_p) \to W(K)) \subseteq \text{Nil}(W(A_p)).$$

Thus $\langle E \rangle_{A_p} = \langle E \otimes_A A_p \rangle$ is nilpotent for all nonzero prime ideals $p$ in $A$. By Dress’ local-global principle, the class $\langle E \rangle$ is a nilpotent element of the ring $W(A)$.

If $A = \mathcal{O}$ is any order in a nonreal global field $K$, then (b) is satisfied, and so $\ker(W(\mathcal{O}) \to W(K))$ is a nilideal. Thus the following theorem, which is the first part of our main result, is a corollary to Theorem 2.3.

**Theorem 2.4.** Let $\mathcal{O}$ be an order in a nonreal global field $K$. Then the kernel of the natural ring homomorphism

$$\varphi : W(\mathcal{O}) \to W(K)$$

is a nilideal.

3. $K$ formally real

In this section $K$ is a formally real global field, that is, a formally real number field. As in the previous section we will use the local-global principle for nilpotency of elements of the Witt ring. Hence we first establish the following result for the local case.

**Theorem 3.1.** Let $p$ be a nonzero prime ideal in an order $\mathcal{O}$ of a formally real number field $K$. Then the kernel of the ring homomorphism $\varphi : W(\mathcal{O}_p) \to W(K)$ is a nilideal.
The set of all orderings of \( \mathbb{Q} \) fields have no orderings. This proves that \( \text{supp} \)

We collect the definitions we need following [6]. For an arbitrary commutative ring \( A \) we write \( \tilde{X}_A \) for the set of all signatures of the ring \( W(A) \), i.e., the set of all ring homomorphisms \( \sigma : W(A) \to \mathbb{Z} \). Recall that \( P \subset A \) is said to be an ordering in \( A \) if

1. \( P + P \subset P \), \( P \cdot P \subset P \),
2. \( P \cup -P = A \),
3. \( P \cap -P =: \text{supp} P \), called the support of \( P \), is a prime ideal in \( A \).

The set of all orderings of \( A \) is denoted by \( X_A \). Since \( K \) is assumed formally real, there is an ordering \( P \in X_K \). One can associate a signature \( \sigma_P : W(K) \to \mathbb{Z} \) with the ordering \( P \) by setting

\[
\sigma_P(a) = 1 \quad \text{for all} \quad a \in P, \ a \neq 0.
\]

More generally, if \( A \) is a local ring and \( P \in X_A \), then there is a signature \( \sigma_P : W(A) \to \mathbb{Z} \) satisfying (3.1). This follows from the fact that \( W(A) \) is generated by the rank one elements \( \langle a \rangle \) with \( a \in A^* \). We need a stronger converse statement associating orderings of \( K \) to signatures of \( \mathcal{O}_p \).

**Lemma 3.2.** For every signature \( \sigma \in \tilde{X}_{\mathcal{O}_p} \) there exists an ordering \( P \) of the field \( K \) such that

\[
\sigma \langle E \rangle = \sigma_P \langle E \rangle_K
\]

for all \( \langle E \rangle \in W(\mathcal{O}_p) \).

**Proof.** Let \( \sigma \in \tilde{X}_{\mathcal{O}_p} \). Then by [6] Prop. 1.5, p. 145] there exists an ordering \( Q \in X_{\mathcal{O}_p} \) such that \( \sigma \langle a \rangle = \sigma_Q \langle a \rangle \) for all \( a \in \mathcal{O}_p^* \). We must show that the ordering \( Q \) on \( \mathcal{O}_p \) can be extended to an ordering \( P \) on \( K \). To this end we show that \( \text{supp} \ Q = (0) \). If \( \text{supp} \ Q \neq (0) \), then \( \text{supp} \ Q = p\mathcal{O}_p \) since \( p\mathcal{O}_p \) is the only nonzero prime ideal in \( \mathcal{O}_p \). By a general rule (see [6] Prop. 1.1, p. 96]), there exists an ordering on \( \mathcal{O}_p/p\mathcal{O}_p \) whose inverse image under the canonical homomorphism \( \mathcal{O}_p \to \mathcal{O}_p/p\mathcal{O}_p \) is the ordering \( Q \). However, this is absurd, since the residue class field \( \mathcal{O}_p/p\mathcal{O}_p \cong \mathcal{O}/p \) is a subfield of the finite field \( R/q \), where \( q \cap \mathcal{O} = p \), and finite fields have no orderings. This proves that \( \text{supp} \ Q = (0) \). Thus, by [6] Prop. 1.1, p. 96], the ordering \( Q \) of \( \mathcal{O}_p \) extends uniquely to an ordering \( P \) of the field \( K \). The associated signature \( \sigma_P \) satisfies

\[
\sigma \langle a \rangle = \sigma_Q \langle a \rangle = \sigma_P \langle a \rangle_K \quad \text{for all} \quad a \in \mathcal{O}_p^*.
\]

Since \( W(\mathcal{O}_p) \) is additively generated by the rank-one classes \( \langle a \rangle \), \( a \in \mathcal{O}_p^* \), we have \( \sigma(E) = \sigma_P \langle E \rangle_K \) for all \( \langle E \rangle \in W(\mathcal{O}_p) \).

**Proof of Theorem 3.1.** Let \( \langle E \rangle \in W(\mathcal{O}_p) \) and \( \langle E \rangle_K = 0 \). Since

\[
\text{Nil} W(\mathcal{O}_p) = \bigcap_{\sigma \in \tilde{X}_{\mathcal{O}_p}} \ker \sigma
\]

(see [1] (7.11), p. 151 and (7.16), p. 152]), it is enough to show that \( \sigma \langle E \rangle = 0 \) for every signature \( \sigma \in \tilde{X}_{\mathcal{O}_p} \). By Lemma 3.2, given such a \( \sigma \), there exists \( P \in X_K \) such that for all \( \langle E \rangle \in W(\mathcal{O}_p) \) we have \( \sigma \langle E \rangle = \sigma_P \langle E \rangle_K \). Since \( \langle E \rangle_K = 0 \) we get \( \sigma \langle E \rangle = 0 \). This finishes the proof.

Theorem 3.1 and Dress’ local-global principle imply the following theorem which is the second part of our main result.
Theorem 3.3. Let $\mathcal{O}$ be an order in a formally real algebraic number field $K$. Then the kernel of the natural ring homomorphism
$$\varphi : W(\mathcal{O}) \rightarrow W(K)$$
is a nilideal.

References


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