

DIRICHLET BOUNDARY CONDITIONS FOR ELLIPTIC OPERATORS WITH UNBOUNDED DRIFT

A. LUNARDI, G. METAFUNE, AND D. PALLARA

(Communicated by David S. Tartakoff)

ABSTRACT. We study the realisation A of the operator $\mathcal{A} = \Delta - \langle D\Phi, D\cdot \rangle$ in $L^2(\Omega, \mu)$ with Dirichlet boundary condition, where Ω is a possibly unbounded open set in \mathbb{R}^N , Φ is a semi-convex function and the measure $d\mu(x) = \exp(-\Phi(x)) dx$ lets \mathcal{A} be formally self-adjoint. The main result is that $A : D(A) = \{u \in H^2(\Omega, \mu) : \langle D\Phi, Du \rangle \in L^2(\Omega, \mu), u = 0 \text{ at } \partial\Omega\}$ is a dissipative self-adjoint operator in $L^2(\Omega, \mu)$.

1. INTRODUCTION

Second-order elliptic operators with unbounded coefficients in \mathbb{R}^N or in unbounded subsets of \mathbb{R}^N have been the object of several recent papers; see e.g. [2, 3, 8, 1, 9]. Since the very first studies it was apparent that operators of the type $\mathcal{A}u = \text{Tr } Q(x)D^2u(x) + \langle F(x)Du(x) \rangle$, without potential terms, are not well settled in L^p spaces with respect to the Lebesgue measure, unless the matrix Q and the vector F satisfy very severe restrictions, such as global Lipschitz continuity (see [9, 7]). It is much more natural and fruitful to work in suitably weighted L^p spaces; see [3, 8]. This is what we do in this paper. We consider the operator \mathcal{A} defined by

$$(1) \quad \mathcal{A}u = \Delta u - \langle D\Phi, Du \rangle = e^{\Phi} \text{div} (e^{-\Phi} Du),$$

where $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}$ is a C^2 semi-convex function, i.e., there is $\alpha \geq 0$ such that

$$(2) \quad \Phi_{\alpha}(x) := \Phi(x) + \alpha|x|^2/2 \text{ is convex,}$$

or, equivalently, the matrix $D^2\Phi(x) + \alpha I$ is nonnegative definite at each x . We emphasize that we do not assume any growth restriction on Φ or on its derivatives. The natural weight is then $\rho(x) = e^{-\Phi(x)}$ because, as it is easy to check, if Ω is any open set in \mathbb{R}^N ,

$$\int_{\Omega} \mathcal{A}u v d\mu = - \int_{\Omega} \langle Du, Dv \rangle d\mu, \quad \forall u, v \in C_0^{\infty}(\Omega),$$

if $\mu(dx) = e^{-\Phi(x)} dx$, so that \mathcal{A} is associated to a nice Dirichlet form and it is formally self-adjoint in $L^2(\Omega, \mu)$. The aim of this paper is to study the realisation of \mathcal{A} in $L^2(\Omega, \mu)$ with Dirichlet boundary condition, i.e., the operator

$$(3) \quad A : D(A) = \{u \in H^2(\Omega, \mu) \cap H_0^1(\Omega, \mu) : \mathcal{A}u \in L^2(\Omega, \mu)\} \rightarrow L^2(\Omega, \mu); \quad Au = \mathcal{A}u.$$

Received by the editors April 19, 2004.

2000 *Mathematics Subject Classification.* Primary 35J70; Secondary 47D07.

Key words and phrases. Elliptic operators, boundary value problems, unbounded coefficients.

Note that for $u \in H^2(\Omega, \mu)$, condition $\mathcal{A}u \in L^2(\Omega, \mu)$ is equivalent to requiring $\langle D\Phi, Du \rangle \in L^2(\Omega, \mu)$. Our main result is that A is self-adjoint and dissipative, provided $\partial\Omega$ is smooth enough and the normal derivative $\partial\Phi/\partial n$ is bounded from above on $\partial\Omega$. A lot of consequences then follow; see Section 3.

A natural approach to the study of \mathcal{A} consists in defining an operator $A_0 : C_0^\infty(\Omega) \rightarrow L^2(\Omega, \mu)$, $A_0u = \mathcal{A}u$, in showing that A_0 is closable, and that its closure is self-adjoint and dissipative. But the problem of the characterisation of the domain of the closure still remains. So, we follow a more direct approach, solving the resolvent equation $\lambda u - \mathcal{A}u = f$ for all $\lambda > 0$ and $f \in C_0^\infty(\Omega)$, which is dense in $L^2(\Omega, \mu)$. Proving the existence of a solution to $\lambda u - \mathcal{A}u = f$ that vanishes on $\partial\Omega$ is not hard, thanks to the regularity of the data. Estimates of its $H^1(\Omega, \mu)$ -norm, and uniqueness of the solution in $D(A)$, are easy consequences of the integration formula (5) proved in Lemma 2.2 below. Estimating the second-order derivatives of u is much more delicate, and here the assumptions of semi-convexity and of upper boundedness of $\partial\Phi/\partial n$ are used and play a fundamental role.

This paper is in some sense parallel to the paper [3], where the operator \mathcal{A} was studied in the whole space \mathbb{R}^N and in any convex regular open set Ω with Neumann boundary condition. The conclusions of [3] are similar to the ones of the present paper, but the assumptions on Φ and Ω are a bit different, i.e., Φ is just convex, with no further regularity assumption, and Ω is convex, too.

2. THE DOMAIN OF \mathcal{A} WITH DIRICHLET BOUNDARY CONDITION

Throughout the paper we assume that Ω is an open set in \mathbb{R}^N with sufficiently smooth (at least C^2) boundary. By $L^2(\Omega)$ and $H^k(\Omega)$, $k \in \mathbb{N}$, we mean the usual L^2 and Sobolev spaces with respect to the Lebesgue measure. The spaces $H^k(\Omega, \mu)$, $k = 1, 2$, are defined as the set of all $u \in H_{loc}^k(\Omega)$ such that the function u and its partial derivatives up to the order k belong to $L^2(\Omega, \mu)$. They are Hilbert spaces with the standard inner products $\langle u, v \rangle = \int_\Omega (uv + \sum_{|\alpha|=1}^k D^\alpha u D^\alpha v) e^{-\Phi} dx$. $H_0^1(\Omega, \mu)$ is the subspace of $H^1(\Omega, \mu)$ consisting of the functions with null trace on the boundary. By $C_b^k(\mathbb{R}^N)$ we denote the space of bounded functions with bounded derivatives up to order k . We say that $\partial\Omega$ is *uniformly C^k* if there exist $r > 0$, $m \in \mathbb{N}$ and a (at most countable) family $\{B_j = B_r(x_j), j \in J\}$ of balls covering $\partial\Omega$ with at most m overlapping and C^k -diffeomorphisms $\phi_j : B_j \rightarrow B_1(0)$ such that $\phi_j(B_j \cap \Omega) = B_1(0) \cap \{y_N > 0\}$ and $\sup_j \|\phi_j\|_{C^k} + \|\phi_j^{-1}\|_{C^k} < \infty$.

Lemma 2.1. $C_0^\infty(\Omega)$ is dense in $L^2(\Omega, \mu)$ and in $H_0^1(\Omega, \mu)$.

Proof. Let $u \in L^2(\Omega, \mu)$, or $u \in H_0^1(\Omega)$. Let $\theta : \mathbb{R}^N \rightarrow \mathbb{R}$ be a smooth function such that $0 \leq \theta(x) \leq 1$ for each x , $\theta \equiv 1$ in $B(0, 1)$, $\theta \equiv 0$ outside $B(0, 2)$, and set $u_n(x) = u(x)\theta(x/n)$. Then $u_n \rightarrow u$ in $L^2(\Omega, \mu)$. Indeed,

$$\int_\Omega |u_n - u|^2 d\mu \leq \int_{\{x \in \Omega, |x| \geq n\}} |u|^2 d\mu$$

which goes to 0 as $n \rightarrow \infty$. If $u \in H_0^1(\Omega)$, then $u_n \rightarrow u$ in $H^1(\Omega, \mu)$, because $Du_n(x) = \theta(x/n)Du(x) + D\theta(x/n)u(x)/n$. Since each u_n has bounded support, it may be approximated in $L^2(\Omega)$ (respectively, in $H^1(\Omega)$) by a sequence of $C_0^\infty(\Omega)$ functions. Such a sequence also approximates u_n in $L^2(\Omega, \mu)$ (respectively, in $H^1(\Omega, \mu)$) because μ is equivalent to the Lebesgue measure on each compact subset of \mathbb{R}^N . \square

The realisation A of \mathcal{A} in $L^2(\Omega, \mu)$ with Dirichlet boundary condition is defined by (3). The following integration formulae will be very useful in what follows.

Lemma 2.2. *Let $\psi \in H_0^1(\Omega, \mu)$, $u \in H^2(\Omega, \mu)$ be such that $\mathcal{A}u \in L^2(\Omega, \mu)$. Then*

$$(4) \quad \int_{\Omega} \mathcal{A}u \psi \, d\mu = - \int_{\Omega} \langle Du, D\psi \rangle \, d\mu.$$

More generally, if $\psi \in H^1(\Omega, \mu)$ and $u \in H^2(\Omega, \mu)$ is such that $\mathcal{A}u \in L^2(\Omega, \mu)$, then

$$(5) \quad \int_{\Omega} \mathcal{A}u \psi \, d\mu = - \int_{\Omega} \langle Du, D\psi \rangle \, d\mu + \int_{\partial\Omega} \frac{\partial u}{\partial n} \psi e^{-\Phi} \, d\sigma,$$

where $d\sigma$ denotes the usual Lebesgue surface measure, the last integral is understood as $\lim_{R \rightarrow \infty} \int_{\partial\Omega} \frac{\partial u}{\partial n} \psi \theta(x/R) e^{-\Phi} \, d\sigma$, and θ is the function used in Lemma 2.1.

Proof. The proof of (4) is immediate if $\psi \in C_0^\infty(\Omega)$, and the statement follows by approximation in the general case. Equality (5) is obtained by approximating ψ by $\psi(x)\theta(x/R)$. \square

Let us state a consequence of Lemma 2.2.

Lemma 2.3. *If $\partial\Omega$ is uniformly C^2 and $u \in H^2(\Omega, \mu)$ is such that $\mathcal{A}u \in L^2(\Omega, \mu)$, then $\partial u/\partial n$ is in $L^2(\partial\Omega, \exp(-\Phi) \, d\sigma)$. Moreover, there exists $C > 0$ such that for every $\varepsilon \in (0, 1)$ the following estimate holds:*

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial n}\right)^2 e^{-\Phi} \, d\sigma \leq \varepsilon \left(\|\mathcal{A}u\|_{L^2(\Omega, \mu)}^2 + \| |D^2u| \|_{L^2(\Omega, \mu)}^2 \right) + \frac{C}{\varepsilon} \| |Du| \|_{L^2(\Omega, \mu)}^2.$$

Proof. It is sufficient to take $\psi = \langle Du, \mathcal{N} \rangle$ in (5), where \mathcal{N} is any C_b^1 extension to \mathbb{R}^N of the normal vector field n , and then to use the Hölder inequality. \square

Lemma 2.2 implies that the operator A is symmetric. In the next theorem we prove that it is self-adjoint if Φ is smooth enough, and

$$(6) \quad \frac{\partial \Phi}{\partial n} \leq 0 \quad \text{on } \partial\Omega.$$

Theorem 2.4. *Assume that $\partial\Omega \in C^3$ and that Φ satisfies (2) and (6). Then $(A, D(A))$ is self-adjoint and dissipative in $L^2(\Omega, \mu)$. Moreover, the map $u \mapsto \langle (D^2\Phi)Du, Du \rangle$ is continuous from $D(A)$ to $L^1(\Omega, \mu)$.*

Proof. We have to show that, for $\lambda > 0$ and $f \in L^2(\Omega, \mu)$, the equation $\lambda u - Au = f$ has a unique solution $u \in D(A)$. Uniqueness is an immediate consequence of Lemma 2.2, taking $\psi = u$ in (5). Concerning existence, we first assume that $f \in C_0^\infty(\Omega)$ and we show that there is a solution $u \in D(A)$ satisfying

$$(7) \quad \begin{cases} \text{(a)} & \|u\|_{L^2(\Omega, \mu)} \leq \frac{1}{\lambda} \|f\|_{L^2(\Omega, \mu)}, \\ \text{(b)} & \| |Du| \|_{L^2(\Omega, \mu)} \leq \frac{1}{\sqrt{\lambda}} \|f\|_{L^2(\Omega, \mu)}, \\ \text{(c)} & \| |D^2u| \|_{L^2(\Omega, \mu)} + \| \langle (D^2\Phi_\alpha)Du, Du \rangle \|_{L^1(\Omega, \mu)} \leq \left(2 + \frac{\alpha}{\lambda}\right) \|f\|_{L^2(\Omega, \mu)}, \end{cases}$$

where Φ_α is defined in (2). Using the Lax-Milgram lemma, we find $u \in H_0^1(\Omega, \mu)$ such that

$$\lambda \int_{\Omega} u\psi \, d\mu + \int_{\Omega} \langle Du, D\psi \rangle \, d\mu = \int_{\Omega} f\psi \, d\mu, \quad \forall \psi \in H_0^1(\Omega, \mu).$$

By local elliptic regularity, $u \in H_{\text{loc}}^2(\Omega)$ and $\lambda u - \mathcal{A}u = f$. In particular, $\mathcal{A}u \in L^2(\Omega, \mu)$. Again, by classical elliptic regularity,

$$u \in C^{2,\beta}(\Omega \cap B(0, R)) \cap H^3(\Omega \cap B(0, R))$$

for every $R > 0$ and $\beta < 1$.

Now we can prove (7). To prove estimates (a) and (b), we multiply the identity $\lambda u - \mathcal{A}u = f$ by u , we integrate over Ω and we use (4) to get

$$\int_{\Omega} (\lambda u^2 + |Du|^2) d\mu = \int_{\Omega} f u d\mu \leq \|f\|_{L^2(\Omega, \mu)} \|u\|_{L^2(\Omega, \mu)}$$

which implies that (a) and (b) hold. To prove (c) we differentiate the equation $\lambda u - \mathcal{A}u = f$ with respect to x_h , $h = 1, \dots, N$, and we get

$$\lambda D_h u - \Delta(D_h u) + \langle D(D_h \Phi), Du \rangle + \langle D\Phi, D(D_h u) \rangle = D_h f,$$

that is,

$$\lambda D_h u - \mathcal{A}D_h u + \sum_{k=1}^N D_{hk} \Phi D_k u = D_h f.$$

Set $\theta_R(x) = \theta(x/R)$. Multiplying by $\theta_R^2 D_h u$, summing over h , and integrating by parts, from (5) we get, since $u \in H^3(\Omega \cap B(0, R))$ for every R ,

$$(8) \quad \int_{\Omega} \left\{ \theta_R^2 (\lambda |Du|^2 + |D^2 u|^2 + \langle D^2 \Phi Du, Du \rangle) + 2 \sum_{h=1}^N \theta_R \langle D(D_h u), D\theta_R \rangle D_h u \right\} d\mu \\ = \int_{\partial\Omega} \theta_R^2 \sum_{h=1}^N \frac{\partial D_h u}{\partial n} D_h u e^{-\Phi} d\sigma + \int_{\Omega} \theta_R^2 \langle Df, Du \rangle d\mu.$$

Since f has compact support, for R large enough $\theta_R \equiv 1$ on the support of f . Using (4) again in the last integral, we write it as $-\int_{\Omega} f(\lambda u - f) d\mu$. Thanks to the assumption $D^2 \Phi \geq -\alpha I$, we obtain

$$(9) \quad \int_{\Omega} \theta_R^2 (\lambda |Du|^2 + |D^2 u|^2 + \langle (D^2 \Phi_{\alpha}) Du, Du \rangle) d\mu \\ \leq \int_{\Omega} (\alpha \theta_R^2 |Du|^2 + CR^{-1} \theta_R |D^2 u| |Du| + f(\lambda u - f)) d\mu \\ + \int_{\partial\Omega} \theta_R^2 \langle (D^2 u)n, Du \rangle e^{-\Phi} d\sigma,$$

for a suitable $C > 0$, independent of R . Using (a) and (b) we get

$$\int_{\Omega} (\alpha \theta_R^2 |Du|^2 + f(\lambda u - f)) d\mu \leq \left(2 + \frac{\alpha}{\lambda}\right) \|f\|_{L^2(\Omega, \mu)}^2.$$

Moreover,

$$\int_{\Omega} CR^{-1} \theta_R |D^2 u| |Du| d\mu \leq \frac{C}{2R} \int_{\Omega} \theta_R^2 |D^2 u|^2 d\mu + \frac{C}{2R} \int_{\Omega} |Du|^2 d\mu.$$

Let us now show that the boundary integral in (9) is negative. Since $u = 0$ on $\partial\Omega$, we have $\langle Du, \tau \rangle = 0$ and $\langle (D^2 u)\tau, \tau \rangle = 0$ for every tangent vector τ to $\partial\Omega$. Then $Du = (\partial u / \partial n)n$ and $\langle (D^2 u)n, Du \rangle = \langle (D^2 u)n, n \rangle \partial u / \partial n$ at $\partial\Omega$. Therefore

$\Delta u = \text{trace } D^2u = \langle (D^2u)n, n \rangle$ at $\partial\Omega$, and the equality $\lambda u - \mathcal{A}u = f$ (which is satisfied also at $\partial\Omega$, since $u \in C^2(\bar{\Omega} \cap B(0, R))$ for every R) yields

$$\Delta u = \frac{\partial\Phi}{\partial n} \frac{\partial u}{\partial n}, \quad \langle (D^2u)n, Du \rangle = \frac{\partial\Phi}{\partial n} \left(\frac{\partial u}{\partial n} \right)^2, \quad \text{at } \partial\Omega,$$

hence

$$\int_{\partial\Omega} \theta_R^2 \langle (D^2u)n, Du \rangle e^{-\Phi} d\sigma = \int_{\partial\Omega} \theta_R^2 \frac{\partial\Phi}{\partial n} \left(\frac{\partial u}{\partial n} \right)^2 e^{-\Phi} d\sigma \leq 0,$$

thanks to (6). Thus, we have proved that

$$\int_{\Omega} \left(1 - \frac{C}{2R} \right) \theta_R^2 |D^2u|^2 + \theta_R^2 \langle (D^2\Phi_\alpha)Du, Du \rangle d\mu \leq \left(2 + \frac{\alpha}{\lambda} + \frac{C}{2R\lambda} \right) \|f\|_{L^2(\Omega, \mu)}^2,$$

and statement (c) follows by letting $R \rightarrow \infty$.

The general case $f \in L^2(\Omega, \mu)$ is easily handled by approximation. Let $(f_n) \subset C_0^\infty(\Omega)$ be such that $f_n \rightarrow f$ in $L^2(\Omega, \mu)$ and let $u_n \in D(A)$ be such that $\lambda u_n - \mathcal{A}u_n = f_n$. The above estimates imply that the sequence (u_n) converges to a function u in $H^2(\Omega, \mu)$ and it is readily seen that $u \in D(A)$, $\lambda u - \mathcal{A}u = f$ and that (a), (b), and (c) hold. \square

Condition (6) can be relaxed assuming some more regularity on $\partial\Omega$.

Theorem 2.5. *Assume that $\partial\Omega \in C^3$ and that it is uniformly C^2 . Let Φ be a C^2 function satisfying (2) and*

$$(10) \quad \frac{\partial\Phi}{\partial n} \leq k \quad \text{at } \partial\Omega,$$

for some $k \in \mathbb{R}$. Then $(A, D(A))$ is self-adjoint and dissipative in $L^2(\Omega, \mu)$. Moreover, the map $u \mapsto \langle (D^2\Phi)Du, Du \rangle$ is continuous from $D(A)$ to $L^1(\Omega, \mu)$.

Proof. The proof is similar to the proof of Theorem 2.4. For $f \in C_0^\infty(\Omega)$, $\lambda > 0$, let $u \in H_0^1(\Omega, \mu)$ be the variational solution of the equation $\lambda u - \mathcal{A}u = f$. As in Theorem 2.4 we get estimates (7)(a), (b) and

$$(11) \quad \begin{aligned} & \int_{\Omega} \left(1 - \frac{C}{2R} \right) \theta_R^2 |D^2u|^2 d\mu + \int_{\Omega} \theta_R^2 \langle (D^2\Phi_\alpha)Du, Du \rangle d\mu \\ & \leq \left(2 + \frac{\alpha}{\lambda} + \frac{C}{2R\lambda} \right) \|f\|_{L^2(\Omega, \mu)}^2 + \int_{\partial\Omega} \theta_R^2 \frac{\partial\Phi}{\partial n} \left(\frac{\partial u}{\partial n} \right)^2 e^{-\Phi} d\sigma. \end{aligned}$$

The boundary integral does not exceed

$$k \int_{\partial\Omega} \theta_R^2 \left(\frac{\partial u}{\partial n} \right)^2 e^{-\Phi} d\sigma,$$

and it can be estimated as follows (see also Lemma 2.3).

Let us take $\psi = \theta_R^2 \langle Du, \mathcal{N} \rangle$ in (5), where \mathcal{N} is any C_b^1 extension to \mathbb{R}^N of the normal vector field n , so that, using Hölder inequality, we obtain for every $0 < \varepsilon < 1$

$$\int_{\partial\Omega} \theta_R^2 \left(\frac{\partial u}{\partial n} \right)^2 e^{-\Phi} d\sigma \leq \varepsilon (\|\mathcal{A}u\|_{L^2(\Omega, \mu)}^2 + \|\theta_R |D^2u|\|_{L^2(\Omega, \mu)}^2) + \frac{C}{\varepsilon} \|Du\|_{L^2(\Omega, \mu)}^2.$$

Since $\mathcal{A}u = \lambda u - f$, writing the last inequality with $\varepsilon k \leq 1/2$ and combining it with (11) and with estimates (a), (b), we arrive at

$$\begin{aligned} \int_{\Omega} \left(\frac{1}{2} - \frac{C}{2R}\right) \theta_R^2 |D^2 u|^2 d\mu + \int_{\Omega} \theta_R^2 \langle (D^2 \Phi_{\alpha}) Du, Du \rangle d\mu \\ \leq \left(2 + \frac{\alpha}{\lambda} + \frac{C}{2R\lambda} + C_1\right) \|f\|_{L^2(\Omega, \mu)}^2, \end{aligned}$$

with C_1 independent of R . Letting $R \rightarrow \infty$ we obtain estimate (7)(c) of Theorem 2.4 (with different constants), and from now on the proof follows the same lines as in Theorem 2.4. \square

Remark 2.6. If $D^2 \Phi$ is bounded from above, then the mapping $u \mapsto \langle (D^2 \Phi) Du, Du \rangle$ is bounded from $H^1(\Omega, \mu)$ to $L^1(\Omega, \mu)$ and the last statement of Theorems 2.4 and 2.5 is obvious. But, if $D^2 \Phi$ is not bounded, the statement is not obvious, and it will be used in the next section to obtain a quantitative Poincaré inequality.

We end this section by showing that $D(A)$ can be strictly contained in $H^2(\Omega, \mu) \cap H_0^1(\Omega, \mu)$.

Example 2.7. We construct a convex function $\phi : [0, \infty) \rightarrow \mathbb{R}$ such that $e^{-\phi}$ and $x^2 e^{-\phi}$ are in $L^1(0, \infty)$ but $\phi'^2 e^{-\phi} \notin L^1(0, \infty)$. Then $u(x) = x$ belongs to $H^2(\mu) \cap H_0^1(\mu)$ but not to $D(A)$. For simplicity, ϕ will be nonsmooth. However, smooth versions are easily obtained using straightforward arguments.

Let $0 = a_1 < b_1 < a_2 < b_2 < \dots$ be points in $[0, \infty)$ such that $b_j - a_j = 1$. Set $1/l_j = a_{j+1} - b_j$, $l_1 = 1$ and define $\phi' = 1$ in (a_1, b_1) , $\phi' = l_j$ in (b_j, a_{j+1}) and $\phi' = l_{j-1}$ in (a_j, b_j) . We have to choose $1 = l_1 < l_2 < \dots$ in such a way that ϕ satisfies the properties above. First observe that ϕ is convex, $\phi' \geq 1$, hence $\phi(x) \geq x$ and then $e^{-\phi}, x^2 e^{-\phi} \in L^1(0, \infty)$. Moreover, if $x \in (b_j, a_{j+1})$, then $\phi(x) \leq j + 1 + \sum_{i=1}^{j-1} l_i$ and therefore

$$\int_{b_j}^{a_{j+1}} \phi'^2 e^{-\phi} dx \geq l_j^2 \exp\left(-\left(j + 1 + \sum_{i=1}^{j-1} l_i\right)\right) \geq l_j \exp\left(-\left(j + 1 + \sum_{i=1}^{j-1} l_i\right)\right).$$

Choosing (inductively) $l_j = e^{(j+1+\sum_{i=1}^{j-1} l_i)}$ the above integral is bigger than 1, hence, summing over j , ϕ'^2 does not belong to $L^1(\mu)$.

3. FURTHER PROPERTIES OF A

Under the assumptions of either Theorem 2.4 or Theorem 2.5, since the operator A is self-adjoint and dissipative in $L^2(\Omega, \mu)$, it is the infinitesimal generator of an analytic contraction semigroup $T(t)$ in $L^2(\Omega, \mu)$. In this section we prove further properties of $T(t)$ and of A .

The characterisation of the domain of $(-A)^{1/2}$ is a standard consequence of the integration formula (4), as the following proposition shows. Recall that the norm in $H_0^1(\Omega, \mu)$ is given by $\|u\|_{H_0^1(\Omega, \mu)} = \|u\|_{L^2(\Omega, \mu)} + \|Du\|_{L^2(\Omega, \mu)}$.

Proposition 3.1. *The domain of $(-A)^{1/2}$ is $H_0^1(\Omega, \mu)$. Therefore, the restriction of $T(t)$ to $H_0^1(\Omega, \mu)$ is an analytic semigroup in $H_0^1(\Omega, \mu)$.*

Proof. Any $u \in D((-A)^{1/2})$ is the $L^2(\Omega, \mu)$ -limit of a sequence of functions $u_n \in D(A) \subset H_0^1(\Omega, \mu)$ which is a Cauchy sequence with respect to the norm $\|u\|_{L^2} + \langle -Au, u \rangle_{L^2}$. From (4) it follows that (Du_n) is a Cauchy sequence in $L^2(\Omega, \mu)$,

hence $u \in H_0^1(\Omega, \mu)$. Conversely, let $u \in H_0^1(\Omega, \mu)$ and let $u_n \in C_0^\infty(\Omega) \subset D(A)$ converge to u in $H^1(\Omega, \mu)$. Formula (4) implies that (u_n) is a Cauchy sequence in $D((-A)^{1/2})$, hence $u \in D((-A)^{1/2})$. \square

Corollary 3.2. *Under the assumptions of either Theorem 2.4 or Theorem 2.5, $T(t)$ is a symmetric Markov semigroup, that is, a semigroup of self-adjoint positivity preserving operators in $L^2(\Omega, \mu)$ that satisfy $\|T(t)f\|_\infty \leq \|f\|_\infty$ for each $f \in L^2(\Omega, \mu) \cap L^\infty(\Omega, \mu)$ and $t > 0$.*

Proof. Since A is self-adjoint, each $T(t)$ is self-adjoint. To prove that each $T(t)$ preserves positivity and that it is a contraction in L^∞ , we use the Beurling-Deny criteria; see e.g. [4, Theorems 1.3.2, 1.3.3].

As $D((-A)^{1/2}) = H_0^1(\Omega, \mu)$, then $u \in D((-A)^{1/2})$ implies $|u| \in D((-A)^{1/2})$, and

$$\|(-A)^{1/2}(|u|)\|^2 = \int_\Omega |D(|u|)|^2 d\mu \leq \int_\Omega |Du|^2 d\mu = \|(-A)^{1/2}u\|^2,$$

so that $T(t)$ is positivity-preserving for all $t > 0$. Again, since $D((-A)^{1/2}) = H_0^1(\Omega, \mu)$, if $0 \leq u \in D((-A)^{1/2})$, then $u \wedge 1 \in D((-A)^{1/2})$, and

$$\|(-A)^{1/2}(u \wedge 1)\|^2 = \int_\Omega |D(u \wedge 1)|^2 d\mu \leq \int_\Omega |Du|^2 d\mu = \|(-A)^{1/2}u\|^2.$$

This implies that $\|T(t)f\|_\infty \leq \|f\|_\infty$ for each $f \in L^2(\Omega, \mu) \cap L^\infty(\Omega, \mu)$. \square

Another immediate consequence of the integration formula (4) is that A is injective: if $u \in D(A)$ and $Au = 0$, then $Au \cdot u = 0$, and integrating over Ω we obtain $Du = 0$ so that u is constant on each connected component of Ω ; since u vanishes at $\partial\Omega$, then $u = 0$.

A natural question is now whether 0 is in the resolvent set of A . This is true if $D(A)$ is compactly embedded in $L^2(\Omega, \mu)$, because in this case the spectrum of A consists of a sequence of isolated eigenvalues. But in general $D(A)$ is not compactly embedded in $L^2(\Omega, \mu)$, as the following counterexample shows.

Example 3.3. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be any convex C^2 function such that $\varphi(x) = x$ for $x \geq 0$. Set $\Phi(x, y) = \varphi(x) + y^2$, and let Ω be the half-plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$. Then $D(A)$ is not compactly embedded in $L^2(\Omega, \mu)$.

Proof. Let $\theta \in C_0^\infty(0, \infty)$ be such that $\int_0^\infty (\theta(y))^2 \exp(-y^2) dy = 1$, and set for each $n \in \mathbb{N}$, $n \geq 3$,

$$u_n(x, y) = \frac{x^n}{\sqrt{(2n)!}} \theta(y), \quad x, y \geq 0, \quad u_n(x, y) = 0 \text{ otherwise.}$$

Since $d\mu = \exp(-\varphi(x) - y^2) dx dy$, then $\|u_n\|_{L^2(\Omega, \mu)} = 1$ for each n . Moreover,

$$\begin{aligned} D_x u_n(x, y) &= \frac{n x^{n-1}}{\sqrt{(2n)!}} \theta(y), & D_y u_n(x, y) &= \frac{x^n}{\sqrt{(2n)!}} \theta'(y), & x > 0, \\ D_{xx} u_n(x, y) &= \frac{n(n-1)x^{n-2}}{\sqrt{(2n)!}} \theta(y), & D_{yy} u_n(x, y) &= \frac{x^n}{\sqrt{(2n)!}} \theta''(y), & x > 0, \end{aligned}$$

and every derivative vanishes for $x \leq 0$. Therefore, $u_n \in D(A)$ and $\|Au_n\|_{L^2(\Omega, \mu)}$ is bounded by a constant independent of n . But no subsequence may converge in

$L^2(\Omega, \mu)$ because

$$\|u_n - u_m\|_{L^2(\Omega, \mu)}^2 = \int_0^\infty \left(\frac{x^{2n}}{(2n)!} + \frac{x^{2m}}{(2m)!} - 2 \frac{x^{n+m}}{\sqrt{(2n)!(2m)!}} \right) e^{-x} dx = 2 - 2 \frac{(n+m)!}{\sqrt{(2n)!(2m)!}}$$

and for any fixed n we have

$$\lim_{m \rightarrow \infty} \frac{(n+m)!}{\sqrt{(2n)!(2m)!}} = 0, \text{ so that } \lim_{m \rightarrow \infty} \|u_n - u_m\|_{L^2(\Omega, \mu)}^2 = 2. \quad \square$$

In the above example $D_x \Phi(x, y)$ is bounded for $x > 0$, and the question of whether $D(A)$ is compactly embedded in $L^2(\Omega, \mu)$ if $|D\Phi|$ goes to ∞ as $|x| \rightarrow \infty$ remains open. In the next proposition we show that the answer is positive if Φ satisfies an additional (mild) nonoscillation condition.

Proposition 3.4. *Assume that $\Phi \in C^2(\mathbb{R}^N)$ satisfies $\Delta\Phi \leq a|D\Phi|^2 + b$ for some $a < 1, b \in \mathbb{R}$. Then the map $u \mapsto |D\Phi|u$ is bounded from $H_0^1(\Omega, \mu)$ to $L^2(\Omega, \mu)$. If, in addition, $|D\Phi| \rightarrow \infty$ at infinity, the embedding of $H_0^1(\Omega, \mu)$ (hence that of $D(A)$) in $L^2(\Omega, \mu)$ is compact.*

Proof. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega, \mu)$ it is sufficient to show that

$$\| |D\Phi|u \|_{L^2(\Omega, \mu)} \leq C \|u\|_{H^1(\Omega, \mu)}$$

for some $C > 0$ and every $u \in C_0^\infty(\Omega, \mu)$. Integrating by parts and using Young’s inequality we get for every $\varepsilon > 0$ and for a suitable C_ε

$$\begin{aligned} \int_\Omega |u|^2 |D\Phi|^2 d\mu &= - \int_\Omega |u|^2 \langle D\Phi, De^{-\Phi} \rangle dx \\ &= \int_\Omega |u|^2 \Delta\Phi e^{-\Phi} dx + 2 \int_\Omega u \langle D\Phi, Du \rangle e^{-\Phi} dx \\ &\leq (a + \varepsilon) \int_\Omega |u|^2 |D\Phi|^2 d\mu + C_\varepsilon \int_\Omega |Du|^2 d\mu + b \int_\Omega |u|^2 d\mu. \end{aligned}$$

Choosing ε such that $a + \varepsilon < 1$, the first statement follows. Concerning the second one, we observe that for each $\varepsilon > 0$ there is $R > 0$ such that $|D\Phi| \geq 1/\varepsilon$ in $\Omega \setminus B(0, R)$. Hence for every u in the unit ball B of $H_0^1(\Omega)$ we have

$$\frac{1}{\varepsilon^2} \int_{\Omega \setminus B(0, R)} |u|^2 d\mu \leq \int_{\Omega \setminus B(0, R)} |u|^2 |D\Phi|^2 d\mu \leq C^2.$$

Since the embedding of $H^1(\Omega \cap B(0, R))$ into $L^2(\Omega \cap B(0, R))$ is compact, we can find $\{f_1, \dots, f_k\} \subset L^2(\Omega \cap B(0, R))$ such that the balls $B(f_i, \varepsilon) \subset L^2(\Omega \cap B(0, R))$ cover the restrictions of the functions of B to $\Omega \cap B(0, R)$. Denoting by \tilde{f}_i the zero-extension of f_i to the whole of Ω , it follows that $B \subset \bigcup_{i=1}^k B(\tilde{f}_i, (C + 1)\varepsilon)$, and the proof is complete. \square

The compactness of the resolvent is a consequence of the logarithmic Sobolev inequality

$$(12) \quad \int_\Omega u^2 \log(|u|) d\mu \leq \frac{1}{\omega} \int_\Omega |Du|^2 d\mu + \|u\|_{L^2(\Omega, \mu)}^2 \log(\|u\|_{L^2(\Omega, \mu)}),$$

for all $u \in H_0^1(\Omega, \mu)$ and some $\omega > 0$ (where we set $0 \log 0 = 0$).

In what follows we give sufficient conditions for the validity of (12).

Proposition 3.5. *Let us denote by $\lambda(x)$ the smallest eigenvalue of the matrix $D^2\Phi(x)$. Then:*

- (i) *if $\lambda(x) \geq \omega_0$ for all $x \in \mathbb{R}^N$ then (12) holds with $\omega = \omega_0$;*
- (ii) *if $\liminf_{|x| \rightarrow \infty} \lambda(x) > 0$, then (12) holds for some $\omega > 0$.*

Proof. (i) Let $u \in H_0^1(\Omega, \mu)$ and extend u outside Ω by setting $u(x) = 0$ for $x \notin \Omega$. Then the extension is in $H^1(\mathbb{R}^N, \nu)$, where $d\nu(x) = c \exp(-\Phi(x)) dx$, $c^{-1} = \int_{\mathbb{R}^N} \exp(-\Phi) dx \geq 1$. By [3], for each $u \in H^1(\mathbb{R}^N, \nu)$ we have

$$\int_{\mathbb{R}^N} |u|^2 \log |u| d\nu \leq \frac{1}{\omega_0} \int_{\mathbb{R}^N} |Du|^2 d\nu + \|u\|_{L^2(\mathbb{R}^N, \nu)}^2 \log(\|u\|_{L^2(\mathbb{R}^N, \nu)}).$$

Since u vanishes outside Ω we easily get

$$\int_{\Omega} |u|^2 \log |u| d\mu \leq \frac{1}{\omega_0} \int_{\Omega} |Du|^2 d\mu + \|u\|_{L^2(\Omega, \mu)}^2 \left(\frac{1}{2} \log c + \log(\|u\|_{L^2(\Omega, \mu)}) \right)$$

and (12) follows since $c \leq 1$.

- (ii) The proof is similar to (i), using [11, Theorem 1.3] instead of [3]. □

Corollary 3.6. *Under the assumptions of Proposition 3.5, $H_0^1(\Omega, \mu)$ is compactly embedded in $L^2(\Omega, \mu)$. Therefore, $\sup \sigma(A) < 0$. Moreover $T(t)$ maps $L^2(\Omega, \mu)$ into $L^{q(t)}(\Omega, \mu)$ with $q(t) = 1 + e^{\omega t}$, and*

$$(13) \quad \|T(t)f\|_{L^{q(t)}(\Omega, \mu)} \leq \|f\|_{L^2(\Omega, \mu)}, \quad t > 0, f \in L^2(\Omega, \mu).$$

Proof. Let B be the unit ball of $H_0^1(\Omega, \mu)$. Inequality (12) yields the existence of a positive constant C such that $\int_{\Omega} |u|^2 d\mu \leq C$ for every $u \in B$. Given $t \geq 1$, let $E = \{|u| < t\}$. Then for $R > 0$

$$\begin{aligned} \int_{\Omega \setminus B(0, R)} |u|^2 d\mu &\leq \int_{(\Omega \setminus B(0, R)) \cap E} t^2 d\mu + \frac{1}{\log t} \int_{(\Omega \setminus B(0, R)) \setminus E} |u|^2 \log |u| d\mu \\ &\leq t^2 \mu(\Omega \setminus B(0, R)) + \frac{C}{\log t} \end{aligned}$$

hence, given $\varepsilon > 0$, there exists $R > 0$ such that $\int_{\Omega \setminus B(0, R)} |u|^2 d\mu \leq \varepsilon$ for every $u \in B$. As in Proposition 3.4, this proves that $H_0^1(\Omega, \mu)$ is compactly embedded in $L^2(\Omega, \mu)$. The fact that $T(t)$ maps $L^2(\Omega, \mu)$ into $L^{q(t)}(\Omega, \mu)$, as well as estimate (13), follow from [5, 6]. □

A necessary and sufficient condition in order that 0 be in the resolvent of A is that the Poincaré inequality holds, i.e.,

$$(14) \quad \int_{\Omega} |u|^2 d\mu \leq \frac{1}{\omega} \int_{\Omega} |Du|^2 d\mu, \quad u \in H_0^1(\Omega, \mu),$$

for some $\omega > 0$. More precisely, since A is self-adjoint, then $\langle (-A - \omega I)u, u \rangle \geq 0$ for each $u \in D(A)$ if and only if $\sigma(A + \omega I) \subset (-\infty, 0]$. In other words, (14) holds for each $u \in D(A)$ (or, equivalently, for each $u \in H_0^1(\Omega, \mu) = D((-A)^{1/2})$) if and only if $\sigma(A) \subset (-\infty, -\omega]$. In this case we have

$$(15) \quad \|T(t)f\|_{L^2(\Omega, \mu)} \leq e^{-\omega t} \|f\|_{L^2(\Omega, \mu)}, \quad t > 0, f \in L^2(\Omega, \mu).$$

Indeed, for each $t > 0$ and $f \in L^2(\Omega, \mu)$,

$$\frac{d}{dt} \|T(t)f\|^2 = \int_{\Omega} 2AT(t)f \cdot T(t)f d\mu = -2\|DT(t)f\|^2 \leq -2\omega \|T(t)f\|^2.$$

If $\Omega = \mathbb{R}^N$, the Poincaré inequality for functions having zero mean is a consequence of the logarithmic Sobolev inequality (in which case $D(A)$ is compactly embedded in $L^2(\Omega, \mu)$) and the constant ω in (14) is the same as in (12); see [10]. This is not true in our setting; see Example 3.9 below. However, in the next proposition we show how to get an explicit estimate of ω in (14) when (6) holds.

Proposition 3.7. *Assume that (6) holds and that there exists $\omega_0 > 0$ such that the map $x \mapsto \Phi(x) - \omega_0|x|^2/2$ is convex. Then (14) holds with $\omega = \omega_0$.*

Proof. We have only to show that $\sigma(A) \subset (-\infty, -\omega_0]$. Corollary 3.6 yields that the resolvent of A is compact, hence $\sigma(A)$ consists of eigenvalues. If $\lambda u - Au = 0$ for some $\lambda \in \mathbb{R}$ and $0 \neq u \in D(A)$, we write (8) with $f = 0$ and let $R \rightarrow \infty$. Since the boundary integral is nonpositive and $D^2\Phi \geq \omega_0 I$ we get $(\lambda + \omega_0) \int_{\Omega} |Du|^2 \leq 0$. Since u is not a constant, then $Du \neq 0$ and $\lambda \leq -\omega_0$. This concludes the proof. \square

Let us again consider Example 3.3 and show that, in general, the Poincaré inequality does imply that the embedding $D(A) \subset L^2(\Omega, \mu)$ is compact.

Example 3.8. We use the same notation as in Example 3.3. Proposition 3.7, applied to the one-dimensional function $y \mapsto y^2$, $y > 0$, yields

$$\int_0^\infty |u(x, y)|^2 e^{-y^2} dy \leq \frac{1}{2} \int_0^\infty |D_y u(x, y)|^2 e^{-y^2} dy, \quad \text{a.e. } x \in \mathbb{R}, u \in H_0^1(\Omega, \mu).$$

Multiplying by $e^{-\phi(x)}$ and integrating with respect to $x \in \mathbb{R}$, we deduce

$$\int_{\Omega} |u(x, y)|^2 d\mu \leq \frac{1}{2} \int_{\Omega} |D_y u(x, y)|^2 d\mu$$

so that the Poincaré inequality holds, even if $D(A)$ is not compactly embedded in $L^2(\Omega, \mu)$, as we have shown in Example 3.3.

If assumption (6) is replaced by the boundedness of $\partial\Phi/\partial n$ at $\partial\Omega$ and still $\Phi(x) - \omega^2|x|^2$ is convex, the constant ω in (14) may also depend on the constant k in (10), as we show in the following example.

Example 3.9. Let $N = 1$ and let $\mathcal{A}u = u'' - xu'$ be the Ornstein-Uhlenbeck operator. Here $\Phi(x) = x^2/2$, hence $D^2\Phi \equiv 1$ and (12) holds with $\omega = 1$. Let $\Omega_a = (-\infty, a)$ and set $u(x) = a - x$. Then $u \in D(A)$ and

$$\int_{-\infty}^a |u'|^2 d\mu \left(\int_{-\infty}^a |u|^2 d\mu \right)^{-1} \rightarrow 0$$

as $a \rightarrow \infty$. This shows that the spectrum of A in $L^2(\Omega_a, \mu)$ is not contained in $(-\infty, -1]$ for large a , hence the constant ω in (14) is smaller than 1.

REFERENCES

- [1] M. BERTOLDI, S. FORNARO: *Gradient estimates in parabolic problems with unbounded coefficients*, *Studia Math.* **165** (2004), 221–254. MR2109509
- [2] S. CERRAI: *Second order PDE's in finite and infinite dimensions. A probabilistic approach*, *Lecture Notes in Mathematics* **1762**, Springer-Verlag, Berlin (2001). MR1840644 (2002j:35327)
- [3] G. DA PRATO, A. LUNARDI: *Elliptic operators with unbounded drift coefficients and Neumann boundary condition*, *J. Diff. Eqns.* **198** (2004), 35–52. MR2037749 (2004k:35066)
- [4] E. B. DAVIES: *Heat Kernels and Spectral Theory*, Cambridge U. P. (1989). MR0990239 (90e:35123)

- [5] L. GROSS: *Logarithmic Sobolev Inequalities*, Amer. J. Math. **97** (1975), 1061–1083. MR0420249 (54:8263)
- [6] L. GROSS: *Logarithmic Sobolev Inequalities and Contractivity Properties of Semigroups*, in: Dirichlet Forms (G. Dell’Antonio and U. Mosco Eds.), Lecture Notes in Mathematics **1563**, Springer-Verlag, Berlin (1993), 54–88. MR1292277 (95h:47061)
- [7] G. METAFUNE, D. PALLARA, V. VESPRI: *L^p -estimates for a class of elliptic operators with unbounded coefficients in \mathbb{R}^N* , Houston J. Math. **31** (2005), 605–620.
- [8] G. METAFUNE, A. RHANDI, J. PRÜSS, R. SCHNAUBELT: *L^p regularity for elliptic operators with unbounded coefficients*, preprint.
- [9] P. RABIER: *Elliptic problems on \mathbb{R}^N with unbounded coefficients in classical Sobolev spaces*, Math. Z. **249** (2005), 1–30. MR2106968
- [10] O.S. ROTHHAUS: *Logarithmic Sobolev Inequalities and the Spectrum of Schrödinger operators*, J. Funct. Anal. **42** (1981), 110–120. MR0620582 (83f:58080b)
- [11] FENG-YU WANG: *Logarithmic Sobolev inequalities: conditions and counterexamples*, J. Operator Theory **46** (2001), 183–197. MR1862186 (2003g:58060)

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PARMA, PARCO AREA DELLE SCIENZE 53, 43100 PARMA, ITALY

E-mail address: `lunardi@unipr.it`

DIPARTIMENTO DI MATEMATICA “ENNIO DE GIORGI”, UNIVERSITÀ DI LECCE, C.P.193, 73100, LECCE, ITALY

E-mail address: `giorgio.metafune@unile.it`

DIPARTIMENTO DI MATEMATICA “ENNIO DE GIORGI”, UNIVERSITÀ DI LECCE, C.P.193, 73100, LECCE, ITALY

E-mail address: `diego.pallara@unile.it`