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STRONG CONVERSE INEQUALITY FOR POISSON SUMS

F. DAI AND Z. DITZIAN

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ABSTRACT. The rate of convergence of Poisson sums and their combinations are shown to be equivalent to appropriate K-functionals.

For a function f that has the expansion

$$f(x) \sim \sum_{k=0}^{\infty} P_k(f) \tag{1}$$

the Poisson sum is given by

$$A_r f = \sum_{k=0}^{\infty} r^k P_k(f), \quad 0 < r < 1. \tag{2}$$

For $A_r f$ to be written as a semi-group, we set $r = e^{-t}$ and obtain

$$T(t)f = \sum_{k=0}^{\infty} e^{-kt} P_k(f). \tag{3}$$

For the purpose of this paper, H_k are eigenspaces of a given self-adjoint operator $P(D)$ with eigenvalues λ_k such that λ_k are either all positive or all negative, and $(-1)^j \lambda_k = |\lambda_k| < |\lambda_{k+1}|$. Moreover, λ_k is a polynomial in k. We deal with a Banach space of functions B such that $H_k \subset B$, $H_k \subset B^*$, span $\bigcup_{k=0}^{\infty} H_k$ is dense in B and $P_k(f)$ is the projection of f onto H_k (see [Ch-Di]).

We further assume that the Cesàro summability of order ℓ, for some ℓ, is a contraction on B, that is,

$$\|C_n^{\ell} f\|_B \le \|f\|_B. \tag{4}$$

We note that if $C_n^{\ell} f$ is a contraction, so is $C_n^m f$ for $m > \ell$. We also note that in all cases known to us $C_n^{\ell} f$ is a contraction when it is a positive operator. If C_n^{ℓ} is a contraction on B, so is the Poisson sum $A_r f$, which can be written (see [Ha, p. 108] or simply summation by parts) as

$$A_r f = (1-r)^{1+\ell} \sum_{k=0}^{\infty} A_k^{\ell} r^k \sigma_k^{\ell}(f), \quad A_k^{\ell} = \binom{k+\ell}{k}, \tag{5}$$

and clearly $T(t)f$ is also a contraction. (Positivity is also inherited.)

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The infinitesimal generator of $T(t)f$ is formally given by

$$\mathcal{A}f \sim \sum_{k=1}^{\infty} -kP_k(f), \tag{6}$$

and the domain of $\mathcal{A}$, $D(\mathcal{A})$ in B is all f in B such that $\mathcal{A}f \in B$ where $\mathcal{A}f$ is given by (6).

We now have the following result.

Theorem 1. *If for a given space B, $C_n^\ell f$ satisfies* (4)*, then $\mathcal{A}T(t)f \in B$ for all $t > 0$ and*

$$t\|\mathcal{A}T(t)f\|_B \le 2(\ell+2)(\ell+1)\|f\|_B; \tag{7}$$

that is, $T(t)$ is a holomorphic semigroup of contractions.

Proof. Clearly, $\mathcal{A}C_n^\ell(f)$ is in B. A simple calculation yields

$$\mathcal{A}C_n^\ell(f) = (n+\ell+1)\big(C_n^{\ell+1}(f) - C_n^\ell(f)\big),$$

and hence

$$\begin{aligned}\mathcal{A}T(t)f &= (1-e^{-t})^{\ell+1}\sum_{k=0}^{\infty} A_k^\ell e^{-tk}\mathcal{A}C_k^\ell(f)\\ &= (1-e^{-t})^{\ell+1}\sum_{k=1}^{\infty} A_k^\ell e^{-tk}(k+\ell+1)\big(C_n^{\ell+1}(f) - C_n^\ell(f)\big).\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathcal{A}T(t)f\|_B &\le (1-e^{-t})^{\ell+1}\sum_{k=1}^{\infty} A_k^\ell e^{-tk}(k+\ell+1)\big(\|C_n^{\ell+1}(f)\|_B + \|C_n^\ell(f)\|_B\big)\\ &\le 2\|f\|_B(1-e^{-t})^{\ell+1}\sum_{k=1}^{\infty} A_k^\ell e^{-tk}k(\ell+2)\\ &\le 2(\ell+2)\|f\|_B(1-e^{-t})^{\ell+1}\left(-\frac{d}{dt}\,\frac{1}{(1-e^{-t})^{\ell+1}}\right)\\ &\le 2(\ell+2)\|f\|_B(\ell+1)\,\frac{e^{-t}}{(1-e^{-t})^{\ell+2}}\,(1-e^{-t})^{\ell+1}\\ &\le 2(\ell+2)(\ell+1)\,\frac{e^{-t}}{1-e^{-t}}\,\|f\|_B,\end{aligned}$$

and as $\frac{te^{-t}}{1-e^{-t}} \le 1$ for $t > 0$, (7) follows. □

We can now follow [Di-Iv, Section 5, Theorem 5.1] and deduce the following result as an immediate corollary.

Theorem 2. *Let $T(t)f$ and $A_r f$ be given by* (3) *and* (2) *respectively, let $\mathcal{A}$ be given by* (6) *and suppose* (4) *is satisfied for some integer ℓ. Then*

$$\|(T(t)-I)^m f\|_B \approx \inf_{g\in D(\mathcal{A}^m)}\big(\|f-g\|_B + t^m\|\mathcal{A}^m g\|_B\big), \quad t>0, \tag{8}$$

and

$$\left\| \sum_{k=0}^{m-1} \binom{m}{k} (-1)^k A_{r^{m-k}} f + (-1)^m f \right\|_B \tag{9}$$
$$\approx \inf_{g \in D(\mathcal{A}^m)} \Big(\|f-g\|_B + (1-r)^m \|\mathcal{A}^m g\|_B \Big), \quad 0 < r < 1,$$

where $D(\mathcal{A}^m)$ *is the domain of* $\mathcal{A}^m$ *in* B.

Remark 3. If $\lambda(k)$ is a polynomial of degree b in k, a simple calculation or Lemma 3.2 of [Da] shows that domain $\mathcal{A}$ is the same as the domain of $\big((-1)^j P(D)\big)^{1/b}$ and $\|\mathcal{A}^m g\|_B \approx \|(-1)^j P(D)^{m/b} g\|_B$ with

$$\big((-1)^j P(D)\big)^{\alpha} g \sim \sum_{k=0}^{\infty} \lambda(k)^{\alpha} P_k(g) \tag{10}$$

as defined in [Di].

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Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

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