PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 9, Pages 2621–2623
S 0002-9939(05)07932-3
Article electronically published on April 19, 2005

COMPLEMENTED SPACES OF OPERATORS

IOANA GHENCIU

(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. Classical results of Kalton are used to study the complementation of the space $W(X,Y)$ of weakly compact operators and the space $K(X,Y)$ of compact operators in the space $L(X,Y)$ of bounded linear operators from X to Y.

Throughout this paper X and Y denote Banach spaces. Notation is consistent with that used in Diestel [2]. Let (e_n) be the canonical base of c_0 and let (e_n^*) be the canonical base of l_1. Let η denote the canonical embedding of X in X^{**}.

Numerous authors have studied the complementation of the spaces of weakly compact operators $W(X,Y)$ and compact operators $K(X,Y)$ in the space $L(X,Y)$ of all continuous operators from X to Y. See Bator and Lewis [1], Kalton [9], Emmanuele [4], [5], Emmanuele and John [6], and Feder [7]. The following theorem generalizes the results in [5], [1], and [7].

Theorem 1. (i) *Let X and Y be Banach spaces with the following properties: There exists a Banach space G with an unconditional basis (g_n), biorthogonal coefficients (g_n^*) and operators $R : G \to Y$ and $S : X \to G$ such that $(R(g_i))$ is a seminormalized basic sequence in Y and $\{S^*(g_i^*) : i \in \mathbb{N}\}$ is not relatively weakly compact. Then $W(X,Y)$ is not complemented in $L(X,Y)$.*

(ii) *Let X and Y be Banach spaces with the following properties: There exists a Banach space G with an unconditional basis (g_n), biorthogonal coefficients (g_n^*) and operators $R : G \to Y$ and $S : X \to G$ such that $(R(g_i))$ is a seminormalized basic sequence in Y and $\{S^*(g_i^*) : i \in \mathbb{N}\}$ is not relatively compact. Then $K(X,Y)$ is not complemented in $L(X,Y)$.*

Proof. (i) Let $(g_{i_j}^*)$ be a subsequence of (g_i^*) such that the sequence $(S^*(g_{i_j}^*))$ has no weakly convergent subsequence. Let $E = [g_{i_j}]$ be the closed linear span of $\{g_{i_j} : j \in \mathbb{N}\}$, $y_j = R(g_{i_j})$, for $j \geq 1$, and let (y_j^*) be a sequence of biorthogonal coefficients corresponding to (y_j). Let $p : G \to E$ be a projection. Define $T : X \to E$ by $T = pS$ and $B : E \to Y$ by $B = R_{|E}$.

Since $\sum g_n^*(g)g_n$ converges unconditionally to g for all $g \in G$,

$$B(\sum_j g_{i_j}^*(Tx)g_{i_j}) = \sum_j g_{i_j}^*(Tx)B(g_{i_j}) = \sum_j T^*(g_{i_j}^*)(x)y_j$$

Received by the editors April 15, 2004.
2000 *Mathematics Subject Classification.* Primary 46B20, 46B28.
Key words and phrases. Spaces of operators, complemented subspaces, compact operators, weakly compact operators, basic sequence.

converges unconditionally for all $x \in X$. Let $D = \{T^*(g^*_{i_j}) : j \in \mathbb{N}\}$. Let X_0 be a separable subspace of X such that $[D]_{|X_0}$ is an isometry and let R_1 denote the restriction map to X_0. Define $\Psi : l_\infty \to L(X,Y)$ by

$$\Psi(b)(x) = \sum_j b_j T^*(g^*_{i_j})(x) y_j$$

for $b \in l_\infty$ and $x \in X$. Let $J : [y_j] \to l_\infty$ be a linear isometry and let $A : Y \to l_\infty$ be a continuous linear extension of J.

Suppose that $W(X,Y) \overset{c}{\hookrightarrow} L(X,Y)$, and let $P : L(X,Y) \to W(X,Y)$ be a projection. Consider the operators $R_1AP\Psi : l_\infty \to W(X_0, l_\infty)$ and $R_1A\Psi : l_\infty \to L(X_0, l_\infty)$. Since $\Psi(e_j) = (S^*(g^*_{i_j})) \otimes y_j$, $\Psi(e_j)$ is a rank one operator, thus compact and weakly compact. Then

$$R_1AP\Psi(e_j) = R_1A\Psi(e_j) \text{ for each } j.$$

An application of Proposition 5 of Kalton [9] produces an infinite subset M of $\mathbb{N}$ such that

$$R_1AP\Psi(b) = R_1A\Psi(b), b \in l_\infty(M).$$

Therefore $R_1A\Psi(\chi_M)$ is weakly compact. But $\Psi(\chi_M)^*(y^*_j) = S^*(g^*_{i_j}), j \in M$, and $\{S^*(g^*_{i_j}) : j \in M\}$ is not relatively weakly compact. Therefore $\Psi(\chi_M)_{|X_0}$ is not weakly compact. However this is a contradiction since $A_{|[y_n]}$ is an isometry and $R_1A\Psi(\chi_M)$ is weakly compact.

(ii) The proof is essentially the same as the proof of (i), replacing "relatively weakly compact" with "relatively compact". □

Corollary 2 ([5], Theorems 2 and 3; [1], Theorem 4)**.** *If $c_0 \hookrightarrow Y$ and X^* contains a w^*-null sequence (x^*_n) which is not w-null, then $W(X,Y)$ is not complemented in $L(X,Y)$.*

Proof. Let $G = c_0$ and $R : c_0 \to Y$ be an embedding. Assume without loss of generality that (x^*_n) has no weakly convergent subsequence. Define $S : X \to c_0$ by $S(x) = (x^*_n(x))$. Then $S^*(e^*_n) = x^*_n$ for each n. Apply Theorem 1 now. □

Corollary 3 ([5], Corollary 4)**.** *Assume that X contains a complemented copy of c_0 and $c_0 \hookrightarrow Y$. Then $W(X,Y)$ is not complemented in $L(X,Y)$.*

Proof. If $c_0 \overset{c}{\hookrightarrow} X$ and $W(X,Y) \overset{c}{\hookrightarrow} L(X,Y)$, then $W(c_0,Y) \overset{c}{\hookrightarrow} L(c_0,Y)$. An application of Corollary 2 concludes the proof. □

Corollary 4 ([7], Corollary 4)**.** *If $c_0 \hookrightarrow Y$ and X is an infinite-dimensional Banach space, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

Proof. Let (x^*_n) be a w^*-null sequence of norm one elements in X^*. Define $S : X \to c_0$ by $S(x) = (x^*_n(x))$. Clearly $(S^*(e^*_n)) = (x^*_n)$ is not relatively compact. Let $G = c_0$ and $R : c_0 \to Y$ be an embedding. Theorem 1 now gives the conclusion. □

Corollary 5 ([5], Theorem 5)**.** *Assume that $L(X,l_1) \neq K(X,l_1)$ and that Y contains a copy of l_1. Then $W(X,Y)$ is not complemented in $L(X,Y)$.*

Proof. Since l_1 has the Schur property, $W(X,l_1) = K(X,l_1)$. Let $T : X \to l_1$ be an operator that is not weakly compact. Note that $T^*_{|c_0}$ is not weakly compact, otherwise $T^{**} : X^{**} \to l_1$ would be weakly compact. But $T^{**}_{|X} = T^{**}\eta = T$ is not weakly compact. Let $G = l_1$ and $R : l_1 \to Y$ be an embedding. By Theorem 1, $W(X,Y)$ is not complemented in $L(X,Y)$. □

Corollary 6 ([9], Lemma 3)**.** *If X contains a complemented copy of l_1 and Y is infinite-dimensional, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

Proof. Let P be a projection from X to l_1. Take $G = l_1$. Since P restricted to l_1 is the identity, P is not compact. We claim that $P^*_{|c_0}$ is not compact. If it were, its adjoint $P^{**} : X^{**} \to l_1$ would be compact. But then $P^{**}\eta = P$ would be compact, a contradiction.

Therefore $(P^*(e_n))$ is not relatively compact. Let (y_n) be a normalized basic sequence in Y. Define $R : l_1 \to Y$ by $R(b_n) = \sum b_n y_n$. Then $(R(e_n^*)) = (y_n)$ is basic and normalized. Apply Theorem 1 now. □

We conclude by presenting a new and very short proof of the main result in Emmanuele [4].

Corollary 7 ([4], Theorem 2)**.** *If c_0 embeds in $K(X,Y)$, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

Proof. An application of Corollaries 3 and 6 shows that we may assume that c_0 embeds in neither X^* nor Y. Thus, by Theorem 4 of Kalton [9], l_∞ does not embed in $K(X,Y)$. If (T_i) is a copy of the unit vector basis of c_0 in $K(X,Y)$, the Diestel-Faires Theorem ([3]) produces a subsequence (T_{i_j}) so that

$$\sum_j T_{i_j} \qquad \text{(strong operator topology)}$$

produces a noncompact operator. Apply the main result of Feder [7]. □

References

[1] E. Bator, P. Lewis, Complemented spaces of operators, Bull. Polish Acad. Sci. Math. 50 (no. 4) (2002), 413-416. MR1948406 (2005b:46043)

[2] J. Diestel, Sequences and Series in Banach Spaces, Springer (1984). MR0737004 (85i:46020)

[3] J. Diestel and J.J. Uhl, Jr., Vector Measures, American Mathematical Society (1977). MR0453964 (56:12216)

[4] G. Emmanuele, A remark on the containment of c_0 in spaces of compact operators, Math. Proc. Camb. Phil. Soc. 111 (1992), 331-335. MR1142753 (92j:47078)

[5] G. Emmanuele, Remarks on the uncomplemented subspace $W(E,F)$, J. Funct. Anal. 99 (1991), 125-130. MR1120917 (92f:46019)

[6] G. Emmanuele, K. John, Uncomplementability of spaces of compact operators in larger spaces of operators, Czechoslovak Math. J. 47 (1997), 19-31. MR1435603 (98e:46017)

[7] M. Feder, On the non-existance of a projection onto the space of compact Operators, Canad. Math. Bull. 25 (1982), 78-81. MR0657655 (83f:47032)

[8] K. John, On the uncomplemented subspace $K(X,Y)$, Czechoslovak Math. J. 42 (1992), 167-173. MR1152178 (93b:47085)

[9] N. Kalton, Spaces of compact operators, Math. Ann. 208 (1974), 267-278. MR0341154 (49:5904)

Department of Mathematics, University of North Texas, Box 311430, Denton, Texas 76203-1430

E-mail address: ig0007@unt.edu

Current address: Department of Mathematics, University of Wisconsin–River Falls, 204 D North Hall, River Falls, Wisconsin 54022-5001