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AN ESTIMATE FOR A CERTAIN AVERAGE OF THE SPECIAL VALUES OF CHARACTER TWISTS OF MODULAR L-FUNCTIONS

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Abstract. We prove the Lindelöf hypothesis in weight aspect for the average of the special values of the twisted L-functions of newforms, which generalizes the work of W. Kohnen and J. Sengupta (2002).

1. Introduction

In this note, we prove the Lindelöf hypothesis in weight aspect for the average of the special values of the twisted L-functions of newforms. More precisely, we prove the following:

Theorem. *Let D be a negative fundamental discriminant, $(D, M) = 1$, and let $\epsilon > 0$. Then*

$$\sum_f L\big(f, \chi\left(\frac{D}{\cdot}\right), k\big) \ll_{\epsilon, D, M} k^{1+\epsilon} \quad (k \to \infty), \tag{1}$$

where the sum varies over all newforms which are normalized cusp Hecke eigenforms of weight $2k$, level M and character $\overline{\chi}^2$.

Recently, W. Kohnen and J. Sengupta [2] proved the above result for the case χ trivial with the assumption that D is a square modulo $4M$ and the root number is 1 in the functional equations of the corresponding L-functions. Their method uses an explicit form of Waldspurger's result expressing the special value of L-functions of a newform of weight $2k$ in terms of the square of the (n, r)-th Fourier coefficient of the corresponding Jacobi form of weight $k + 1$ under the Shimura correspondence. Then they use the bounds for the Petersson norm for newforms of weight $2k$ and use the Petersson formula, which finally reduces to estimating the (n, r)-th Fourier coefficient of the (n, r)-th Jacobi Poincaré series of weight $k + 1$.

Our method of proof uses the existence of Shimura and Shintani maps. Even though the multiplicity one theorem is not known in general, in [3] we have shown that the existence of these maps gives a precise form of Waldspurger's Theorem (see [3, Theorem 5.7]) for the sum over the eigensubspace corresponding to each newform of integral weight. From this we observe that each $L\big(f, \chi\left(\frac{D}{\cdot}\right), k\big) \geq 0$. Indeed, we obtain our result by invoking Theorem 3.4 of [3], namely the identity (2) below,

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which expresses the holomorphic kernel function of the periods of cusp forms as the image of the (D, r)-th Poincaré series of Jacobi forms under the Shimura map, and then estimating the norm of the (D, r)-th Poincaré series.

2. Proof

Let $\{f_1, f_2, \cdots, f_\ell\}$ be the orthogonal basis of normalized Hecke eigenforms of $S_{2k}^{\mathrm{new}}(M, \overline{\chi}^2)$. By substituting $D_0 = D$, $D \equiv r^2 \pmod 4$ in Theorem 3.4 of [3], we get

$$P_{(D,r)}\big|\mathcal{S}_{D,r}(w) = \frac{(-i)^{k-1}2^{k-2}(M_1|D|)^{k-1/2}\Gamma(k-1/2)R(\overline{\chi}, D)}{\pi^{k+1/2}\binom{2k-2}{k-1}} \cdot \sum_{t|M_2} \mu(t)\overline{\chi}(t)\left(\tfrac{D}{t}\right) t^{k-1} f_{k+1,M/t,\overline{\chi},M_1^2D^2,-r^2M_1,D}\,\big|K(tw), \tag{2}$$

where

$$R(\chi, D) = (DM_1)^{-1/2} \sum_{a \bmod (M_1|D|)} \chi(a)\left(\tfrac{D}{a}\right) e^{2\pi i a/(M_1|D|)},$$

M_1 is the conductor of χ, $M_2 = M/M_1$ and K is the operator mapping $f(\tau)$ into $\overline{f(-\overline{\tau})}$. In the above, $P_{(D,r)}$ is the (D, r)-th Poincaré series in the space of Jacobi forms of integral weight and $\mathcal{S}_{D,r}$ is the Shimura map. Further, the holomorphic kernel function that appears on the right-hand side of (2) is a cusp form in $S_{2k}(M, \chi^2)$, which is defined as follows (see [3, p. 2604]):

$$f_{k+1,M,\overline{\chi},M_1^2D_0D,-r_0rM_1,D_0}(w) = \sum_Q \overline{\chi}(c)\chi_{D_0}(Q)Q(w,1)^{-k}, \tag{3}$$

where the summation varies over all binary quadratic forms $Q = [a, b, c]$ such that $b^2 - 4ac = M_1^2D_0D$, $a \equiv 0 \pmod{MM_1}$ and $b \equiv -r_0rM_1 \pmod 2$, and $\chi_{D_0}(Q)$ is the generalized genus character.

Using the norm estimate of Iwaniec and Michel [1],

$$\langle f_i, f_i\rangle \ll_{\epsilon,M} \frac{\Gamma(2k)}{(4\pi)^{2k}}k^{\epsilon} \quad (i = 1, 2, \ldots, \ell), \tag{4}$$

we get

$$\sum_{i=1}^{\ell} L\big(f_i, \chi\left(\frac{D}{\cdot}\right), k\big) \ll \frac{\Gamma(2k)}{(4\pi)^{2k}}k^{\epsilon} \sum_{i=1}^{\ell} \frac{L\left(f_i, \chi\left(\frac{D}{\cdot}\right), k\right)}{\langle f_i, f_i\rangle}. \tag{5}$$

Using equations (5) and (6) of [3], the right-hand side of (5) equals

$$\frac{2^{2k-2}\pi^{k-2}(M_1|D|)^{k-1/2}k^{1+\epsilon}}{\Gamma(k)\binom{2k-2}{k-1}R(\overline{\chi}, D)(-i^{k+1})} \cdot \langle P_1, \sum_{t|M_2} \mu(t)\overline{\chi}(t)\left(\tfrac{D}{t}\right) t^{k-1} f^{\mathrm{new}}_{k+1,M/t,\overline{\chi},M_1^2D^2,-r^2M_1,D}\,\big|K(tw)\rangle, \tag{6}$$

where P_1 is the first Poincaré series in the space of integral weight cusp forms.

Since the Shimura map $\mathcal{S}_{D,r}$ preserves the space of oldforms and newforms, from equation (2) and the above inequality, we get

$$\sum_{i=1}^{\ell} L\big(f_i, \chi\left(\frac{D}{\cdot}\right), k\big) \ll \frac{2^{2k-2}\pi^{k-2}(M_1|D|)^{k-1/2}k^{1+\epsilon}}{\Gamma(k)\binom{2k-2}{k-1}R(\overline{\chi}, D)(-i^{k+1})}\langle P_{(D,r)}, P_{(D,r)}\rangle \tag{7}$$
$$\ll_{\epsilon,D,M} k^{1+\epsilon},$$

which follows by estimating the norm of the Poincaré series $P_{(D,r)}$ as done in [2]. This proves the theorem.

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