UNIFORMLY BOUNDED LIMIT OF FRACTIONAL HOMOMORPHISMS

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Abstract. We show that a bounded homomorphism \( T : L_1^\omega(\mathbb{R}^+) \to \mathcal{A} \) is equivalent to a uniformly bounded family of fractional homomorphisms \( T_\alpha : AC_\omega^\alpha(\mathbb{R}^+) \to \mathcal{A} \) for any \( \alpha > 0 \). We add this characterization to the Widder-Arendt-Kisyński theorem and relate it to \( \alpha \)-times integrated semigroups.

1. INTRODUCTION

Let \( \mathcal{A} \) be a Banach algebra (with or without identity). For \( \Omega \subset \mathbb{C} \), a family \((r_\lambda)_{\lambda \in \Omega}\) of elements of \( \mathcal{A} \) is called a pseudo-resolvent if the equation \( r_\lambda - r_\mu = (\mu - \lambda)r_\lambda r_\mu \) holds for \( \lambda, \mu \in \Omega \). An example of a pseudo-resolvent is the following. Let \( \mathbb{R}, \mathbb{R}^+, \text{and } \mathbb{C} \) be the sets of real, positive real and complex numbers, respectively. For each \( \lambda \in \mathbb{R} \), denote by \( \epsilon_\lambda \) the function \( \epsilon_\lambda(t) = e^{\lambda t}, t \in \mathbb{R}^+ \).

Take \( \omega \in \mathbb{R}^+ \cup \{0\} \) and let \( L_1^\omega(\mathbb{R}^+) \) be the usual Banach algebra with norm given by

\[
\|f\|_w := \int_0^\infty |f(t)|e^{\omega t}dt < +\infty,
\]

and the convolution \( f \ast g(t) := \int_0^t f(t-s)g(s)ds \), with \( t \geq 0 \), as its product. Then \((\epsilon_\lambda)_{\lambda \in (\omega, +\infty)}\) is a pseudo-resolvent in \( L_1^\omega(\mathbb{R}^+) \) and verifies

\[
\|\epsilon_{\lambda_1} \ast \cdots \ast \epsilon_{\lambda_n}\|_\omega = \frac{1}{(\lambda - \omega)^n}, \quad \lambda \in (\omega, +\infty), \ n \in \mathbb{N}.
\]

Moreover, the set \((\epsilon_{\lambda})_{\lambda \in (\omega, +\infty)}\) is linearly dense in \( L_1^\omega(\mathbb{R}^+) \).

The next result shows the equivalence between a homomorphism \( T : L_1^\omega(\mathbb{R}^+) \to \mathcal{A} \) and a class of pseudo-resolvents; in other words, the family \((\epsilon_{\lambda})_{\lambda \in (\omega, +\infty)}\) is universal for this class of pseudo-resolvents. We present here an early version; see \([8]\) and \([3]\) which is included in \([9, \text{Theorem 5.1}]\). Recently, this result has been called the Widder-Arendt-Kisyński theorem; see for example \([5, \text{Theorem 1.1}]\).
Theorem 1.1 (8). Let $A$ be a Banach algebra, let $\omega \geq 0$, let $(r_\lambda)_{\lambda \in (\omega, +\infty)}$ be a pseudo-resolvent in $A$ and let

$$M = \sup \{ (\lambda - \omega)^n \| r_\lambda^n \| : n \in \mathbb{N}, \lambda \in (\omega, +\infty) \}.$$  

Then the following conditions are equivalent:

(i) $M < +\infty$,

(ii) There exists a continuous homomorphism $T : L^1_\omega(\mathbb{R}^+) \to A$ such that $T(\epsilon - \lambda) = r_\lambda$ for each $\lambda \in (\omega, +\infty)$.

Furthermore, if a continuous homomorphism $T : L^1_\omega(\mathbb{R}^+) \to A$ satisfying $T(\epsilon - \lambda) = r_\lambda$ for each $\lambda \in (\omega, +\infty)$ exists, then it is unique and $\| T \| = M$.

This theorem has many interesting applications, as W. Chojnacki pointed out, and it is equivalent to Hille-Yosida theorem; see [3] and [4]. Another different proof of this result is given in [2], where the author uses Yosida approximation in a more general setting. A generalization of [9, Theorem 5.1] is presented in [5] improving Kisyński’s arguments.

In this paper, we arrive at Theorem 1.1 with an original point of view. We suppose $\omega > 0$; see for example [11]. This operator $W^{-\alpha} : D_+ \to D_+$ is one-to-one and onto. Its inverse, $W^\alpha$, is called a Weyl fractional derivate (of $f$) of order $\alpha$ and $W^{\alpha} = (-1)^n \frac{d^n}{dx^n} W^{-\alpha} W^{\alpha}$ with $n \in \mathbb{N}$ and $n > \alpha$.

Theorem 2.1. Suppose $\alpha > 0$ and $\omega \geq 0$. If $f \in D_+$, then the formula

$$\| f \|_{(\alpha, \omega)} := \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty t^\alpha e^{\omega t} |W^\alpha f(t)| dt$$
defines a norm in $D_+$ such that
\[ \|f \ast g\|_{(\alpha, \omega)} \leq (2^{\alpha+1} - 1)\|f\|_{(\alpha, \omega)}\|g\|_{(\alpha, \omega)}, \]
with $f, g \in D_+$. We denote by $AC_{\omega}^{(\alpha)}(\mathbb{R}^+) \to AC_{\omega}^{(\alpha)}(\mathbb{R}^+)$ holds with $0 \leq \alpha \leq \beta$.

**Remark 2.2.** For $\alpha = 0$, it is clear that $AC_{\omega}^{(\alpha = 0)}(\mathbb{R}^+) = L^1_\alpha(\mathbb{R}^+)$ and $AC_{\omega}^{(\alpha = 1)}(\mathbb{R}^+) \to AC_{\omega}^{(\alpha = 0)}(\mathbb{R}^+)$ holds with $\alpha \geq 0$ and (\alpha, \omega) \} belongs to $AC_{\omega}^{(\alpha)}(\mathbb{R}^+)$ with $\alpha \geq 0$ and
\[ (2.1) \quad \|\epsilon-\lambda\|_{(\alpha, \omega)} = \frac{\lambda^\alpha}{(\lambda - \omega)^{\alpha+1}}, \quad \lambda \in (\omega, +\infty). \]

Note that $\|\epsilon-\lambda\|_{(\alpha, \omega)} \to \|\epsilon-\lambda\|_\omega$ when $\alpha \to 0^+$ and $\lambda \in (\omega, +\infty)$. Moreover, $(\epsilon-\lambda)_{\lambda \in (\omega, +\infty)}$ is a pseudo-resolvent in $AC_{\omega}^{(\alpha)}(\mathbb{R}^+)$ and $(n\epsilon-n)_{n \in \mathbb{N}}$ is a bounded approximate identity in $AC_{\omega}^{(\alpha)}(\mathbb{R}^+)$ for any $\alpha \geq 0$.

**Proposition 2.3.** The set $(\epsilon-\lambda)_{\lambda \in (\omega, +\infty)}$ is linearly dense in $AC_{\omega}^{(\alpha)}(\mathbb{R}^+)$ with $\alpha, \omega \geq 0$.

**Proof.** It is a consequence of $W^\alpha(\epsilon-\lambda) = \lambda^\alpha \epsilon-\lambda$ with $\alpha \geq 0$, $\lambda > \omega$ and [1, Proposition 2.2].

3. Main Result

**Theorem 3.1.** Let $\mathcal{A}$ be a Banach algebra.

(i) If there exists a continuous homomorphism $T : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$, then there exists a family of continuous homomorphisms $T_\alpha : AC_{\omega}^{(\alpha)}(\mathbb{R}^+) \to \mathcal{A}$ for each $\alpha > 0$ such that $T_\alpha(\epsilon-\lambda) = T(\epsilon-\lambda)$ for each $\lambda \in (\omega, +\infty)$ and $\|T_\alpha\| \leq \|T\|$ for any $\alpha > 0$.

(ii) Conversely, if for each $\alpha > 0$ there exist a continuous homomorphisms $T_\alpha : AC_{\omega}^{(\alpha)}(\mathbb{R}^+) \to \mathcal{A}$ such that $T_\alpha(\epsilon-\lambda)$ does not depend on $\alpha$ for each $\lambda \in (\omega, +\infty)$ and $\limsup_{\alpha \to 0^+} \|T_\alpha\| < +\infty$, then there exists a unique continuous homomorphism $T : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ such that $T(\epsilon-\lambda) = T_\alpha(\epsilon-\lambda)$ for each $\lambda \in (\omega, +\infty)$ and $\|T\| \leq \limsup_{\alpha \to 0^+} \|T_\alpha\|$.

**Proof.** The first part follows from Remark 2.2. To prove (ii), we use Theorem 1.1. We define $r_\lambda := T_\alpha(\epsilon-\lambda)$ for each $\lambda > \omega$. The family $(r_\lambda)_{\lambda \in (\omega, +\infty)}$ is well defined and is a pseudo-resolvent in $\mathcal{A}$. For $n \in \mathbb{N}$ and $\lambda \in (\omega, +\infty)$,
\[ \|r_\lambda^n\| \leq \|T_\alpha\| \|\epsilon_{\lambda} * \cdots \epsilon_{\lambda}\|_{(\alpha, \omega)}, \]
where $n$ times

We apply Theorem 2.1 and get that
\[ \|\epsilon_{\lambda} * \cdots \epsilon_{\lambda}\|_{(\alpha, \omega)} \leq (2^{\alpha+1} - 1)^n\|\epsilon_{\lambda}\|_{(\alpha, \omega)} = (2^{\alpha+1} - 1)^n - 1 \frac{\lambda^{n\alpha}}{(\lambda - \omega)^{n(\alpha+1)}}, \]
where we use (1). Now we obtain
\[ \|r_\lambda^n\| \leq \lim_{\alpha \to 0^+} \left( 2^{\alpha+1} - 1 \right)^n - 1 \frac{\lambda^{n\alpha}}{(\lambda - \omega)^{n(\alpha+1)}}, \]
\[ \limsup_{\alpha \to 0^+} \|T_\alpha\|, \]

and we get
\[ \sup\{ (\lambda - \omega)^n \| r^\alpha_n \| : n \in \mathbb{N}, \lambda \in (\omega, +\infty) \} \leq \limsup_{\alpha \to +\infty} \| T_\alpha \| < +\infty. \]

By Theorem 1.1, there exists a unique continuous homomorphism \( T : L^1_\omega(\mathbb{R}^+) \to A \) such that \( T(\epsilon - \lambda) = r_\lambda = T_\alpha(\epsilon - \lambda) \) for each \( \lambda \in (\omega, +\infty) \) and \( \| T \| \leq \limsup_{\alpha \to +\infty} \| T_\alpha \| \).

**Remark 3.2.** Since the set \((\epsilon - \lambda)_{\lambda \in (\omega, +\infty)}\) is linearly dense in \( AC_\omega^{(\alpha)}(\mathbb{R}^+) \) (Proposition 2.3), \( T_\alpha(f) = T(f) \) for any \( f \in AC_\omega^{(\alpha)}(\mathbb{R}^+) \) and \( \alpha > 0 \). Also, we may combine (i) and (ii) to get
\[ \| T \| = \limsup_{\alpha \to +\infty} \| T_\alpha \| = \sup_{\alpha > 0} \| T_\alpha \|, \]
where we use \( \| T_\beta \| \leq \| T_\alpha \| \) with \( \beta \geq \alpha > 0 \).

4. Connection with integrated semigroups

Widder’s characterization of Laplace transforms of real-valued bounded functions, which states that for \( r \in C^{(\infty)}(0, \infty) \) there exists \( f \in L^\infty(0, \infty) \) such that
\[ r(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt, \quad \lambda > 0, \]
if and only if
\[ \sup\{ \lambda^{n+1} \frac{|r^{(n)}(\lambda)|}{n!} : \lambda > 0, n \in \mathbb{N} \} < +\infty, \]
(see [12]) was considered by W. Arendt on arbitrary Banach space \( X \). An integrated version of this result is given in terms of a function \( F : [0, \infty) \to X \) such that \( F(0) = 0 \), \( \| F(t + h) - F(t) \| \leq Mh_t, (t, h \geq 0) \) and
\[ r(\lambda) = \lambda \int_0^\infty e^{-\lambda t} F(t) dt, \quad \lambda > 0; \]
see [11, Theorem 1.1]. This result was improved by M. Hieber who showed that for any \( \alpha \in (0, 1) \) there exists a function \( F_\alpha : [0, \infty) \to X \) such that \( F_\alpha(0) = 0 \), \( \| F_\alpha(t + h) - F_\alpha(t) \| \leq Mh_\alpha, (t, h \geq 0) \) and
\[ r(\lambda) = \lambda^\alpha \int_0^\infty e^{-\lambda t} F(t) dt, \quad \lambda > 0; \]
see [11, Theorem 3.2]. If \( r \) is a pseudo-resolvent in a Banach algebra \( A \), then the function \( F_\alpha \) is an \( \alpha \)-times integrated semigroup in \( A \). We now give here the definition of \( \alpha \)-times integrated semigroups in a Banach algebra \( A \), since they are usually considered in the setting of linear and bounded operators on a Banach space \( X \); see for example [11].

**Definition 4.1.** For any \( \alpha > 0 \), an \( \alpha \)-times integrated semigroup \( s_\alpha(\cdot) : [0, \infty) \to A \) is a continuous mapping with \( s_\alpha(0) = 0 \) and
\[ s_\alpha(t) s_\alpha(s) = \frac{1}{\Gamma(\alpha)} \int_t^{t+s} (t + s - r)^{\alpha-1} s_\alpha(r) dr - \int_0^s (t + s - r)^{\alpha-1} s_\alpha(r) dr, \]
with \( t, s \geq 0 \).
If \( \|s_\alpha(t)\| \leq Ce^{\omega t} \) for \( C, t \geq 0 \) and \( \omega \in \mathbb{R} \), then condition (2) is equivalent (by Laplace transform) to \( r_\lambda := \lambda^\alpha \int_0^\infty e^{-\lambda t}s_\alpha(t)dt \), with \( \lambda > \omega \) a pseudo-resolvent in \( \mathcal{A} \); see the analogous proof in Banach space in [2]. If \((s_\alpha(t))_{t \geq 0}\) is an \( \alpha \)-times integrated semigroup in \( \mathcal{A} \), then \((s_\alpha(t))_{t \geq 0}\) with

\[
s_\nu(t) := \frac{1}{\Gamma(\nu - \alpha)} \int_0^t (t - s)^{\nu - \alpha - 1} s_\alpha(s)ds, \quad t \geq 0,
\]
is a \( \nu \)-times integrated semigroup with \( \nu > \alpha \) in \( \mathcal{A} \). The set of Bochner-Riesz functions \((R^\nu_t)_{t \geq 0}\), i.e.,

\[
R^\nu_t(s) := (t - s)^{\nu - 1} \frac{1}{\Gamma(\nu)} \chi((0,t))(s), \quad t \geq 0,
\]
is an example of a \( \nu \)-times integrated semigroup in \( AC^\omega(\mathbb{R}^+) \) for \( \nu > \alpha, \omega \geq 0 \) and

\[
\|R^\nu_t\|_{(\alpha, \omega)} \leq \frac{t^\nu \epsilon^\omega}{\Gamma(\nu + 1)}, \quad t \geq 0.
\]

Now, we may add in the Widder-Arendt-Kisyński theorem more conditions related to fractional homomorphisms and \( \alpha \)-times integrated semigroups.

**Theorem 4.2.** Let \( \mathcal{A} \) be a Banach algebra, let \( \omega \) be a non-negative number, let \( (\epsilon_\lambda)_{\lambda \in (\omega, +\infty)} \) be a pseudo-resolvent in \( \mathcal{A} \) and let

\[
M = \sup\{(\lambda - \omega)^n\|r_\lambda^n\| : n \in \mathbb{N}, \lambda \in (\omega, +\infty)\}.
\]

Then the following conditions are equivalent:

(i) \( M < +\infty \).

(ii) There exists a continuous homomorphism \( T : L^1_\omega(\mathbb{R}^+) \rightarrow \mathcal{A} \) such that \( T(\epsilon_\lambda) = r_\lambda \) for each \( \lambda \in (\omega, +\infty) \).

(iii) For any \( \alpha > 0 \), there exists an \( \alpha \)-times integrated semigroup \((s_\alpha(t))_{t \geq 0} \subset \mathcal{A} \) such that \( \|s_\alpha(t)\| \leq \frac{C}{(\alpha + 1)} t^\alpha e^{\omega t} \) with \( C, t \geq 0 \) and \( r_\lambda = \lambda^\alpha \int_0^\infty e^{-\lambda t}s_\alpha(t)dt \), with \( \lambda > \omega \).

(iv) For any \( \alpha > 0 \), there exists a continuous homomorphism \( T_\alpha : AC^\omega(\mathbb{R}^+) \rightarrow \mathcal{A} \) such that \( T_\alpha(\epsilon_\lambda) = r_\lambda \) for each \( \lambda \in (\omega, +\infty) \) and \( T_\alpha(\epsilon_\lambda) = r_\lambda \) for each \( \lambda \in (\omega, +\infty) \) exists, then it is unique, \( T(f) = T_\alpha(f) \) for \( f \in AC^\omega(\mathbb{R}^+) \) and

\[
M = \|T\| = \sup_{\alpha > 0} \|T_\alpha\| = \inf\{C : \|s_\alpha(t)\| \leq \frac{C}{\Gamma(\alpha + 1)} t^\alpha e^{\omega t}, \ t \geq 0\}.
\]

**Proof.** (ii) \( \Rightarrow \) (iii) Take \((R^\alpha_t)_{t \geq 0}\) to be the family of Bochner-Riesz functions of order \( \alpha > 0 \). Then \( (R^\alpha_t)_{t \geq 0} \subset L^1_\omega(\mathbb{R}^+) \) and we define \( s_\alpha(t) := T(R^\alpha_t) \), for \( t > 0 \) and \( s_\alpha(0) := 0 \). Since \((R^\alpha_t)_{t \geq 0}\) is an \( \alpha \)-times integrated semigroup in \( L^1_\omega(\mathbb{R}^+) \), \((s_\alpha(t))_{t \geq 0}\) is an \( \alpha \)-times integrated semigroup in \( \mathcal{A} \) and

\[
\|s_\alpha(t)\| \leq \|T\| \|R^\alpha_t\|_\omega \leq \|T\| \frac{t^\alpha}{\Gamma(\alpha + 1)} e^{\omega t}.
\]

By continuity of \( T \), we have for \( \lambda > \omega \),

\[
\lambda^\alpha \int_0^\infty e^{-\lambda t}s_\alpha(t)dt = \lambda^\alpha T \left( \int_0^\infty e^{-\lambda t}R^\alpha_tdt \right) = \lambda^\alpha T(W^{\alpha}(\epsilon_\lambda) = T(\epsilon_\lambda) = r_\lambda.
\]
(iii) \(\Rightarrow\) (iv) We define \(T_\alpha : AC_\omega^{(\alpha)}(\mathbb{R}^+) \to \mathcal{A}\) by
\[
T_\alpha(f) := \int_0^\infty W^\alpha f(t)s_\alpha(t)dt, \quad f \in \mathcal{D}_+.
\]
Following the same arguments as in [10, Theorem 3.1], we prove that \(T_\alpha\) are continuous homomorphisms, \(\|T_\alpha\| \leq C\) for any \(\alpha > 0\) and \(\sup_{\alpha > 0} \|T_\alpha\| \leq C < +\infty\).
Since \(\epsilon_\lambda \in AC_\omega^{(\alpha)}(\mathbb{R}^+)\) for \(\lambda > \omega\) and \(W^\alpha(\epsilon_\lambda) = \lambda^\alpha\epsilon_\lambda\), we have \(T_\alpha(\epsilon_\lambda) = r_\lambda\) for \(\lambda > \omega\).

(iv) \(\Rightarrow\) (ii) It is proved by Theorem 3.1(ii).

If a continuous homomorphism \(T : L_\omega^1(\mathbb{R}^+) \to \mathcal{A}\) satisfying \(T(\epsilon^-_\lambda) = r_\lambda\) for each \(\lambda \in (\omega, +\infty)\) exists, then \(AC_\omega^{(\alpha)}(\mathbb{R}^+) \hookrightarrow L_\omega^1(\mathbb{R}^+)\) and \(T(f) = T_\alpha(f)\) for \(f \in AC_\omega^{(\alpha)}(\mathbb{R}^+)\). Moreover, if we collect in the proof the following inequalities,
\[
\|T\| = M = \sup_{\alpha > 0} \|T_\alpha\| \leq \inf \{ C : \|s_\alpha(t)\| \leq C\frac{t^\alpha e^{\omega t}}{\Gamma(\alpha + 1)}, \quad t \geq 0 \} \leq \|T\|,
\]
we finally get the equality. \(\square\)

Remark 4.3. If \(0 < \alpha \leq 1\), we use \(s_\alpha(t) = T(R_\alpha t)\) \((t > 0)\) to obtain
\[
\|s_\alpha(t + h) - s_\alpha(t)\| \leq 2\|T\|\frac{h^\alpha}{\Gamma(\alpha + 1)} e^{\omega(t+h)},
\]
with \(t, h > 0\); see also [7, Corollary 3.3].

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References
