A CAUCHY-SCHWARZ TYPE INEQUALITY
FOR BILINEAR INTEGRALS ON POSITIVE MEASURES

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Abstract. If \( W : \mathbb{R}^n \to [0, \infty] \) is Borel measurable, define for \( \sigma \)-finite positive Borel measures \( \mu, \nu \) on \( \mathbb{R}^n \) the bilinear integral expression
\[
I(W; \mu, \nu) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} W(x - y) \, d\mu(x) \, d\nu(y).
\]
We give conditions on \( W \) such that there is a constant \( C \geq 0 \), independent of \( \mu \) and \( \nu \), with
\[
I(W; \mu, \nu) \leq C \sqrt{I(W; \mu, \mu) I(W; \nu, \nu)}.
\]
Our results apply to a much larger class of functions \( W \) than known before.

1. Introduction and results

Given a Borel function \( W : \mathbb{R}^n \to [0, \infty] \), for \( \sigma \)-finite positive measures \( \mu, \nu \) on \( \mathbb{R}^n \) define the integral
\[
I(W; \mu, \nu) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} W(x - y) \, d\mu(x) \, d\nu(y).
\]
Denote for \( C \geq 0 \) by \( \mathcal{W}(n, C) \) the class of Borel functions \( W : \mathbb{R}^n \to [0, \infty] \) such that for all \( \sigma \)-finite positive measures \( \mu, \nu \) on \( \mathbb{R}^n \)
\[
I(W; \mu, \nu) \leq C \sqrt{I(W; \mu, \mu) I(W; \nu, \nu)} \tag{1.1}
\]
holds. Moreover, denote
\[
\mathcal{W}(n) := \bigcup_{C \geq 0} \mathcal{W}(n, C).
\]
If \( W \) is an even function and the symmetric bilinear form \( I(W; \cdot, \cdot) \) is positive semidefinite, then \( W \in \mathcal{W}(n, 1) \) (Cauchy-Schwarz inequality). Hence we may regard (1.1) as a generalized form of the Cauchy-Schwarz inequality.

An even function \( W \) such that \( I(W; \cdot, \cdot) \) is positive semidefinite is called positive definite. Roughly speaking, positive definiteness of a function corresponds to nonnegativity of its Fourier transform \([0, \infty]\). The only result regarding (1.1) the author is aware of that goes beyond positive definite functions is given by Mattner \([\mathbf{4} \text{ Sect. 5.1}]\): If \( \| \cdot \| \) is any norm on \( \mathbb{R}^n \), \( h : [0, \infty) \to [0, \infty] \) is decreasing, and \( W \) is given by \( W(x) := h(\|x\|) \), then \( W \in \mathcal{W}(n) \). Theorem \([\mathbf{5}]\) below recovers this...
statement and extends it by allowing $h$ to be nonmonotone. Theorem 1.2, the main result of the present paper, yields a criterion for membership in $W(n)$ for functions $W$ that cannot be written as $h \circ p$ with a seminorm $p$ on $\mathbb{R}^n$.

The study of property (1.1) is motivated by the partial differential equation
\begin{equation}
-\Delta u + Vu = (W * u^2)u, \quad u \in H^1(\mathbb{R}^n).
\end{equation}
Here $*$ denotes convolution, $V$ in $L^\infty(\mathbb{R}^n)$ is periodic, and $0$ lies in a gap of the spectrum of $(-\Delta + V)$; cf. [1]. One is interested in the existence of nontrivial solutions to (1.2). For the special case $n = 3$ and $W(x) = 1/\|x\|_2$ the problem was settled in [2] by using the fact that this particular function $W$ is positive definite. In [1] it is shown that $W \in W(n)$ (together with appropriate growth conditions) is sufficient to obtain a nontrivial solution of (1.2).

1.1. Main results. The statement of our theorems requires us to introduce some notation and definitions. For a topological space $X$ denote by $\mathcal{P}(X)$ the set of Borel functions $f : X \to [0, \infty]$. For $n \in \mathbb{N}$ denote by $\mathcal{C}(n)$ the class of subsets of $\mathbb{R}^n$ that are closed, convex, and symmetric (i.e. $-A = A$). The dimension $\dim A$ of a convex subset $A$ of $\mathbb{R}^n$ is the dimension of the affine hull of $A$.

**Definition 1.1.** For $X, A \subseteq \mathbb{R}^n$, $X \neq \emptyset$, put
\[ \kappa(X, A) := \inf\{ m \in \mathbb{N} \mid X \text{ can be covered by } m \text{ translates of } A \} \]
and
\[ \alpha(X) := \inf\{ m \in \mathbb{N} \mid \exists A \in \mathcal{C}(n) : \dim A = n, \ A \subseteq X \text{ and } \kappa(X, A) = m \} . \]
For $X = \emptyset$ set $\kappa(\emptyset, A) := 0$ and $\alpha(\emptyset) := 0$.

Given a set $X$, a map $W : X \to \mathbb{R}$ and $t \in \mathbb{R}$ denote
\[ [W]_t := \{ x \in X \mid W(x) \geq t \} . \]
Furthermore, define the class $\mathcal{A}(n)$ by
\[ \mathcal{A}(n) := \left\{ W \in \mathcal{P}(\mathbb{R}^n) \mid \limsup_{t \to 0} \alpha([W]_t) + \limsup_{t \to \infty} \alpha([W]_t) < \infty \right\} . \]

Our main result then reads:

**Theorem 1.2.** For every $n$ in $\mathbb{N}$ the inclusion $\text{conv}(\mathcal{A}(n)) \subseteq W(n)$ holds.

**Remark 1.3.** It will be shown in the proof of Theorem 1.2 that $W(n)$ is a convex cone. Obviously, $\mathcal{A}(n)$ is a cone. The Example 1.6 given below demonstrates that $\mathcal{A}(n)$ is not convex.

The present author does not know whether a function in $W(n)$ that is sufficiently regular, say continuous, must necessarily belong to $\text{conv}(\mathcal{A}(n))$.

A simpler criterion for membership in $W(n)$ can be formulated in the case of the composition of a map with a seminorm. To state it we introduce further concepts and notation.

**Definition 1.4.** For a subset $Y$ of $[0, \infty)$ put $\lambda(Y) := \sup\{ t > 0 \mid [0, t] \subseteq Y \}$ and
\[ \beta(Y) := \begin{cases} 0, & Y = \emptyset, \\ \infty, & \lambda(Y) = -\infty \text{ and } Y \neq \emptyset, \\ \sup(Y)/\lambda(Y), & \text{otherwise}. \end{cases} \]
Here we set $\infty/a := \infty$ if $a > 0$ and $\infty/\infty := 1$. 
We introduce

\[ B := \left\{ h \in \mathcal{P}([0, \infty)) \ \middle| \ \limsup_{t \to 0} \beta([h]_t) + \limsup_{t \to \infty} \beta([h]_t) < \infty \right\}. \]

Our second result then reads:

**Theorem 1.5.** Suppose that \( h \in \mathcal{P}([0, \infty)) \) and that \( p \) is a seminorm on \( \mathbb{R}^n \). If \( h \in B \), then \( h \circ p \in \mathcal{A}(n) \). If \( h \circ p \in \mathcal{A}(n) \) and \( \text{codim}(\ker p) \geq 2 \), then \( h \in B \).

We provide some examples to illustrate the concepts introduced so far:

**Example 1.6.** Denote by \( h \) the characteristic function of \([0, 1]\), taken as a map from \([0, \infty)\) into \([0, \infty]\). Then \( h \in B \). For \( i = 1, 2 \) define \( W_i \) as a map in \( \mathcal{P}(\mathbb{R}^2) \) by \( W_i(x_1, x_2) := h(|x_i|) \). Theorem 1.5 implies that \( W_i \in \mathcal{A}(2) \) for \( i = 1, 2 \), but clearly \( W := W_1 + W_2 \notin \mathcal{A}(2) \). Since \( \mathcal{A}(2) \) is a cone this implies that \( \mathcal{A}(2) \) is not convex. Nevertheless, \( W \in \mathcal{W}(2) \) by Theorem 1.7 and since \( \mathcal{W}(2) \) is a convex cone.

**Example 1.7.** We construct a function \( W \) in \( \mathcal{A}(n) \) that is not even, and hence is neither positive definite nor of the form \( h \circ p \) with \( h \) in \( \mathcal{P}([0, \infty)) \) and \( p \) a seminorm on \( \mathbb{R}^n \). Pick \( x_0 \) in \( \mathbb{R}^n \setminus \{0\} \) and set

\[ W_0(x) := \frac{1}{||x||_2}, \]

\[ W(x) := W_0(x) + W_0(x - x_0). \]

Denoting by \( D(r, x) \) the closed ball of radius \( r > 0 \) with center \( x \), it follows easily that

\[ D(1/t, 0) \subseteq [W]_t \subseteq D(2/t, 0) \cup D(2/t, x_0) \]

for all \( t > 0 \). This implies that \( W \in \mathcal{A}(n) \).

**Example 1.8.** We show that the assumption on \( \text{codim}(\ker p) \) used in Theorem 1.5 is not purely technical. If \( p \) is a seminorm on \( \mathbb{R}^n \) with \( \text{codim}(\ker p) = 0 \), then trivially \( h \circ p \in \mathcal{A}(n) \) for arbitrary \( h \) in \( \mathcal{P}([0, \infty)) \). Given the seminorm \( p(x) := ||x|| \) in \( \mathbb{R} \) with \( \text{codim}(\ker p) = 1 \), we construct \( h \) in \( \mathcal{P}([0, \infty)) \) such that \( W := h \circ p \in \mathcal{A}(1) \) but \( h \notin B \). Put

\[ h(s) := \begin{cases} \infty, & s = 0, \\ \exp(-(k - 1)^2), & s = \exp(k^2) \text{ for some } k \in \mathbb{N}, \\ 1/s, & \text{otherwise}. \end{cases} \]

For \( t > 1 \) we obtain \( [h]_t = [0, 1/t] \), and for \( 0 < t \leq 1 \) we obtain

\[ (1.3) \quad [h]_t = [0, 1/t] \cup \left\{ \exp\left(1 + \left\lfloor \sqrt{-\log t}\right\rfloor^2\right) \right\}. \]

Recall that \( \lfloor a \rfloor \) denotes the largest integer less than or equal to \( a \) if \( a \in \mathbb{R} \). From (1.3) it is clear that \( \alpha([W]_t) \leq 3 \) for all \( t \geq 0 \), so \( W \in \mathcal{A}(1) \). On the other hand, for \( t_k := \exp(-k^2) \) we find

\[ \beta([h]_{t_k}) = \exp((1 + k)^2) \exp(-k^2) = \exp(1 + 2k) \]

and therefore \( \limsup_{t \to 0} \beta([h]_t) = \infty \). Hence \( h \notin B \).
1.2. **General notation.** In $\mathbb{R}^n$ denote by $\|\cdot\|_p$ for $p$ in $[1, \infty]$ the standard $l^p(n)$-norm. In the case of $p = 2$ we write $x \cdot y$ for the standard Euclidean scalar product of elements $x, y$ in $\mathbb{R}^n$. If $V$ is a subspace of $\mathbb{R}^n$, denote by $V^\perp$ the orthogonal subspace with respect to the standard scalar product.

The power set of a set $X$ will be written $2^X$. The cardinality of $X$ is denoted by $|X|$. Some operators used are: $\text{conv} A$ for the convex hull of $A$, $\text{cl} A$, $\text{int} A$, and $\partial A$ for closure, interior, and boundary of a subset $A$ of a topological space.

A parallelotope is a rectangular parallelepiped.

2. **Some convex geometry**

The next lemma allows us to deal with unbounded sets in $C(n)$ in a convenient manner.

**Lemma 2.1.** If $A \in C(n)$, then there is a unique subspace $V$ of $\mathbb{R}^n$ such that $B := A \cap V^\perp \in C(n)$ is compact and $A = B + V$.

**Proof.** First we remark: If a set $A$ in $C(n)$ includes a ray (a set $\{x + ty \mid t \geq 0\}$ for some $x, y$ in $\mathbb{R}^n$), then it includes the 1-dimensional subspace parallel to that ray. If $A$ includes a translate of a subspace $V$ of $\mathbb{R}^n$, then $A$ includes $V$.

Now fix $A$ in $C(n)$. From \[3\] Lemma 2.5.4 we obtain a unique subspace $V$ of $\mathbb{R}^n$ of maximal dimension such that a translate of $V$ and thus $V$ is included in $A$. Moreover, by that lemma it also holds that $B := A \cap V^\perp \in C(n)$ does not include a line (the translate of a 1-dimensional subspace) and $A = B + V$. If $B$ was not bounded, then it included a ray by \[3\] Lemma 2.5.1. Since $B$ is symmetric it therefore included a line also. Contradiction. Since $A$ is closed $B$ must therefore be compact.

If, on the other hand, for some subspace $V$ of $\mathbb{R}^n$, $B = A \cap V^\perp$ is compact and $A = B + V$, then $V$ is included in $A$. If $A$ includes a translate of another subspace $W$, and thus includes $W$, then $W \subseteq V$. Hence $V$ has maximal dimension among the subspaces included in $A$, and it is unique, again by Lemma 2.5.4 loc. cit. \[\square\]

**Definition 2.2.** We call the pair $(B, V)$ given for $A$ in $C(n)$ by Lemma 2.1 the **splitting** of $A$.

**Definition 2.3.** Denote for $X \subseteq \mathbb{R}^n$ by

$$\text{ccs} X := \text{cl} (\text{conv} \frac{1}{2}(X - X)) \in C(n)$$

the closed convex hull of the symmetrization of $X$.

**Remark 2.4.** For $A, B \subseteq \mathbb{R}^n$ we have $\text{conv}(A + B) = \text{conv} A + \text{conv} B$. Thus

$$\text{ccs} X = \text{cl} \left(\frac{1}{2}(\text{conv} X - \text{conv} X)\right).$$

From this it also follows that $\text{ccs}(X + Y) = \text{ccs} X + \text{ccs} Y$ if one of $X$ and $Y$ is relative compact. Moreover, $\text{ccs} A = A$ if $A \in C(n)$.

**Definition 2.5.** If $X \subseteq \mathbb{R}^n$ and $(A, V)$ is the splitting of $\text{ccs} X$, put $\gamma(X) := \dim V$.

**Lemma 2.6.** The map $\gamma : 2^{\mathbb{R}^n} \to \{0, 1, 2, \ldots, n\}$ is monotone increasing with respect to the partial order induced on $2^{\mathbb{R}^n}$ by inclusion. If $X \subseteq Y \subseteq \mathbb{R}^n$ and $\gamma(X) = \gamma(Y)$, then from $A \in C(n)$ with $\dim A = n$ and $\kappa(X, A) < \infty$ it follows that $\kappa(Y, A) < \infty$. 
Proof. Monotonicity of $\gamma$ is obvious. Fix $X \subseteq Y$ with $\gamma(X) = \gamma(Y)$, and suppose we are given $A$ in $\mathcal{C}(n)$ with $\dim A = n$ and $\kappa(X, A) < \infty$. Let $(B, V)$ be the splitting of $A$ and let $I \subseteq \mathbb{R}^n$ be finite with $X \subseteq I + A = I + B + V$. Since $I + B$ is compact, in view of Remark 2.4 we obtain

\[(2.1) \quad \text{ccs} X \subseteq \text{ccs}(I + B + V) = \text{ccs}(I + B) + V.
\]

Since $\text{ccs} X \subseteq \text{ccs} Y$ and $\gamma(X) = \gamma(Y)$ there is a subspace $W$ of $\mathbb{R}^n$ with $\dim W = \gamma(X)$ and there are splittings $(B_1, W)$ and $(B_2, W)$ of $\text{ccs} X$ and $\text{ccs} Y$, respectively, with $B_1 \subseteq B_2$. Put $A_1 := A \cap W^\perp$. Now (2.1) implies $W \subseteq V$, and hence $A_1 + W = A$. Therefore $\dim A = n$ yields $dim A_1 = dim W^\perp = n - \gamma(X)$, and relint $A_1$ (the interior of $A_1$ relative to the smallest subspace including $A_1$) is open in $W^\perp$. Since $B_2 \subseteq W^\perp$ is compact there is a finite set $J \subseteq W^\perp$ with $B_2 \subseteq J + A_1$.

It follows that

\[Y \subseteq \text{ccs} Y = B_2 + W \subseteq J + A_1 + W = J + A\]

and thus $\kappa(Y, A) < \infty$. \hfill $\Box$

Lemma 2.7. For all $n$ in $\mathbb{N}$ there is a constant $C_1(n) \geq 0$ such that for all $A$ in $\mathcal{C}(n)$ with $\dim A = n$ the following hold:

a) $\kappa(A, \frac{1}{2}A) \leq C_1(n),$

b) there is a discrete subgroup $G$ of the additive group of $\mathbb{R}^n$ such that $\mathbb{R}^n = G + A$ and $\sup_{x \in \mathbb{R}^n} |(x + 3A) \cap G| \leq C_1(n).$

Proof. From [7, Lemma 2.4] we obtain for all $m$ in $\mathbb{N}$ a constant $C_2(m)$, monotone increasing in $m$, such that for every $m$-dimensional compact $B$ in $\mathcal{C}(m)$ there is a parallelotope $P \subseteq \mathbb{R}^m$, centered at the origin, with

\[(2.2) \quad P \subseteq B \subseteq C_2(m)P.
\]

Now set $C_1(n) := [3C_2(n) + 1]^n$, where $[a]$ denotes the largest integer below or equal to $a$ if $a \in \mathbb{R}$.

Fix $A$ in $\mathcal{C}(n)$ and let $(B, V)$ be the splitting of $A$. Since $\dim A = n$ we have $\dim B + \dim V = n$. We may assume $\dim B = m$ and $V = \{0\} \times \mathbb{R}^{n-m}$ as a subspace of $\mathbb{R}^n$. We identify $\mathbb{R}^m$ with $\mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$ so that $B \subseteq \mathbb{R}^n$, and we choose a parallelootope $P \subseteq \mathbb{R}^m$ for $B$ as in (2.2). Then from $2C_2(m) \leq 3C_2(n)$ and the definition of $C_1(n)$ we obtain

\[\kappa(A, \frac{1}{2}A) = \kappa(B, \frac{1}{2}B) \leq \kappa(C_2(m)P, \frac{1}{2}P) \leq \kappa(3C_2(n)P, P) \leq C_1(n).
\]

For the second assertion we use for $P$ from above the representation

\[P = [-r_1, r_1] \times [-r_2, r_2] \times \cdots \times [-r_m, r_m]
\]

with some $r_1, r_2, \ldots, r_m > 0$ and put $G_0 := 2r_1\mathbb{Z} \times 2r_2\mathbb{Z} \times \cdots \times 2r_m\mathbb{Z} \subseteq \mathbb{R}^m$. Then $G_0$ is an additive subgroup of $\mathbb{R}^m$ with $G_0 + B \supseteq G_0 + P = \mathbb{R}^m$. Now set $G := G_0 \times \{0\} \subseteq \mathbb{R}^n$. Then $G + A = G + B + V = \mathbb{R}^n$. On the other hand we have for every $x$ in $\mathbb{R}^n$

\[(x + 3A) \cap G = (x + 3B) \cap G \subseteq (x + 3C_2(m)P) \cap G \subseteq (x + 3C_2(n)P) \cap G
\]

and hence

\[|(x + 3A) \cap G| \leq |(x + 3C_2(n)P) \cap G| \leq C_1(n).
\]

This completes the proof. \hfill $\Box$
Lemma 2.8. Suppose that $p$ is a seminorm on $\mathbb{R}^n$ and that $Y \subseteq [0, \infty)$. Put $X := p^{-1}(Y)$. Then $\alpha(X) \leq C_3(n)\beta(Y)^n$ for some constant $C_3(n)$. If $\text{codim}(\ker p) \geq 2$, then $\alpha(X) \geq \beta(Y)/2$.

Proof. For $r > 0$ put $A(r) := \{ x \in \mathbb{R}^n \mid p(x) \leq r \} \in \mathcal{C}(n)$. Let $(B(1), V)$ be the splitting of $A(1)$ and put $B(r) := rB(1)$ for $r > 0$. Then $(B(r), V)$ is the splitting of $A(r)$. Moreover, $V = \ker p$. Set $m := \text{dim } V$, so $\dim B(1) = m$.

Define $f, g: [0, \infty) \to \mathbb{N}$ by setting $f(0) := g(0) := 1$ and, for $t > 0$, $f(t) := \kappa(\partial A(t), A(1)) = \kappa(\partial B(t), B(1))$ and $g(t) := \kappa(A(t), A(1)) = \kappa(B(t), B(1))$. Then $f$ and $g$ are monotone increasing, $f \leq g$, and

$$
\kappa(\partial A(r), A(s)) = f(r/s),
\kappa(A(r), A(s)) = g(r/s)
$$

for $r, s > 0$. As in the beginning of the proof of Lemma 2.7 we obtain

$$
g(t) = \kappa(B(t), B(1)) \leq \kappa(tC_2(m)P, P) = [tC_2(m) + 1]^m.
$$

Here $P \subseteq B(1)$ is a parallelotope chosen as for (2.2). If $m \geq 2$, then

$$
f(t) = \kappa(\partial B(t), B(1)) \geq t.
$$

This can be seen as follows: Consider $B(1)$ as a subset of $\mathbb{R}^m$. Fix $x_0$ in $\partial B(1)$ such that $2\|x_0\|_2 = \text{diam } B(1)$. Let $Q$ be the orthogonal projection in $\mathbb{R}^m$ onto $\text{span } \{x_0\}$ and $L := \ker Q$. Then $\dim L \geq 1$. It follows that for every $x$ in $[-tx_0, tx_0]$ (the segment joining $-tx_0$ and $tx_0$) the set $\{x + L \cap \partial B(t)\}$ is not empty. Moreover, from $B(1) \in \mathcal{C}(n)$ it follows that $x_0 + L$ is a supporting hyperplane for $B(1)$. If $x_1, x_2, \ldots, x_k \in \mathbb{R}^m$ are such that

$$
\partial B(t) \subseteq \bigcup_{l=1}^{k}(x_l + B(1)),
$$

then from the above it is clear that

$$
[-tx_0, tx_0] \subseteq \bigcup_{l=1}^{k}(Qx_l + B(1))
$$

and therefore $k \geq [t + 1] \geq t$. This yields (2.4).

Let us consider the case $0 < \lambda(Y) \leq \text{sup } Y < \infty$. There is

$$
\varepsilon \in [0, \lambda(Y)/2]
$$

such that $[0, \lambda(Y) - \varepsilon] \subseteq Y$. It follows that

$$
A(\lambda(Y) - \varepsilon) \subseteq X \subseteq A(\text{sup } Y).
$$

Thus, using (2.3), we obtain

$$
\alpha(X) \leq \kappa(A(\text{sup } Y), A(\lambda(Y) - \varepsilon)) = g\left(\frac{\text{sup } Y}{\lambda(Y) - \varepsilon}\right) \leq g(2\beta(Y)) \leq C_3(n)\beta(Y)^n
$$

for some constant $C_3(n) \geq 1$.

There is $\varepsilon \in [0, \text{sup } Y/2]$ such that $\text{sup } Y - \varepsilon \in Y$ and therefore

$$
(2.5) \quad \partial A(\text{sup } Y - \varepsilon) \subseteq X.
$$
Every \( A \) in \( \mathcal{C}(n) \) with \( A \subseteq X \) is path connected and satisfies \( 0 \in A \). Since \( p \) is continuous, \( p(A) \) is included in the path component of \( Y \) containing \( 0 \). Therefore \( p(A) \subseteq [0, \lambda(Y)] \) and \( A \subseteq A(\lambda(Y)) \). This shows that

\[
\kappa(X, A) \geq \kappa(X, A(\lambda(Y)))
\]

for all \( A \) in \( \mathcal{C}(n) \). Hence we find for \( m \geq 2 \), applying (2.4) and (2.5):

\[
\alpha(X) \geq \kappa(\partial A(\sup Y - \varepsilon), A(\lambda(Y))) = f\left(\frac{\sup Y - \varepsilon}{\lambda(Y)}\right) \geq f(\beta(Y)/2) \geq \beta(Y)/2.
\]

The case \( \lambda(Y) > 0 \), \( \sup(Y) = \infty \) is handled similarly, and in all other cases the assertion is trivial. \( \square \)

3. PROOF OF THE THEOREMS

Let us first prove Theorem 1.2. Suppose that we are given \( h \in \mathcal{P}([0, \infty)) \) and a seminorm \( p \) on \( \mathbb{R}^n \). Set \( W := h \circ p \). Then \( [W]_t = p^{-1}([h]_t) \) for every \( t > 0 \). Now Lemma 2.8 yields \( \alpha([W]_t) \leq C\beta([h]_t)^n \) with some positive constant \( C \). Moreover, if \( \text{codim}(\ker p) \geq 2 \) Lemma 2.8 implies that \( \beta([h]_t) \leq 2\alpha([W]_t) \). From these facts the theorem follows.

The proof of Theorem 1.2 taken up next, is divided into the following steps:

(i) \( W(n, C) \) is closed under increasing pointwise limits for every \( C \geq 0 \).

(ii) \( W(n, C) \) is a convex cone for every \( C \geq 0 \).

Now suppose that \( W \in \mathcal{P}(\mathbb{R}^n) \).

(iii) If \( A \in \mathcal{C}(n) \) has dimension \( n \), if \( \kappa(\sup W, A) < \infty \), if there is \( a > 0 \) such that \( W \geq a \) on \( 2A \), and if \( W \) is bounded with \( b := \sup W(\mathbb{R}^n) \), then \( W \in W(n, C) \) for \( C := C_1(n)^{3}k(\sup W, A)b/a \), where \( C_1(n) \) is the constant given in Lemma 2.7.

(iv) If \( \sup_{t \geq 0} \alpha([W]_t) < \infty \), then \( W \in W(n, C) \) for some \( C \geq 0 \).

(v) If \( \lim_{t \to \infty} \sup \alpha([W]_t) + \lim_{t \to \infty} \alpha([W]_t) < \infty \), then \( W \in W(n, C) \) for some \( C \geq 0 \).

Theorem 1.2 is then a consequence of (iii) and (v).

Statements (i) and (ii) were proven in [4, Sect. 5.1]. For completeness we repeat the argument here. Suppose that \( C \geq 0 \). Fix two \( \sigma \)-finite positive Borel measures \( \mu, \nu \) on \( \mathbb{R}^n \). If \( W \) is the pointwise limit of an increasing sequence of functions in \( W(n, C) \), then (1.1) follows from Lebesgue’s Monotone Convergence Theorem. This proves (i) since \( \mu, \nu \) were chosen arbitrarily.

Consider the implication

\[
(3.1) \quad \left( u \leq C\sqrt{vw} \quad \text{and} \quad x \leq C\sqrt{yz} \right) \quad \Rightarrow \quad (u + x)^2 \leq C^2(v + y)(w + z)
\]

for \( u, v, w, x, y, z \) in \( [0, \infty) \), which is a consequence of \( 2\sqrt{vwyz} \leq vz + yw \). If \( W_1, W_2 \in W(n, C) \), then (3.1) implies that \( W_1 + W_2 \in W(n, C) \). Since \( W(n, C) \) is a cone, \( W(n, C) \) is convex.

To show (iii) choose a discrete additive subgroup \( G \) of \( \mathbb{R}^n \) for \( A \) as in Lemma 2.4. Let \( I \) be a finite subset of \( \mathbb{R}^n \) with \( \sup W \subseteq I + A \) and \( |I| = \kappa(\sup W, A) \). Put \( J := (I + 3A) \cap G \). From the choice of \( G \) it follows that

\[
(3.2) \quad |J| \leq C_1(n)|I|.
\]
Define \( \overline{W} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) by \( \overline{W}(x, y) := W(x - y) \). Then \( \overline{W} \) is a Borel function. We claim that

\[
\text{supp } \overline{W} \subseteq \bigcup_{u, v \in G} (u + A) \times (v + A).
\]

To see this, suppose that \((x, y) \in \text{supp } \overline{W} \), or equivalently \(x - y \in \text{supp } W\). There is \(w\) in \(I\) such that \(x - y \in w + A\), and there are \(u, v \in G\) such that \(x \in u + A\) and \(y \in v + A\). It follows that \(u - v \in x - y + 2A \subseteq w + 3A \subseteq I + 3A\). Also \(u - v \in G\) because \(G\) is a subgroup. This proves the claim.

Now the Cauchy-Schwarz inequality for sums yields

\[
(3.3) \quad I(W; \mu, \nu) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \overline{W} \, d(\mu \times \nu) \leq b \int_{\text{supp } \overline{W}} d(\mu \times \nu)
\]

\[
\leq b \sum_{u, v \in G} \mu(u + A) \nu(v + A) \leq b \left( \sum_{u, v \in G} \mu(u + A)^2 \sum_{u, v \in G} \nu(v + A)^2 \right)^{\frac{1}{2}}.
\]

We need to estimate the sums in the last term. For every \(x\) in \(\mathbb{R}^n\), from \(A \in C(n)\) it follows that the statement \((u \in G \text{ and } x \in u + A)\) is equivalent to the statement \((u \in (x + A) \cap G)\). By the choice of \(G\) this leads to

\[
|\{ u \in G \mid x \in u + A \}| = |(x + A) \cap G| \leq |(x + 3A) \cap G| \leq C_1(n)
\]

and thus for all \(x, y\) in \(\mathbb{R}^n\)

\[
(3.4) \quad |\{ u \in G \mid (x, y) \in (u + A) \times (u + A) \}| \leq C_1(n)^2.
\]

We also have

\[
(3.5) \quad \bigcup_{u \in G} (u + A) \times (u + A) \subseteq \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x - y \in 2A \} =: D
\]

and \(\overline{W} \geq a\) on \(D\). Using (3.2), (3.4) and (3.5) we calculate

\[
\sum_{u, v \in G} \mu(u + A)^2 = |J| \sum_{u \in G} \mu(u + A)^2 = |J| \sum_{u \in G} \int_{(u + A) \times (u + A)} d(\mu \times \mu)
\]

\[
\leq C_1(n)^2 |J| \int_D d(\mu \times \mu) \leq C_1(n)^2 |J| \frac{C_2(n) |I|}{a} \int_{\mathbb{R}^n \times \mathbb{R}^n} \overline{W} \, d(\mu \times \mu)
\]

\[
= C_1(n)^2 |I| \frac{C_2(n) |I|}{a} I(W; \mu, \mu),
\]

a similar estimate holding for the sum over \(\nu(v + A)^2\). This proves (iii) in view of (3.3).

To show (iv) suppose that \(M := \sup_{t \geq 0} \alpha(|W|) < \infty\). For \(m \in \mathbb{N}\) and \(1 \leq i \leq m2^m\) define \(W_{m,i}\) and \(W_m\) in \(P(\mathbb{R}^n)\) by setting

\[
W_{m,i} := \frac{1}{2m} \chi_{[W]_{i/2m}},
\]

\[
W_m := \sum_{i=1}^{m2^m} W_{m,i}.
\]
Here $\chi_A$ denotes for $A \subseteq \mathbb{R}^n$ the characteristic function of $A$. The sequence $(W_m)$ is increasing and converges pointwise to $W$. Fix $m$ and $i$. There is $A$ in $C(n)$ such that $\dim A = n$, $A \subseteq [W_i]_{1/2^m}$, and $\kappa([W_i]_{1/2^m}, A) \leq M$. Since $A$ is closed supp $W_{m,i} = \text{cl}([W_i]_{1/2^m})$ can be covered by the same number of translates of $A$ as $[W_i]_{1/2^m}$, i.e. $\kappa(\text{supp} W_{m,i}, A) = \kappa([W_i]_{1/2^m}, A)$. Using Lemma 2.7, we thus obtain

$$\kappa(\text{supp} W_{m,i}, \frac{1}{2} A) \leq C_1(n) \kappa(\text{supp} W_{m,i}, A) \leq C_1(n) M.$$  

Moreover, $W_{m,i} = 1/2^m$ on $A$ and $W_{m,i} \leq 1/2^m$ on $\mathbb{R}^n$. By (iii) $W_{m,i} \in W(n, C)$ for $C = C_1(n)^4 M$, independently of $m$ and $i$. By (ii) $W_m \in W(n, C)$ for every $m$, and thus (1) yields the desired result.

The remaining case (v) is handled as follows: We can assume that $W \neq 0$, otherwise there is nothing to do. By our assumptions there are $M > 0$ and $0 < t_1 < t_0$ such that $\alpha([W]_t) \leq M$ for $t$ in $(0, t_1] \cup [t_0, \infty)$ and $[W]_t \neq \emptyset$ for $t$ in $(0, t_1]$. Consider $\gamma([W]_t)$ as a function of $t$ sending $(0, \infty)$ into $\{0, 1, 2, \ldots, n\}$ ($\gamma$ is given in Definition 2.5). We can choose $0 < t_3 < t_2 < t_1$ with $\gamma([W]_{t_2}) = \gamma([W]_{t_3})$. For $x$ in $\mathbb{R}^n$ put $W_1(x) := \min\{t_2, W(x)\}$ and $W_2(x) := \min\{t_0 - t_3, W(x) - W_1(x)\}$. Also put $W_3 := W - W_1 - W_2$. Then $W_1 \leq t_3$, $W_2 \leq t_0 - t_3$, and $W_i \geq 0$ for $i = 1, 2, 3$. Moreover, we have

$$[W_1]_t = \begin{cases} [W]_t, & 0 \leq t \leq t_3, \\ \emptyset, & t_3 < t, \end{cases}$$

$$[W_2]_t = \begin{cases} [W]_{t-t_3}, & 0 \leq t \leq t_0 - t_3, \\ \emptyset, & t_0 - t_3 < t, \end{cases}$$

$$[W_3]_t = [W]_{t-t_0}.$$  

From (iv) it follows that $W_1, W_3 \in W(n, C)$ for some $C \geq 0$. Since $[W]_{t_2} \neq \emptyset$ and $\alpha([W]_{t_2}) < \infty$ there is $A$ in $C(n)$ with $\dim A = n$, $A \subseteq [W]_{t_2}$, and $\kappa([W]_{t_2}, A) < \infty$. By Lemma 2.6, $\kappa([W]_{t_3}, A) < \infty$ also, and by Lemma 2.4, $\kappa([W]_{t_3}, \frac{1}{2} A) < \infty$. Hence the closedness of $A$ and supp $W_2 \subseteq \text{cl}([W]_{t_3})$ imply that $\kappa(\text{supp} W_2, \frac{1}{2} A) < \infty$. We also have $W_2 \geq \frac{1}{2} t_2 - t_3$ on $A$ and $W_2 \leq t_0 - t_3$ on $\mathbb{R}^n$. Now (iii) implies that $W_2 \in W(n, C)$ for some $C$, and by (ii) the same holds for $W = W_1 + W_2 + W_3$. This finishes the proof of (v).

REFERENCES


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