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DECAY AND GROWTH FOR A NONLINEAR PARABOLIC DIFFERENCE EQUATION

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ABSTRACT. We prove a difference equation analogue of the decay-of-mass result for the nonlinear parabolic equation $u_t = \Delta u + \mu|\nabla u|$ when $\mu < 0$, and a new growth result when $\mu > 0$.

1. Introduction

Consider the following difference equation:

$$\begin{aligned} u_i^{n+1} - u_i^n \quad &= \quad \alpha\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \\ &\quad + \mu\left(|u_i^n - u_{i-1}^n| + |u_i^n - u_{i+1}^n|\right),\ i \in \mathbb{Z},\ n \in \mathbb{Z}_+, \end{aligned} \tag{1}$$

starting with some $u^0 = (u_i^0)_{i\in\mathbb{Z}}$ such that $u^0 \geq 0$ and $\sum_{i\in\mathbb{Z}} u_i^0 < \infty$, where the parameters μ and α satisfy

$$0 < |\mu| \leq \alpha \ \text{ and } \ \alpha + |\mu| \leq \frac{1}{2}. \tag{2}$$

This scheme corresponds (after appropriate rescaling) to the following partial differential equation for $u(x,t)$:

$$u_t = u_{xx} + \mu|u_x|,\ x \in \mathbb{R},\ t \in \mathbb{R}_+, \tag{3}$$

with initial condition $u(x,0) = u^0(x)$ such that $u^0 \geq 0$ and $\int_{\mathbb{R}} u^0(x)\,\mathrm{d}x < \infty$ (as usual, u_i^n in (1) corresponds to $u(i\Delta x, n\Delta t)$). The behavior of the total mass $\int_{\mathbb{R}} u(x,t)\,\mathrm{d}x$ as $t \to \infty$ is as follows:

- (**D**) When $\mu < 0$ the *mass decays to zero*: $\int_{\mathbb{R}} u(x,t)\,\mathrm{d}x \to 0$ as $t \to \infty$; see Ben-Artzi, Goodman and Levy [1, Theorem 5.1].
- (**G**) When $\mu > 0$ the *mass grows to infinity*: $\int_{\mathbb{R}} u(x,t)\,\mathrm{d}x \to \infty$ as $t \to \infty$ (for $u^0 \neq 0$); see Laurençot and Souplet [4, Theorem 1(i)].

Here we prove, first, that the difference equation (1) satisfies a decay-of-mass result that is analogous to (D); and second, that it satisfies a growth result stronger than (G):

- (**$\boldsymbol{\Delta}$**) When $\mu < 0$ the *mass decays to zero*: $\sum_{i\in\mathbb{Z}} u_i^n \to 0$ as $n \to \infty$; see Theorem 3.
- (**$\boldsymbol{\Gamma}$**) When $\mu > 0$ there is *convergence to a constant*: for each $u^0 \neq 0$ there is a constant $c > 0$ such that $\lim_{n\to\infty} u_i^n = c$ for *all* i; see Theorem 6.

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Moreover, the result (Γ) applies to any *bounded* (not necessarily summable) initial condition u^0. Finally, both results (Δ) and (Γ) (like (D) and (G)) extend to the multi-dimensional case; see Theorems 5 and 8.

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2. Preliminaries

Let $\ell^\infty(\mathbb{Z}) = \{u = (u_i)_{i\in\mathbb{Z}} : \sup_{i\in\mathbb{Z}} |u_i| < \infty\}$ be the space of doubly infinite bounded sequences, and let $\ell^1(\mathbb{Z}) = \{u = (u_i)_{i\in\mathbb{Z}} : \|u\| < \infty\}$ be the subspace of summable sequences, where $\|\cdot\|$ denotes the ℓ^1-norm $\|u\| = \sum_{i\in\mathbb{Z}} |u_i|$. Put $\ell^\infty_+(\mathbb{Z}) = \{u \in \ell^\infty(\mathbb{Z}) : u \geq 0\}$ (all inequalities $u \geq v$ are meant coordinatewise: $u_i \geq v_i$ for all i); similarly for $\ell^1_+(\mathbb{Z})$.

Given parameters μ and α that satisfy (2), define $F : \ell^\infty_+(\mathbb{Z}) \to \ell^\infty_+(\mathbb{Z})$ by

$$\begin{aligned} F_i(u) \quad &:= \quad (1-2\alpha)u_i + \alpha(u_{i-1} + u_{i+1}) \\ &\qquad + \mu\left(|u_i - u_{i-1}| + |u_i - u_{i+1}|\right) \end{aligned}$$

for each $i \in \mathbb{Z}$, and $F(u) = (F_i(u))_{i\in\mathbb{Z}}$. The conditions on μ and α guarantee that indeed $F(u) \in \ell^\infty_+(\mathbb{Z})$ when $u \in \ell^\infty_+(\mathbb{Z})$; moreover, $F(u) \in \ell^1_+(\mathbb{Z})$ when $u \in \ell^1_+(\mathbb{Z})$ (see Lemma 1 below). We write $F^{(n)}(u)$ for the n-th iterate of F, i.e., $F^{(1)}(u) = F(u)$ and $F^{(n)}(u) = F(F^{(n-1)}(u))$. Then (1) is just $u^{n+1} = F(u^n)$, and so $u^n = F^{(n)}(u^0)$.

Lemma 1. *F satisfies:*

(i) *$F(u) \in \ell^\infty_+(\mathbb{Z})$ for all $u \in \ell^\infty_+(\mathbb{Z})$.*
(ii) *$F(u) \in \ell^1_+(\mathbb{Z})$ for all $u \in \ell^1_+(\mathbb{Z})$.*
(iii) *$\|F(u)\| \leq \|u\|$ when $\mu < 0$, and $\|F(u)\| \geq \|u\|$ when $\mu > 0$, for all $u \in \ell^1_+(\mathbb{Z})$.*
(iv) *F is monotonic: $F(u) \leq F(v)$ for all $u, v \in \ell^\infty_+(\mathbb{Z})$ with $u \leq v$.*

Proof. $F_i(u)$ is a convex combination of u_{i-1}, u_i, u_{i+1} (the coefficients are among $\alpha \pm \mu$, $1 - 2\alpha \pm 2\mu$, and $1 - 2\alpha$, which are all nonnegative by (2)), which proves (i). When $u \in \ell^1_+(\mathbb{Z})$, we have $\sum_i F_i(u) = \sum_i u_i + 2\mu \sum_i |u_i - u_{i-1}| \leq (1+4|\mu|)\sum_i u_i < \infty$, which proves (ii) and (iii). For (iv), $F_i(u)$ is a continuous piecewise linear function of u_{i-1}, u_i, u_{i+1} (there are four regions, determined by the signs of $u_i - u_{i-1}$ and $u_i - u_{i+1}$). In each region $F_i(u)$ is monotonic (it is a convex combination of its arguments), and the continuous "gluing" of these pieces is therefore also monotonic. More precisely, given $u \leq v$, one can find a chain $u = v^0 \leq v^1 \leq ... \leq v^m$ such that v^{k-1} and v^k belong to the same region of linearity of F_i for each $k = 1, ..., m$, and the endpoint v^m satisfies $v^m_j = v_j$ for $j = i-1, i, i+1$ (indeed: increase in turn each one of the three coordinates $j = i-1, i, i+1$ starting from u_j, until either the boundary of a region is crossed — this happens when $w_i = w_{i-1}$ or $w_i = w_{i+1}$ — or v_j is reached). Thus $F_i(v^{k-1}) \leq F_i(v^k)$ (the two points are in the same region) for all $k = 1, ..., m$, and $F_i(v^m) = F_i(v)$, which completes the proof. □

We introduce an auxiliary operator $G : \ell_+^\infty(\mathbb{Z}) \to \ell_+^\infty(\mathbb{Z})$ defined by

$$(4) \qquad G_i(u) := \begin{cases} (\alpha+\mu)u_{i-1} + (1-2\alpha)u_i + (\alpha-\mu)u_{i+1}, & \text{for } i \geq 1, \\ (\alpha-\mu)u_{-1} + (1-2\alpha+2\mu)u_0 + (\alpha-\mu)u_1, & \text{for } i = 0, \\ (\alpha-\mu)u_{i-1} + (1-2\alpha)u_i + (\alpha+\mu)u_{i+1}, & \text{for } i \leq -1, \end{cases}$$

and $G(u) = (G_i(u))_{i\in\mathbb{Z}}$. Thus $G(u)$ is obtained from $F(u)$ when each term $|u_j - u_{j+1}|$ is replaced by $u_j - u_{j+1}$ for $j \geq 0$, and by $u_{j+1} - u_j$ for $j \leq -1$. Note that $F(u) = G(u)$ whenever u is unimodal with mode at 0 ("centered unimodal"), i.e., $u_i \geq u_{i+1}$ for $i \geq 0$ and $u_i \geq u_{i-1}$ for $i \leq 0$.

Lemma 2. *G satisfies:*

(i) *G is a linear monotonic operator.*

(ii) *$\|G(u)\| \leq \|u\|$ when $\mu < 0$, and $\|G(u)\| \geq \|u\|$ when $\mu > 0$, for all $u \in \ell_+^1(\mathbb{Z})$.*

(iii) *$F(u) \leq G(u)$ when $\mu < 0$, and $F(u) \geq G(u)$ when $\mu > 0$, for all $u \in \ell_+^\infty(\mathbb{Z})$.*

(iv) *$F^{(n)}(u) \leq G^{(n)}(u)$ when $\mu < 0$, and $F^{(n)}(u) \geq G^{(n)}(u)$ when $\mu > 0$, for all $u \in \ell_+^\infty(\mathbb{Z})$ and all $n \geq 1$.*

Proof. (i) is immediate. (ii) follows from $\|G(u)\| = \|u\| + 4\mu u_0$. For (iii), let $i \geq 1$; we have

$$\begin{aligned} \frac{1}{\mu}(F_i(u) - G_i(u)) &= |u_i - u_{i-1}| + |u_i - u_{i+1}| \\ &\quad - (u_i - u_{i-1}) - (u_{i+1} - u_i) \geq 0, \end{aligned}$$

so $F_i(u) \leq G_i(u)$ when $\mu < 0$, and $F_i(u) \geq G_i(u)$ when $\mu > 0$; similarly when $i \leq -1$ and $i = 0$. Finally, (iv) follows by induction on n: when $\mu < 0$, from $F^{(n)}(u) \leq G^{(n)}(u)$ and the monotonicity of G follows $G\left(F^{(n)}(u)\right) \leq G\left(G^{(n)}(u)\right)$, and from (iii) follows $F\left(F^{(n)}(u)\right) \leq G\left(F^{(n)}(u)\right)$, which together yield $F^{(n+1)}(u) \leq G^{(n+1)}(u)$; similarly when $\mu > 0$. ◻

3. Decay of mass

We now assume that $\mu < 0$; put $\lambda = |\mu|$. Lemma 1(iii) implies that the total mass $\sum_i u_i^n$ decreases with n; the result below shows that in fact it decays to zero.

Theorem 3. *Let $\mu < 0$ and α satisfy* (2). *Then for all $u^0 \in \ell_+^1(\mathbb{Z})$*

$$\lim_{n\to\infty} \sum_{i\in\mathbb{Z}} u_i^n = 0.$$

To prove the theorem we will show that $\left\|G^{(n)}(u^0)\right\| \to_n 0$ and then use Lemma 2(iv). Take $q = \alpha/(\alpha+\lambda)$, and let $z = (q^{|i|})_{i\in\mathbb{Z}} \in \ell_+^1(\mathbb{Z})$.

Lemma 4. *There exists $0 < \rho < 1$ such that $G(z) \leq (1-\rho)z$.*

Proof. For $i \geq 1$ we have

$$\begin{aligned} G_i(z) &= (\alpha-\lambda)q^{i-1} + (1-2\alpha)q^i + (\alpha+\lambda)q^{i+1} \\ &= \left(1 - \frac{\lambda^2}{\alpha}\right) q^i \end{aligned}$$

(recall that $q = \alpha/(\alpha+\lambda)$). Similarly for $i \leq -1$. Finally, for $i = 0$,

$$G_0(z) = (1 - 2\alpha - 2\lambda) + 2(\alpha+\lambda)q < 1 - \frac{\lambda^2}{\alpha}.$$

Take $\rho = \lambda^2/\alpha$. ◻

There is nothing special about this value of q; we choose it for convenience only (any q close enough to 1, specifically $(\alpha-\lambda)/(\alpha+\lambda) < q < 1$, will do). Also, note that $F(z) = G(z)$ since z is centered unimodal.

Proof of Theorem 3. Let q, z and ρ be as above. Given $u \in \ell^1_+(\mathbb{Z})$, for each $k \geq 0$ let $v^{[k]} \in \ell^1_+(\mathbb{Z})$ be the k-truncation of u, i.e., $v_i^{[k]} := u_i$ for $i = -k, ..., k$ and $v_i^{[k]} := 0$ otherwise, and define $\theta_k := \max_{i=-k,\ldots,k} u_i/q^{|i|}$. Then $v^{[k]} \to_k u$ and $v^{[k]} \leq \theta_k z$. By Lemmata 2(i) and 4 (iterated n times), we get

$$G^{(n)}(v^{[k]}) \leq G^{(n)}(\theta_k z) = \theta_k G^{(n)}(z) \leq \theta_k (1-\rho)^n z.$$

Also, $\left\|G^{(n)}(u - v^{[k]})\right\| \leq \left\|u - v^{[k]}\right\|$ by Lemma 2(ii). Therefore

$$\begin{aligned} \left\|G^{(n)}(u)\right\| &= \left\|G^{(n)}(v^{[k]})\right\| + \left\|G^{(n)}(u - v^{[k]})\right\| \\ &\leq \theta_k (1-\rho)^n \left\|z\right\| + \left\|u - v^{[k]}\right\|. \end{aligned}$$

But $0 < 1-\rho < 1$, so $\limsup_{n\to\infty} \left\|G^{(n)}(u)\right\| \leq \left\|u - v^{[k]}\right\|$. This holds for all k, which together with $\left\|u - v^{[k]}\right\| \to 0$ as $k \to \infty$ shows that $\left\|G^{(n)}(u)\right\| \to 0$ as $n \to \infty$; recalling that $0 \leq F^{(n)}(u) \leq G^{(n)}(u)$ by Lemma 2(iv) completes the proof. □

4. Decay in higher dimensions

Let $d \geq 1$ be an integer. The d-dimensional version of (3) is the differential equation

$$u_t = \Delta u + \mu |\nabla u|, \ x \in \mathbb{R}^d, \ t \in \mathbb{R}_+.$$

The decay-of-mass result of Ben-Artzi, Goodman and Levy [1, Theorem 5.1] for this equation, when $\mu < 0$, holds for any dimension d. Our result of Theorem 3 also generalizes to d dimensions.

Let $\mathbb{Z}^d$, the space of d-dimensional integer vectors $i = (i_1, ..., i_d)$, be endowed with the ℓ^1-norm $\|i\| = \sum_{r=1}^d |i_r|$, and put $\ell^\infty(\mathbb{Z}^d) = \{u = (u_i)_{i\in\mathbb{Z}^d} : \sup_{i\in\mathbb{Z}^d} |u_i| < \infty\}$ and $\ell^1(\mathbb{Z}^d) = \{u = (u_i)_{i\in\mathbb{Z}^d} : \|u\| < \infty\}$, where $\|u\| = \sum_{i\in\mathbb{Z}^d} |u_i|$. Given μ and α such that

$$0 < |\mu| \leq \alpha \ \text{ and } \ \alpha + |\mu| \leq \frac{1}{2d}, \tag{5}$$

define $F : \ell^\infty_+(\mathbb{Z}^d) \to \ell^\infty_+(\mathbb{Z}^d)$ by $F(u) = (F_i(u))_{i\in\mathbb{Z}^d}$ and

$$F_i(u) := (1 - 2d\alpha)u_i + \alpha \sum_{j\in V(i)} u_j \ + \ \mu \sum_{j\in V(i)} |u_i - u_j|$$

for each $i \in \mathbb{Z}^d$, where $V(i) := \{j \in \mathbb{Z}^d : \|j - i\| = 1\}$ denotes the 1-neighborhood of i (i.e., those j that are obtained from i by increasing or decreasing one coordinate by 1). Put $u^n := F^{(n)}(u^0)$.

To define the auxiliary operator G, for each $i \in \mathbb{Z}^d$ we partition $V(i)$ into $V_+(i) := \{j \in \mathbb{Z}^d : \|j\| = \|i\| + 1\}$ and $V_-(i) := \{j \in \mathbb{Z}^d : \|j\| = \|i\| - 1\}$, and put

$$\begin{aligned} G_i(u) &:= (1 - 2d\alpha)u_i + \alpha \sum_{j\in V(i)} u_j \\ &\quad + \mu \sum_{j\in V_+(i)} (u_i - u_j) + \mu \sum_{j\in V_-(i)} (u_j - u_i). \end{aligned}$$

This can be rewritten as

$$\begin{aligned} G_i(u) &= (1 - 2d\alpha + [|V_+(i)| - |V_-(i)|]\,\mu)\,u_i \\ &\quad + (\alpha+\mu)\sum_{j\in V_-(i)} u_j + (\alpha-\mu)\sum_{j\in V_+(i)} u_j, \end{aligned}$$

where $|A|$ denotes the number of elements of a finite set A (compare with (4)).

It is straightforward to check that Lemmata 1 and 2 continue to hold. As for Lemma 4 (for $\lambda = -\mu > 0$), we again take $q = \alpha/(\alpha+\lambda)$ and put $z = \left(q^{\|i\|}\right)_{i\in\mathbb{Z}^d} \in \ell^1_+(\mathbb{Z}^d)$. The set $V_+(i)$ contains $d+m$ elements, where m is the number of coordinates of i that vanish. Increasing z_j from $q^{\|i\|+1}$ to $q^{\|i\|-1}$ for m of the elements j of $V_+(i)$ can only increase $G_i(z)$; hence

$$\begin{aligned} G_i(z) &\le (1-2d\alpha)q^{\|i\|} + d(\alpha-\lambda)q^{\|i\|-1} + d(\alpha+\lambda)q^{\|i\|+1} \\ &= \left(1 - \frac{d\lambda^2}{\alpha}\right) q^{\|i\|} = (1-\rho)z_i. \end{aligned}$$

Therefore the proof of Theorem 3 in the previous section applies to the d-dimensional case as well (with the appropriate trivial adjustments, like $\|i\| \le k$ instead of $i = -k, ..., k$). Thus we have

Theorem 5. *Let $d \ge 1$ be an integer, and let $\mu < 0$ and α satisfy* (5). *Then for all $u^0 \in \ell^1_+(\mathbb{Z}^d)$*

$$\lim_{n\to\infty} \sum_{i\in\mathbb{Z}^d} u_i^n = 0.$$

5. Growth

We now return to the one-dimensional case and assume that $\mu > 0$. Here the total mass $\sum_i u_i^n$ increases (recall Lemma 1(iii)), and we will show that u^n always converges to a constant sequence $(..., c, c, c, ...)$ for some $c > 0$. In fact, this applies starting from any *bounded* (not necessarily summable) initial condition, i.e., for any $u^0 \ne 0$ in $\ell^\infty_+(\mathbb{Z})$. (In the trivial case $u^0 = 0$ we have $u^n = 0$ for all n.)

Theorem 6. *Let $\mu > 0$ and α satisfy* (2). *Then for each $u^0 \in \ell^\infty_+(\mathbb{Z})$, $u^0 \ne 0$, there exists $c > 0$ such that*

$$\lim_{n\to\infty} u_i^n = c \quad \textit{for all} \quad i \in \mathbb{Z}.$$

Let $\pi \in \ell^1_+(\mathbb{Z})$ be given by

$$\pi_i = \frac{\mu}{\alpha}\left(\frac{\alpha-\mu}{\alpha+\mu}\right)^{|i|} \tag{6}$$

for each $i \in \mathbb{Z}$; this is a probability measure on $\mathbb{Z}$, i.e., $\sum_{i\in\mathbb{Z}} \pi_i = 1$. The auxiliary operator G was defined in Section 2. We have

Proposition 7. *For each $u \in \ell^\infty_+(\mathbb{Z})$*

$$\lim_{n\to\infty} G_i^{(n)}(u) = \pi \cdot u \equiv \sum_{k=-\infty}^{\infty} \pi_k u_k \quad \textit{for all} \quad i \in \mathbb{Z}.$$

Proof. The linear operator G corresponds to a Markov chain[1] on $\mathbb{Z}$ with transition probabilities given by a stochastic matrix P, where P_{ik} is the coefficient of u_k in the formula for $G_i(u)$ in (4). It is easy to verify that there is a single irreducible component (the whole space $\mathbb{Z}$ when $\alpha > \mu$, and $\{0\}$ when $\alpha = \mu$), and that π given by (6) has finite mass and satisfies $\pi_k = \sum_{i\in\mathbb{Z}} \pi_i P_{ik}$ for all $k \in \mathbb{Z}$. Therefore (see Feller [2, Theorem XV.7]), π is the unique invariant probability measure of the Markov chain, and $P^n_{ik} \to_n \pi_k$ for all $i, k \in \mathbb{Z}$, where P^n denotes the n-th power of the matrix P. This implies $G^{(n)}_i(u) = \sum_k P^n_{ik} u_k \to_n \sum_k \pi_k u_k$ for any $u \in \ell^\infty_+(\mathbb{Z})$ (since $\pi \in \ell^1_+(\mathbb{Z})$). □

Proposition 7 together with Lemma 2(iv) readily imply that if $u^0 \in \ell^1_+(\mathbb{Z})$, $u^0 \neq 0$, then the total mass $\|u^n\|$ increases to infinity. We now prove the stronger result of Theorem 6.

Proof of Theorem 6. Let $M_n := \sup_{i\in\mathbb{Z}} u^n_i$; the sequence M_n is nonincreasing (since each coordinate of u^{n+1} is an average of coordinates of u^n), and so it converges to a limit M. Assuming without loss of generality that the 0-th coordinate u^0_0 of u^0 is positive yields by Lemma 2(iv) and Proposition 7

$$M_n \geq u^n_i \geq G^{(n)}_i(u^0) \to_n \pi \cdot u^0 \geq \pi_0 u^0_0 = \frac{\mu}{\alpha} u^0_0 > 0, \tag{7}$$

hence $M > 0$.

We will show that $\lim_n u^n_i = M$ for all i. There are three cases.

Case 1: $\alpha = \mu$. Let $\varepsilon > 0$, and assume without loss of generality that $u^0_0 \geq M_0 - \varepsilon$; then (7) implies $\lim_n M_n \geq u^0_0 \geq M_0 - \varepsilon$. The sequence M_n is nonincreasing, hence $M = \lim_n M_n = M_0$, and using (7) again yields $\lim_n u^n_i = M$ for all i.

Case 2: $\alpha > \mu$ and $\alpha + \mu < 1/2$. For large n the supremum M_n stays almost constant (and close to M), from which we will deduce that there must be an appropriate block of consecutive coordinates that are all close to M (see (9)); we will then apply Proposition 7 (see (10)).

Indeed, let $\varepsilon > 0$. Then there exists K such that

$$\sum_{k=-K}^{K} \pi_k \geq 1 - \varepsilon,$$

and there exists N such that

$$M_N \leq M + \varepsilon',$$

where $\varepsilon' := \gamma^K \varepsilon$ and $\gamma := \min\{\alpha - \mu, 1 - 2\alpha - 2\mu\} > 0$. Let $L := K + N$ and assume now without loss of generality[2] that $u^L_0 \geq M_L - \varepsilon'$. Then $u^L_0 = F^{(K)}_0(u^N)$ is a convex combination of the coordinates of u^N that are at a distance of at most K from 0, i.e.,

$$u^L_0 = \sum_{k=-K}^{K} \beta_k u^N_k,$$

[1] A standard reference for Markov chains is Feller [2, Chapter XV].

[2] Note that F is translation-invariant, and so, instead of centering G at 0, we could have centered it at any i_0; this would merely have shifted π by i_0 and left everything unchanged, in particular Lemma 2 and Proposition 7.

where $\sum_k \beta_k = 1$ and $\beta_k \geq 0$. While the coefficients β_k are not fixed (they depend on u^N), they are uniformly bounded away from zero:

$$\beta_k \geq \gamma^K > 0 \text{ for all } k = -K, ..., K \tag{8}$$

(indeed, the nonzero coefficients in $F_i(u)$ — of u_{i-1}, u_i, and u_{i+1} — are all $\geq \gamma$; use induction on K).

For each $k = -K, ..., K$ we have

$$\begin{aligned} M - \varepsilon' \leq M_L - \varepsilon' \leq u_0^L &\leq \beta_k u_k^N + (1 - \beta_k)(M + \varepsilon') \\ &\leq \gamma^K u_k^N + (1 - \gamma^K)(M + \varepsilon') \end{aligned}$$

(the last inequality, which is equivalent to $(\beta_k - \gamma^K)(M + \varepsilon' - u_k^N) \geq 0$, follows from (8) and $u_k^N \leq M_N \leq M + \varepsilon'$ by our choice of N). This implies

$$u_k^N \geq M + \varepsilon' - \frac{2\varepsilon'}{\gamma^K} > M - 2\varepsilon \text{ for all } k = -K, ..., K \tag{9}$$

(recall that $\varepsilon' = \gamma^K \varepsilon$).

Finally, applying Lemma 2(iv) and Proposition 7, and recalling the choice of K yields

$$\begin{aligned} u_i^{n+N} &= F_i^{(n)}(u^N) \geq G_i^{(n)}\left(u^N\right) \\ &\to_n \pi \cdot u^N \geq (M - 2\varepsilon) \sum_{k=-K}^{K} \pi_k \geq (M - 2\varepsilon)(1 - \varepsilon) \end{aligned} \tag{10}$$

for all i, which completes the proof in this case.

Case 3: $\alpha > \mu$ and $\alpha + \mu = 1/2$. The proof here is a modification of the argument in the previous case. Since now $1 - 2\alpha - 2\mu = 0$, some of the coefficients β_k may vanish: instead of (8) and (9) which hold for *all* $k = -K, ..., K$, we only get similar inequalities for *every other* k (indeed: the coefficients of u_{i-1} and u_{i+1} in $F_i(u)$ are positive, whereas the coefficient of u_i may vanish). However, if y is the alternating sequence $y = (..., 1, 0, 1, 0, 1, 0, ...)$, then it is easy to see that $F(y) = (..., 1 - \eta, 1, 1 - \eta, 1, 1 - \eta, 1, ...)$, where $\eta := 2\alpha - 2\mu < 1$, and $F^{(n)}(y) = (..., 1, 1 - \eta^n, 1, 1 - \eta^n, 1, 1 - \eta^n, ...)$ for every $n \geq 1$.

Therefore we proceed as follows: given $\varepsilon > 0$, let R be such that $\eta^R \leq \varepsilon$, let K_0 be such that $\sum_{k=-K_0}^{K_0} \pi_k \geq 1 - \varepsilon$, and take $K := K_0 + R$ and $\gamma := \alpha - \mu > 0$. Continuing as in Case 2, we now get $\beta_k \geq \gamma^K > 0$, and thus $u_k^N > M - 2\varepsilon$, for every other k between $-K$ and K. Therefore, for all $k = -K_0, ..., K_0$, we have by the monotonicity of F (see Lemma 1(iv); only the coordinates between $-K$ and K matter here)

$$u_k^{R+N} = F_k^{(R)}(u^N) \geq F_k^{(R)}((M - 2\varepsilon)y),$$

where y is the alternating sequence above. The homogeneity of degree 1 of F, the computation of $F^{(n)}(y)$ above, and our choice of R imply

$$u_k^{R+N} \geq (M - 2\varepsilon)F_k^{(R)}(y) \geq (M - 2\varepsilon)(1 - \eta^R) \geq (M - 2\varepsilon)(1 - \varepsilon).$$

Applying now Proposition 7 as in (10), with u^{R+N} instead of u^N, yields

$$\liminf_{n\to\infty} u_i^{n+R+N} \geq (M - 2\varepsilon)(1 - \varepsilon)^2$$

for all i (recall the choice of K_0). □

The result of Theorem 6 holds in the multi-dimensional case as well.

Theorem 8. *Let $d \geq 1$ be an integer, and let $\mu > 0$ and α satisfy* (5). *Then for each $u^0 \in \ell^\infty_+(\mathbb{Z}^d)$, $u^0 \neq 0$, there exists $c > 0$ such that*

$$\lim_{n\to\infty} u_i^n = c \quad \textit{for all} \quad i \in \mathbb{Z}^d.$$

Indeed, the same arguments apply; the invariant probability measure π corresponding to G is now given by

$$\pi_i = \left(\frac{\mu}{\alpha}\right)^d \left(\frac{\alpha-\mu}{\alpha+\mu}\right)^{\|i\|}$$

for each $i \in \mathbb{Z}^d$.

References

[1] Ben-Artzi, M., J. Goodman and A. Levy [2000], "Remarks on a Nonlinear Parabolic Equation," *Transactions of the American Mathematical Society*, 352, 731–751. MR1615935 (2000c:35092)

[2] Feller, W. [1968], *An Introduction to Probability Theory and Its Applications*, Volume 1, Third Edition, Wiley. MR0228020 (37:3604)

[3] Gilding, B., M. Guedda and R. Kersner [1998], "The Cauchy Problem for the KPZ Equation," prepublication LAMFA 28, Amiens, December 1998.

[4] Laurençot, P. and P. Souplet [2003], "On the Growth of Mass for a Viscous Hamilton–Jacobi Equation," *Journal d'Analyse Mathématique*, 89, 367–383. MR1981925 (2004c:35188)

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