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UNIFORMLY BOUNDED LIMIT OF FRACTIONAL HOMOMORPHISMS

PEDRO J. MIANA

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ABSTRACT. We show that a bounded homomorphism $T : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ is equivalent to a uniformly bounded family of fractional homomorphisms $T_\alpha : AC^{(\alpha)}_\omega(\mathbb{R}^+) \to \mathcal{A}$ for any $\alpha > 0$. We add this characterization to the Widder-Arendt-Kisyński theorem and relate it to α-times integrated semigroups.

1. INTRODUCTION

Let $\mathcal{A}$ be a Banach algebra (with or without identity). For $\Omega \subset \mathbb{C}$, a family $(r_\lambda)_{\lambda\in\Omega}$ of elements of $\mathcal{A}$ is called a *pseudo-resolvent* if the equation $r_\lambda - r_\mu = (\mu-\lambda)r_\lambda r_\mu$ holds for $\lambda, \mu \in \Omega$. An example of a pseudo-resolvent is the following. Let $\mathbb{R}, \mathbb{R}^+$ and $\mathbb{C}$ be the sets of real, positive real and complex numbers, respectively. For each $\lambda \in \mathbb{R}$, denote by ϵ_λ the function

$$\epsilon_\lambda(t) = e^{\lambda t}, \qquad t \in \mathbb{R}^+.$$

Take $\omega \in \mathbb{R}^+ \cup \{0\}$ and let $L^1_\omega(\mathbb{R}^+)$ be the usual Banach algebra with norm given by

$$\|f\|_w := \int_0^\infty |f(t)|e^{\omega t}dt < +\infty,$$

and the convolution $f * g(t) := \int_0^t f(t-s)g(s)ds$, with $t \geq 0$, as its product. Then $(\epsilon_{-\lambda})_{\lambda\in(\omega,+\infty)}$ is a pseudo-resolvent in $L^1_\omega(\mathbb{R}^+)$ and verifies

$$\|\underbrace{\epsilon_{-\lambda} * \cdots * \epsilon_{-\lambda}}_{n \text{ times}}\|_\omega = \frac{1}{(\lambda-\omega)^n}, \qquad \lambda \in (\omega, +\infty),\ n \in \mathbb{N}.$$

Moreover, the set $(\epsilon_{-\lambda})_{\lambda\in(\omega,+\infty)}$ is linearly dense in $L^1_\omega(\mathbb{R}^+)$.

The next result shows the equivalence between a homomorphism $T : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ and a class of pseudo-resolvents; in other words, the family $(\epsilon_{-\lambda})_{\lambda\in(\omega,+\infty)}$ is universal for this class of pseudo-resolvents. We present here an early version; see [8] and [3] which is included in [9, Theorem 5.1]. Recently, this result has been called the Widder-Arendt-Kisyński theorem; see for example [5, Theorem 1.1].

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Theorem 1.1 ([8]). *Let $\mathcal{A}$ be a Banach algebra, let $\omega \geq 0$, let $(r_\lambda)_{\lambda \in (\omega, +\infty)}$ be a pseudo-resolvent in $\mathcal{A}$ and let*

$$M = \sup\{(\lambda - \omega)^n \|r_\lambda^n\| \ : \ n \in \mathbb{N}, \ \lambda \in (\omega, +\infty)\}.$$

Then the following conditions are equivalent:

(i) $M < +\infty$.

(ii) *There exists a continuous homomorphism $T : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ such that $T(\epsilon_{-\lambda}) = r_\lambda$ for each $\lambda \in (\omega, +\infty)$.*

Furthermore, if a continuous homomorphism $T : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ satisfying $T(\epsilon_{-\lambda}) = r_\lambda$ for each $\lambda \in (\omega, +\infty)$ exists, then it is unique and $\|T\| = M$.

This theorem has many interesting applications, as W. Chojnacki pointed out, and it is equivalent to Hille-Yosida theorem; see [3] and [4]. Another different proof of this result is given in [2], where the author uses Yosida approximation in a more general setting. A generalization of [9, Theorem 5.1] is presented in [5] improving Kisyński's arguments.

In this paper, we arrive at Theorem 1.1 with an original point of wiev. We show that a bounded homomorphism $T : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ is equivalent to a uniformly bounded family of homomorphisms $T_\alpha : AC^{(\alpha)}_\omega(\mathbb{R}^+) \to \mathcal{A}$ for any $\alpha > 0$; see Theorem 3.1. Fractional Banach algebras $AC^{(\alpha)}_\omega(\mathbb{R}^+)$ are included in $L^1_\omega(\mathbb{R}^+)$ and defined in the second section. The proof of this main result is based on Theorem 1.1.

A family of fractional algebras similar to $AC^{(\alpha)}_\omega(\mathbb{R}^+)$ was introduced in [10], working with α-times integrated semigroups. Integrated semigroups are considered in the fourth section to complete a different approach to the Widder-Arendt-Kisyński theoreom in which Lipschitz and Hölder continuous functions appear; see [1, Theorem 1.1], [6, Theorem 2.6], [9, Corollary 7.2], or [7, Theorem 3.2]. Lipschitz functions are considered in the integrated version of the Widder's characterization of Laplace transforms on arbitrary Banach spaces.

Actually, α-times integrated semigroups (in the case $\alpha = 1$) were also used to prove Theorem 1.1 with $\omega = 0$ in [8, Theorem 4.2]. However, we present here a different technique since we consider α-times integrated semigroups for any $\alpha > 0$ and the limit when α goes to zero.

2. Fractional Banach algebras

Suppose $f \in \mathcal{D}_+$, where $C_c^\infty[0, \infty) \equiv \mathcal{D}_+$ is the set of infinitely differentiable functions with compact support on $[0, \infty)$. The *Weyl fractional integral* of f of order $\alpha > 0$, $W^{-\alpha} f$, is defined by

$$W^{-\alpha} f(u) = \frac{1}{\Gamma(\alpha)} \int_u^\infty (t - u)^{\alpha - 1} f(t) dt, \qquad u \geq 0,$$

with $\alpha > 0$; see for example [11]. This operator $W^{-\alpha} : \mathcal{D}_+ \to \mathcal{D}_+$ is one-to-one and onto. Its inverse, W^α, is called a *Weyl fractional derivate* (of f) of order α and $W^\alpha = (-1)^n \frac{d^n}{dt^n} W^{-(n-\alpha)}$ with $n \in \mathbb{N}$ and $n > \alpha$.

Theorem 2.1. *Suppose $\alpha > 0$ and $\omega \geq 0$. If $f \in \mathcal{D}_+$, then the formula*

$$\|f\|_{(\alpha, \omega)} := \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty t^\alpha e^{\omega t} |W^\alpha f(t)| dt$$

defines a norm in $\mathcal{D}_+$ such that

$$\|f*g\|_{(\alpha,\omega)} \le (2^{\alpha+1}-1)\|f\|_{(\alpha,\omega)}\|g\|_{(\alpha,\omega)},$$

with $f, g \in \mathcal{D}_+$. We denote by $AC_\omega^{(\alpha)}(\mathbb{R}^+)$ the Banach algebra obtained by the completion of $\mathcal{D}_+$ in this norm.

Proof. The proof is similar to [10, Theorem 1.2]. □

Remark 2.2. For $\alpha = 0$, it is clear that $AC_\omega^{(0)}(\mathbb{R}^+) = L_\omega^1(\mathbb{R}^+)$ and $AC_\omega^{(\beta)}(\mathbb{R}^+) \hookrightarrow AC_\omega^{(\alpha)}(\mathbb{R}^+)$ holds with $0 \le \alpha \le \beta$.

It is easy to check that $W^\alpha(\epsilon_{-\lambda}) = \lambda^\alpha \epsilon_{-\lambda}$ with $\alpha \in \mathbb{R}$ and $\lambda > 0$. Functions $(\epsilon_{-\lambda})_{\lambda\in(\omega,+\infty)}$ belong to $AC_\omega^{(\alpha)}(\mathbb{R}^+)$ with $\alpha \ge 0$ and

$$\|\epsilon_{-\lambda}\|_{(\alpha,\omega)} = \frac{\lambda^\alpha}{(\lambda-\omega)^{\alpha+1}}, \qquad \lambda \in (\omega,+\infty). \tag{2.1}$$

Note that $\|\epsilon_{-\lambda}\|_{(\alpha,\omega)} \to \|\epsilon_{-\lambda}\|_\omega$ when $\alpha \to 0^+$ and $\lambda \in (\omega,+\infty)$. Moreover, $(\epsilon_{-\lambda})_{\lambda\in(\omega,\infty)}$ is a pseudo-resolvent in $AC_\omega^{(\alpha)}(\mathbb{R}^+)$ and $(n\epsilon_{-n})_{n>\omega}$ is a bounded approximate identity in $AC_\omega^{(\alpha)}(\mathbb{R}^+)$ for any $\alpha \ge 0$.

Proposition 2.3. *The set $(\epsilon_{-\lambda})_{\lambda\in(\omega,+\infty)}$ is linearly dense in $AC_\omega^{(\alpha)}(\mathbb{R}^+)$ with $\alpha, \omega \ge 0$.*

Proof. It is a consequence of $W^\alpha(\epsilon_{-\lambda}) = \lambda^\alpha\epsilon_{-\lambda}$ with $\alpha \ge 0$, $\lambda > \omega$ and [5, Proposition 2.2]. □

3. Main result

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra.*

(i) *If there exists a continuous homomorphism $T : L_\omega^1(\mathbb{R}^+) \to \mathcal{A}$, then there exists a family of continuous homomorphisms $T_\alpha : AC_\omega^{(\alpha)}(\mathbb{R}^+) \to \mathcal{A}$ for each $\alpha > 0$ such that $T_\alpha(\epsilon_{-\lambda}) = T(\epsilon_{-\lambda})$ for each $\lambda \in (\omega,+\infty)$ and $\|T_\alpha\| \le \|T\|$ for any $\alpha > 0$.*

(ii) *Conversely, if for each $\alpha > 0$ there exist a countinuous homomorphisms $T_\alpha : AC_\omega^{(\alpha)}(\mathbb{R}^+) \to \mathcal{A}$ such that $T_\alpha(\epsilon_{-\lambda})$ does not depend on α for each $\lambda \in (\omega,+\infty)$ and $\limsup_{\alpha\to0^+}\|T_\alpha\| < +\infty$, then there exists a unique continuous homomorphism $T : L_\omega^1(\mathbb{R}^+) \to \mathcal{A}$ such that $T(\epsilon_{-\lambda}) = T_\alpha(\epsilon_{-\lambda})$ for each $\lambda \in (\omega,+\infty)$ and $\|T\| \le \limsup_{\alpha\to0^+}\|T_\alpha\|$.*

Proof. The first part follows from Remark 2.2. To prove (ii), we use Theorem 1.1. We define $r_\lambda := T_\alpha(\epsilon_\lambda)$ for each $\lambda > \omega$. The family $(r_\lambda)_{\lambda\in(\omega,\infty)}$ is well defined and is a pseudo-resolvent in $\mathcal{A}$. For $n \in \mathbb{N}$ and $\lambda \in (\omega,+\infty)$,

$$\|r_\lambda^n\| \le \|T_\alpha\| \, \|\underbrace{\epsilon_\lambda * \cdots * \epsilon_\lambda}_{n \text{ times}}\|_{(\alpha,\omega)}.$$

We apply Theorem 2.1 and get that

$$\|\underbrace{\epsilon_\lambda * \cdots * \epsilon_\lambda}_{n \text{ times}}\|_{(\alpha,\omega)} \le (2^{\alpha+1}-1)^{n-1}\|\epsilon_\lambda\|_{(\alpha,\omega)}^n = (2^{\alpha+1}-1)^{n-1}\frac{\lambda^{n\alpha}}{(\lambda-\omega)^{n(\alpha+1)}},$$

where we use (1). Now we obtain

$$\|r_\lambda^n\| \le \lim_{\alpha\to0}\left((2^{\alpha+1}-1)^{n-1}\frac{\lambda^{n\alpha}}{(\lambda-\omega)^{n(\alpha+1)}}\right)\limsup_{\alpha\to0^+}\|T_\alpha\|,$$

and we get

$$\sup\{(\lambda-\omega)^n\|r_\lambda^n\| \ : \ n\in\mathbb{N},\ \lambda\in(\omega,+\infty)\}\le \limsup_{\alpha\to 0^+}\|T_\alpha\|<+\infty.$$

By Theorem 1.1, there exists a unique continuous homomorphism $T: L^1_\omega(\mathbb{R}^+)\to\mathcal{A}$ such that $T(\epsilon_{-\lambda})=r_\lambda=T_\alpha(\epsilon_{-\lambda})$ for each $\lambda\in(\omega,+\infty)$ and $\|T\|\le\limsup_{\alpha\to 0^+}\|T_\alpha\|$. □

Remark 3.2. Since the set $(\epsilon_{-\lambda})_{\lambda\in(\omega,+\infty)}$ is linearly dense in $AC^{(\alpha)}_\omega(\mathbb{R}^+)$ (Proposition 2.3), $T_\alpha(f)=T(f)$ for any $f\in AC^{(\alpha)}_\omega(\mathbb{R}^+)$ and $\alpha>0$. Also, we may combine (i) and (ii) to get

$$\|T\|=\limsup_{\alpha\to 0^+}\|T_\alpha\|=\sup_{\alpha>0}\|T_\alpha\|,$$

where we use $\|T_\beta\|\le\|T_\alpha\|$ with $\beta\ge\alpha>0$.

4. Connection with integrated semigroups

Widder's characterization of Laplace transforms of real-valued bounded functions which states that for $r\in C^{(\infty)}(0,\infty)$ there exists $f\in L^\infty(0,\infty)$ such that

$$r(\lambda)=\int_0^\infty e^{-\lambda t}f(t)dt, \qquad \lambda>0,$$

if and only if

$$\sup\{\lambda^{n+1}\frac{|r^{(n)}(\lambda)|}{n!}:\ \lambda>0, n\in\mathbb{N}\}<+\infty,$$

(see [12]) was considered by W. Arendt on arbitrary Banach space X. An integrated version of this result is given in terms of a function $F:[0,\infty)\to X$ such that $F(0)=0$, $\|F(t+h)-F(t)\|\le Mh$, $(t,h\ge 0)$ and

$$r(\lambda)=\lambda\int_0^\infty e^{-\lambda t}F(t)dt, \qquad \lambda>0;$$

see [1, Theorem 1.1]. This result was improved by M. Hieber who showed that for any $\alpha\in(0,1]$ there exists a function $F_\alpha:[0,\infty)\to X$ such that $F_\alpha(0)=0$, $\|F_\alpha(t+h)-F_\alpha(t)\|\le Mh^\alpha$, $(t,h\ge 0)$ and

$$r(\lambda)=\lambda^\alpha\int_0^\infty e^{-\lambda t}F(t)dt, \qquad \lambda>0;$$

see [7, Theorem 3.2]. If r is a pseudo-resolvent in a Banach algebra $\mathcal{A}$, then the function F_α is an α-times integrated semigroup in $\mathcal{A}$. We now give here the definition of α-times integrated semigroups in a Banach algebra $\mathcal{A}$, since they are usually considered in the setting of linear and bounded operators on a Banach space X; see for example [7].

Definition 4.1. For any $\alpha>0$, an α-times integrated semigroup $s_\alpha(\cdot):[0,\infty)\to\mathcal{A}$ is a continuous mapping with $s_\alpha(0)=0$ and

$$s_\alpha(t)s_\alpha(s)=\frac{1}{\Gamma(\alpha)}(\int_t^{t+s}(t+s-r)^{\alpha-1}s_\alpha(r)dr-\int_0^s(t+s-r)^{\alpha-1}s_\alpha(r)dr), \tag{4.1}$$

with $t,s\ge 0$.

If $\|s_\alpha(t)\| \le Ce^{\omega t}$ for $C, t \ge 0$ and $\omega \in \mathbb{R}$, then condition (2) is equivalent (by Laplace transform) to $r_\lambda := \lambda^\alpha \int_0^\infty e^{-\lambda t} s_\alpha(t)dt$, with $\lambda > \omega$ a pseudo-resolvent in $\mathcal{A}$; see the analogous proof in Banach space in [7]. If $(s_\alpha(t))_{t\ge 0}$ is an α-times integrated semigroup in $\mathcal{A}$, then $(s_\nu(t))_{t\ge 0}$ with

$$(4.2) \qquad s_\nu(t) := \frac{1}{\Gamma(\nu-\alpha)} \int_0^t (t-s)^{\nu-\alpha-1} s_\alpha(s)ds, \qquad t \ge 0,$$

is a ν-times integrated semigroup with $\nu > \alpha$ in $\mathcal{A}$. The set of Bochner-Riesz functions $(R_t^\nu)_{t\ge 0}$, i.e,

$$R_t^\nu(s) := \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} \chi_{(0,t)}(s), \qquad t \ge 0,$$

is an example of a ν-times integrated semigroup in $AC_\omega^{(\alpha)}(\mathbb{R}^+)$ for $\nu > \alpha$, $\omega \ge 0$ and

$$\|R_t^\nu\|_{(\alpha,\omega)} \le \frac{t^\nu e^{t\omega}}{\Gamma(\nu+1)}, \qquad t \ge 0.$$

Now, we may add in the Widder-Arendt-Kisyński theoreom more conditions related to fractional homomorphisms and α-times integrated semigroups.

Theorem 4.2. *Let $\mathcal{A}$ be a Banach algebra, let ω be a non-negative number, let $(r_\lambda)_{\lambda\in(\omega,+\infty)}$ be a pseudo-resolvent in $\mathcal{A}$ and let*

$$M = \sup\{(\lambda-\omega)^n \|r_\lambda^n\| \ : \ n \in \mathbb{N}, \ \lambda \in (\omega, +\infty)\}.$$

Then the following conditions are equivalent:

(i) $M < +\infty$.

(ii) *There exists a continuous homomorphism $T : L_\omega^1(\mathbb{R}^+) \to \mathcal{A}$ such that $T(\epsilon_{-\lambda}) = r_\lambda$ for each $\lambda \in (\omega, +\infty)$.*

(iii) *For any $\alpha > 0$, there exists an α-times integrated semigroup $(s_\alpha(t))_{t\ge 0} \subset \mathcal{A}$ such that $\|s_\alpha(t)\| \le \frac{C}{\Gamma(\alpha+1)} t^\alpha e^{\omega t}$ with $C, t \ge 0$ and $r_\lambda = \lambda^\alpha \int_0^\infty e^{-\lambda t} s_\alpha(t)dt$, with $\lambda > \omega$.*

(iv) *For any $\alpha > 0$, there exists a continuous homomorphism $T_\alpha : AC_\omega^{(\alpha)}(\mathbb{R}^+) \to \mathcal{A}$ such that $T_\alpha(\epsilon_{-\lambda}) = r_\lambda$ for each $\lambda \in (\omega, +\infty)$ and $\sup_{\alpha>0} \|T_\alpha\| < +\infty$.*

Furthermore, if a continuous homomorphism $T : L_\omega^1(\mathbb{R}^+) \to \mathcal{A}$ such that $T(\epsilon_{-\lambda}) = r_\lambda$ for each $\lambda \in (\omega, +\infty)$ exists, then it is unique, $T(f) = T_\alpha(f)$ for $f \in AC_\omega^{(\alpha)}(\mathbb{R}^+)$ and

$$M = \|T\| = \sup_{\alpha>0} \|T_\alpha\| = \inf\{C \ : \|s_\alpha(t)\| \le C \frac{t^\alpha e^{\omega t}}{\Gamma(\alpha+1)}, \ t \ge 0\}.$$

Proof. (ii) $\Rightarrow$ (iii) Take $(R_t^\alpha)_{t>0}$ to be the family of Bochner-Riesz functions of order $\alpha > 0$. Then $(R_t^\alpha)_{t\ge 0} \subset L_\omega^1(\mathbb{R}^+)$ and we define $s_\alpha(t) := T(R_t^\alpha)$, for $t > 0$ and $s_\alpha(0) := 0$. Since $(R_t^\alpha)_{t>0}$ is an α-times integrated semigroup in $L_\omega^1(\mathbb{R}^+)$, $(s_\alpha(t))_{t\ge 0}$ is an α-times integrated semigroup in $\mathcal{A}$ and

$$\|s_\alpha(t)\| \le \|T\| \ \|R_t^\alpha\|_\omega \le \|T\| \frac{t^\alpha}{\Gamma(\alpha+1)} e^{\omega t}.$$

By continuity of T, we have for $\lambda > \omega$,

$$\lambda^\alpha \int_0^\infty e^{-\lambda t} s_\alpha(t)dt = \lambda^\alpha T\left(\int_0^\infty e^{-\lambda t} R_t^\alpha dt\right) = \lambda^\alpha T(W^{-\alpha}\epsilon_\lambda) = T(\epsilon_\lambda) = r_\lambda.$$

(iii) ⇒ (iv) We define $T_\alpha : AC^{(\alpha)}_\omega(\mathbb{R}^+) \to \mathcal{A}$ by

$$T_\alpha(f) := \int_0^\infty W^\alpha f(t) s_\alpha(t)dt, \qquad f \in \mathcal{D}_+.$$

Following the same arguments as in [10, Theorem 3.1], we prove that T_α are continuous homomorphisms, $\|T_\alpha\| \le C$ for any $\alpha > 0$ and $\sup_{\alpha>0} \|T_\alpha\| \le C < +\infty$. Since $\epsilon_\lambda \in AC^{(\alpha)}_\omega(\mathbb{R}^+)$ for $\lambda > \omega$ and $W^\alpha(\epsilon_\lambda) = \lambda^\alpha \epsilon_\lambda$, we have $T_\alpha(\epsilon_\lambda) = r_\lambda$ for $\lambda > \omega$.

(iv) ⇒ (ii) It is proved by Theorem 3.1(ii).

If a continuous homomorphism $T : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ satisfying $T(\epsilon_{-\lambda}) = r_\lambda$ for each $\lambda \in (\omega, +\infty)$ exists, then $AC^{(\alpha)}_\omega(\mathbb{R}^+) \hookrightarrow L^1_\omega(\mathbb{R}^+)$ and $T(f) = T_\alpha(f)$ for $f \in AC^{(\alpha)}_\omega(\mathbb{R}^+)$. Moreover, if we collect in the proof the following inequalities,

$$\|T\| = M = \sup_{\alpha>0} \|T_\alpha\| \le \inf\{C \ : \|s_\alpha(t)\| \le C\frac{t^\alpha e^{\omega t}}{\Gamma(\alpha+1)},\ t \ge 0\} \le \|T\|,$$

we finally get the equality. □

Remark 4.3. If $0 < \alpha \le 1$, we use $s_\alpha(t) = T(R^\alpha_t)$ $(t > 0)$ to obtain

$$\|s_\alpha(t+h) - s_\alpha(t)\| \ \le 2\|T\| \frac{h^\alpha}{\Gamma(\alpha+1)} e^{\omega(t+h)},$$

with $t, h > 0$; see also [7, Corollary 3.3].

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Departamento de Matemáticas, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain
E-mail address: pjmiana@unizar.es