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ASYMPTOTIC EXPANSION OF SOLUTIONS TO NONLINEAR ELLIPTIC EIGENVALUE PROBLEMS

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ABSTRACT. We consider the nonlinear eigenvalue problem

$$-\Delta u + g(u) = \lambda \sin u \text{ in } \Omega, \ \ u > 0 \text{ in } \Omega, \ \ u = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbf{R}^N$ $(N \geq 2)$ is an appropriately smooth bounded domain and $\lambda > 0$ is a parameter. It is known that if $\lambda \gg 1$, then the corresponding solution u_λ is almost flat and almost equal to π inside Ω. We establish an asymptotic expansion of $u_\lambda(x)$ $(x \in \Omega)$ when $\lambda \gg 1$, which is explicitly represented by g.

1. Introduction

We consider the nonlinear eigenvalue problem

$$-\Delta u + g(u) = \lambda \sin u \quad \text{in } \Omega, \tag{1.1}$$

$$u > 0 \quad \text{in } \Omega, \tag{1.2}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbf{R}^N (N \geq 2)$ is an appropriately smooth bounded domain and $\lambda > 0$ is a parameter. We assume the following conditions (A.1)–(A.3):

(A.1) $g \in C^{m,\gamma}(\mathbf{R})$ $(m \geq 1, 0 < \gamma < 1)$ and $g(u) > 0$ for $u > 0$.
(A.2) $g(0) = g'(0) = 0$.
(A.3) $g(u)/u$ is strictly increasing for $0 < u < \pi$.
The typical example of $g(u)$ is $g(u) = |u|^{m-1}u$ $(m > 1)$.

The equation (1.1)–(1.3) is regarded as the equation of a simple pendulum with a nonlinear self-interaction term $g(u)$, and the following (P.1) and (P.2) are well known and easy to show (cf. [1], [2], [4], [5]).

(P.1) For a given $\lambda \in \mathbf{R}$, (1.1)–(1.3) has a unique solution $u_\lambda \in C^3(\bar{\Omega})$ if and only if $\lambda > \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ with Dirichlet zero boundary condition.

(P.2) $\|u_\lambda\|_\infty < \pi$ and $u_\lambda \to \pi$ locally uniformly in Ω as $\lambda \to \infty$.
Therefore, we see from (P.2) that u_λ is almost flat inside Ω, and one common interest to study (1.1)–(1.3) is to investigate precisely the asymptotic behavior of u_λ inside Ω. In other words, we are interested in "how flat u_λ is inside Ω".

In the case when $g \equiv 0$, there are many works concerning the asymptotic behavior of the solutions of (1.1)–(1.3) inside Ω as $\lambda \to \infty$, since the properties (P.1) and

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(P.2) are valid when $g \equiv 0$ (cf. [2], [3] and the references therein). To take a simple example, let us consider the case $N = 1$, $\Omega = (-R, R)$ and $g \equiv 0$. In this case, we denote by $u_{0,\lambda}$ the unique solution associated with given $\lambda > \lambda_1$. Then it is known (cf. [6]) that as $\lambda \to \infty$,

$$\|u_{0,\lambda}\|_\infty = \pi - 8(1 + o(1))e^{-\sqrt{\lambda}(1+o(1))R}. \tag{1.4}$$

We remark that the second term in the right-hand side of (1.4) decays *exponentially* as $\lambda \to \infty$.

However, as far as the author knows, there are no works concerning the precise asymptotic analysis of the interior behavior of u_λ as $\lambda \to \infty$ when $g \not\equiv 0$. So the natural and fundamental questions we have to ask here are as follows:

(Q.1) Does the second term of $\|u_\lambda\|_\infty$ decay exponentially, too?

(Q.2) If the answer to question (Q.1) is in the negative, then what is the second term of $\|u_\lambda\|_\infty$ as $\lambda \to \infty$?

(Q.3) How flat is u_λ inside Ω when $\lambda \gg 1$?

To answer these questions, we establish an asymptotic expansion of $u_\lambda(x)$ as $\lambda \to \infty$, which is explicitly represented by g and show that the second term and the remainder terms decay algebraically as $\lambda \to \infty$.

Now we state our results. Let $G(u) := \int_0^u g(s)ds$.

Theorem 1. *Let $x \in \Omega$ be fixed. Then the following asymptotic formula holds as $\lambda \to \infty$:*

$$u_\lambda(x) = \pi - \sum_{k=1}^{m+1} \frac{b_k}{\lambda^k} + o\left(\frac{1}{\lambda^{m+1}}\right), \tag{1.5}$$

where b_k $(k = 1, 2, \cdots, m+1)$ are constants determined by $\{g^{(j)}(\pi)\}_{j=0}^{k-1}$.

For example,

$$\begin{aligned} b_1 &= g(\pi), \\ b_2 &= -g(\pi)g'(\pi), \\ b_3 &= \frac{1}{6}g(\pi)^3 + g(\pi)^2 g'(\pi) + \frac{1}{2}g(\pi)^2 g^{(2)}(\pi). \end{aligned} \tag{1.6}$$

We explain the idea of the proof of Theorem 1 briefly. We first consider (1.1)–(1.3) for the case $\Omega = B_R := \{x \in \mathbf{R}^N : |x| < R\}$. In this case, (1.1)–(1.3) are equivalent to the ordinary differential equation

$$u''(r) + \frac{N-1}{r}u'(r) - g(u(r)) + \lambda \sin u(r) = 0, \quad 0 < r < R, \tag{1.7}$$

$$u(r) > 0, \quad 0 \le r < R, \tag{1.8}$$

$$u'(0) = u(R) = 0. \tag{1.9}$$

Then we prove (1.5) for $u_\lambda(0)$ $(= \|u_\lambda\|_\infty)$. Then by using the fact that $\|u_\lambda\|_\infty$ does not depend on the radius R of the ball B_R, we prove Theorem 1. Therefore, to prove (1.5) for $\Omega = B_R$ and $x = 0$ is crucial. The difficulty we encounter for the case $\Omega = B_R$ $(N \ge 2)$ is as follows. If we directly follow the argument in [8], in which (1.5) was obtained for the case $N = 1, \Omega = (-R, R)$ and $x = 0$, then we find that the second term $(N-1)u_\lambda'(r)/r$ in (1.7), which does not appear in the case $N = 1$, is quite difficult to treat. We concentrate our attention on treating this characteristic term appropriately and developing new devices to deal with the radial solution of (1.7)–(1.9) for the case $N \geq 2$. Then we obtain the answer to (Q.1)–(Q.3).

2. PROOF OF THEOREM 1 FOR $\Omega = B_R$ AND $x = 0$

In this section, we consider (1.7)–(1.9) and establish (1.5) for $x = 0$. Note that $u_\lambda(0) = \|u_\lambda\|_\infty$. We begin with the fundamental equalities which play important roles in this section. Multiply (1.7) by u'_λ. Then for $r \in [0, R]$,

$$\left\{ u''_\lambda(r) + \frac{N-1}{r} u'_\lambda(r) + \lambda \sin u_\lambda(r) - g(u_\lambda(r)) \right\} u'_\lambda(r) = 0.$$

Since $u_\lambda(0) = \|u_\lambda\|_\infty$, this implies that for $r \in [0, R]$,

$$\begin{aligned} & \frac{1}{2}u'_\lambda(r)^2 + \int_0^r \frac{N-1}{s} u'_\lambda(s)^2 ds - \lambda \cos u_\lambda(r) - G(u_\lambda(r)) \equiv \ \text{constant} \\ & = -\lambda \cos \|u_\lambda\|_\infty - G(\|u_\lambda\|_\infty) \quad (\text{put } r = 0) \\ & = \frac{1}{2}u'_\lambda(R)^2 + \int_0^R \frac{N-1}{s} u'_\lambda(s)^2 ds - \lambda \quad (\text{put } r = R). \end{aligned} \tag{2.1}$$

Let $M_\lambda := \inf\{\theta > 0 : \lambda \sin \theta = g(\theta)\}$. It is clear that $M_\lambda < \pi$ and $\lambda \sin \theta > g(\theta)$ for $0 < \theta < M_\lambda$. We know from [1] that $\|u_\lambda\|_\infty < M_\lambda$. Therefore, for $0 \leq r < R$, we have

$$\lambda \sin u_\lambda(r) > g(u_\lambda(r)). \tag{2.2}$$

Further, it is well known that

$$u'_\lambda(r) < 0 \quad (0 < r \leq R). \tag{2.3}$$

We begin with the following fundamental lemma.

Lemma 2.1. *Let $0 < r_0 < R$ be fixed. Then $u''_\lambda(r) < 0$ for $0 \leq r \leq r_0$ and $\lambda \gg 1$.*

Proof. We know that for $0 \leq r \leq R$,

$$(r^{N-1}u'_\lambda(r))' = r^{N-1}(g(u_\lambda(r)) - \lambda \sin u_\lambda(r)). \tag{2.4}$$

Integrate (2.4) over $[0, r]$ to obtain

$$u'_\lambda(r) = \frac{1}{r^{N-1}} \int_0^r s^{N-1}(g(u_\lambda(s)) - \lambda \sin u_\lambda(s)) ds. \tag{2.5}$$

By this, we obtain

$$\begin{aligned} u''_\lambda(r) \quad = \quad & g(u_\lambda(r)) - \lambda \sin u_\lambda(r) \\ & + (N-1)r^{-N} \int_0^r s^{N-1}(\lambda \sin u_\lambda(s) - g(u_\lambda(s))) ds. \end{aligned} \tag{2.6}$$

Then by integration by parts,

$$\begin{aligned} & \int_0^r s^{N-1}(\lambda \sin u_\lambda(s) - g(u_\lambda(s))) ds \\ & = \frac{r^N}{N}(\lambda \sin u_\lambda(r) - g(u_\lambda(r))) \\ & - \int_0^r \frac{s^N}{N}(\lambda \cos u_\lambda(s) - g'(u_\lambda(s))) u'_\lambda(s) ds. \end{aligned} \tag{2.7}$$

By this, (P.2), (2.2), (2.3) and (2.6), for $\lambda \gg 1$, we obtain

$$\begin{aligned} u_\lambda''(r) &= -\frac{1}{N}(\lambda \sin u_\lambda(r) - g(u_\lambda(r))) \\ &\quad - \frac{N-1}{N}\int_0^r (\lambda \cos u_\lambda(s) - g'(u_\lambda(s)))u_\lambda'(s)s^N ds \\ &< 0. \end{aligned}$$

Thus the proof is complete. □

Lemma 2.2. *Let an arbitrary* $0 < r_0 < R$ *be fixed. Then* $u_\lambda'''(r) < 0$ *for* $0 \le r \le r_0$ *and* $\lambda \gg 1$.

Proof. By (2.6), we have

$$\begin{aligned} -u_\lambda'''(r) &= (\lambda \cos u_\lambda(r) - g'(u_\lambda(r)))u_\lambda'(r) \\ &\quad + N(N-1)r^{-N-1}\int_0^r s^{N-1}(\lambda \sin u_\lambda(s) - g(u_\lambda(s)))ds \\ &\quad - (N-1)r^{-1}(\lambda \sin u_\lambda(r) - g(u_\lambda(r))). \end{aligned}$$

By this and (2.7),

$$\begin{aligned} -u_\lambda'''(r) &= (\lambda \cos u_\lambda(r) - g'(u_\lambda(r)))u_\lambda'(r) \\ &\quad - (N-1)r^{-N-1}\int_0^r s^N(\lambda \cos u_\lambda(s) - g'(u_\lambda(s)))u_\lambda'(s)ds. \end{aligned} \tag{2.8}$$

Since $\lambda \gg 1$, by (P.2) and integration by parts, we obtain

$$\begin{aligned} &\int_0^r s^N(\lambda \cos u_\lambda(s) - g'(u_\lambda(s)))u_\lambda'(s)ds \\ &= -(\lambda + g'(\pi))(1+o(1))\int_0^r s^N u_\lambda'(s)ds \\ &= -(\lambda + g'(\pi))(1+o(1))\left(\frac{1}{N+1}r^{N+1}u_\lambda'(r) - \int_0^r \frac{1}{N+1}s^{N+1}u_\lambda''(s)ds\right). \end{aligned}$$

By this, Lemma 2.1, (2.3) and (2.8), for $0 \le r \le r_0$ and $\lambda \gg 1$, we obtain

$$\begin{aligned} -u_\lambda'''(r) &= \frac{2}{N+1}(\lambda + g'(\pi))(1+o(1))(-u_\lambda'(r)) \\ &\quad + \frac{N-1}{N+1}r^{-N-1}(\lambda + g'(\pi))(1+o(1))\int_0^r s^{N+1}(-u_\lambda''(s))ds \\ &> 0. \end{aligned} \tag{2.9}$$

Thus the proof is complete. □

We put

$$\xi_\lambda := \lambda \sin \|u_\lambda\|_\infty - g(\|u_\lambda\|_\infty). \tag{2.10}$$

By l'Hopital's rule,

$$\lim_{r\to 0}\frac{u_\lambda'(r)}{r} = \lim_{r\to 0} u_\lambda''(r) = u_\lambda''(0).$$

By this and (1.7), we obtain

$$Nu_\lambda''(0) + \lambda \sin \|u_\lambda\|_\infty - g(\|u_\lambda\|_\infty) = 0.$$

This along with (2.10) and Lemma 2.1 implies that

$$\xi_\lambda = -Nu_\lambda''(0) > 0. \tag{2.11}$$

Furthermore, we put

$$\begin{aligned} I_\lambda(r) &:= \lambda(\cos u_\lambda(r) - \cos \|u_\lambda\|_\infty) + G(u_\lambda(r)) - G(\|u_\lambda\|_\infty), && (2.12)\\ II_\lambda(r) &:= \int_0^r \frac{N-1}{s} u_\lambda'(s)^2 ds. && (2.13) \end{aligned}$$

Then for $r \in [0, R]$, by (2.1), we obtain

$$\frac{1}{2} u_\lambda'(r)^2 = I_\lambda(r) - II_\lambda(r). \tag{2.14}$$

Lemma 2.3. *Let $0 < r_0 < R$ be fixed. Then for $0 \le r \le r_0$ and $\lambda \gg 1$*

$$I_\lambda(r) = \xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(r)) + \frac{1}{2}(\lambda + g'(\pi))(1+o(1))(\|u_\lambda\|_\infty - u_\lambda(r))^2. \tag{2.15}$$

Proof. By Taylor expansion and (P.2), for $0 \le r \le r_0$ and $\lambda \gg 1$, we obtain

$$\begin{aligned} \cos u_\lambda(r) &= \cos \|u_\lambda\|_\infty - \sin \|u_\lambda\|_\infty (u_\lambda(r) - \|u_\lambda\|_\infty) && (2.16)\\ &\quad - \frac{1}{2} \cos \|u_\lambda\|_\infty (u_\lambda(r) - \|u_\lambda\|_\infty)^2 (1+o(1)), \\ G(u_\lambda(r)) &= G(\|u_\lambda\|_\infty) + g(\|u_\lambda\|_\infty)(u_\lambda(r) - \|u_\lambda\|_\infty) && (2.17)\\ &\quad + \frac{1}{2}(g'(\|u_\lambda\|_\infty) + o(1))(u_\lambda(r) - \|u_\lambda\|_\infty)^2. \end{aligned}$$

Then by (P.2), (2.12), (2.16) and (2.17), for $0 \le r \le r_0$ and $\lambda \gg 1$, we have

$$\begin{aligned} I_\lambda &= \lambda(\cos u_\lambda(r) - \cos \|u_\lambda\|_\infty) + G(u_\lambda(r)) - G(\|u_\lambda\|_\infty) \\ &= \xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(r)) + \frac{1}{2}(\lambda + g'(\pi))(1+o(1))(\|u_\lambda\|_\infty - u_\lambda(r))^2. \end{aligned}$$

□

Lemma 2.4. *Let $0 < r_0 < R$ be fixed. Then for $0 \le r \le r_0$ and $\lambda \gg 1$,*

$$\begin{aligned} II_\lambda(r) &\le \frac{N-1}{N} \xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(r)) && (2.18)\\ &\quad + \frac{N-1}{2(N+1)}(\lambda + g'(\pi))(1+o(1))(\|u_\lambda\|_\infty - u_\lambda(r))^2. \end{aligned}$$

Proof. Since (2.18) is valid for $r = 0$, let $r > 0$. By l'Hopital's rule and (2.11),

$$\begin{aligned} \lim_{r\to 0} \frac{II_\lambda(r)}{\|u_\lambda\|_\infty - u_\lambda(r)} &= \lim_{r\to 0} \frac{-(N-1)u_\lambda'(r)}{r} = \lim_{r\to 0} (1-N) u_\lambda''(r) && (2.19)\\ &= (1-N)u_\lambda''(0) = \frac{N-1}{N} \xi_\lambda. \end{aligned}$$

Then by this and Cauchy's mean value theorem, for $0 < r \le r_0$ and $\lambda \gg 1$, we obtain

$$
\begin{aligned}
&\frac{II_\lambda(r) - ((N-1)\xi_\lambda/N)(\|u_\lambda\|_\infty - u_\lambda(r))}{(\|u_\lambda\|_\infty - u_\lambda(r))^2} \\
&= \frac{(N-1)u_\lambda'(r_1) + (N-1)\xi_\lambda r_1/N}{-2r_1(\|u_\lambda\|_\infty - u_\lambda(r_1))} \\
&= \frac{(N-1)u_\lambda''(r_2) + (N-1)\xi_\lambda/N}{-2(\|u_\lambda\|_\infty - u_\lambda(r_2)) + 2r_2 u_\lambda'(r_2)} \\
&= \frac{(N-1)u_\lambda'''(r_3)}{4u_\lambda'(r_3) + 2r_3 u_\lambda''(r_3)},
\end{aligned}
\tag{2.20}
$$

where $0 < r_3 < r_2 < r_1 < r$. By Lemma 2.2, we see that $-u_\lambda''(r)$ is increasing for $0 \le r \le r_0$. Then by (2.9), for *any* $0 \le r \le r_0$ and $\lambda \gg 1$, we obtain

$$
\begin{aligned}
-u_\lambda'''(r) &\le \frac{2}{N+2}(\lambda + g'(\pi))(1+o(1))(-u_\lambda'(r)) \\
&\quad + \frac{N-1}{N+1} r^{-N-1}(\lambda + g'(\pi))(1+o(1))(-u_\lambda''(r)) \int_0^r s^{N+1}\,ds \\
&\le \frac{2}{N+2}(\lambda + g'(\pi))(1+o(1))(-u_\lambda'(r)) \\
&\quad + \frac{N-1}{(N+1)(N+2)}(\lambda + g'(\pi))(1+o(1)) r(-u_\lambda''(r)) \\
&\le \frac{1}{2(N+1)}(\lambda + g'(\pi))(1+o(1))(-4u_\lambda'(r) - 2ru_\lambda''(r)).
\end{aligned}
\tag{2.21}
$$

Put $r = r_3$ in (2.21). Then by this and (2.20), we obtain (2.18). Thus the proof is complete. □

Lemma 2.5. *Let* $0 < r_0 < R$ *be fixed. Then* $\xi_\lambda = o\left(\lambda e^{-\sqrt{2\lambda(1+o(1))/(N+1)}r_0}\right)$ *as* $\lambda \to \infty$.

Proof. By (2.14), Lemmas 2.3 and 2.4, for $0 \le r \le r_0$ and $\lambda \gg 1$,

$$
\begin{aligned}
\frac{1}{2}u_\lambda'(r)^2 &= I_\lambda(r) - II_\lambda(r) \\
&\ge \frac{1}{N}\xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(r)) \\
&\quad + \frac{1}{N+1}(1+o(1))(g'(\pi)+\lambda)(\|u_\lambda\|_\infty - u_\lambda(r))^2 \\
&\ge \frac{1}{2}P_\lambda(\|u_\lambda\|_\infty - u_\lambda(r)) + \frac{1}{2}Q_\lambda(\|u_\lambda\|_\infty - u_\lambda(r))^2,
\end{aligned}
\tag{2.22}
$$

where $P_\lambda := 2\xi_\lambda/N, Q_\lambda := 2\lambda(1+o(1))/(N+1)$. We put $t = \sqrt{s/(P_\lambda + Q_\lambda s)}$. Then by (2.14) and (2.22), for $\lambda \gg 1$, we obtain

$$\begin{aligned} r_0 &= \int_0^{r_0} 1dt = \int_0^{r_0} \frac{-u_\lambda'(r)}{\sqrt{2(I_\lambda(r) - II_\lambda(r))}} dr \\ &\le \int_{u_\lambda(r_0)}^{\|u_\lambda\|_\infty} \frac{1}{\sqrt{P_\lambda(\|u_\lambda\|_\infty - \theta) + Q_\lambda(\|u_\lambda\|_\infty - \theta)^2}} d\theta \\ &= \int_0^{\|u_\lambda\|_\infty - u_\lambda(r_0)} \frac{1}{\sqrt{P_\lambda s + Q_\lambda s^2}} ds = \int_0^{A_\lambda} \frac{2}{1 - Q_\lambda t^2} dt \\ &= \frac{1}{\sqrt{Q_\lambda}} \log\left(\frac{1/\sqrt{Q_\lambda} + A_\lambda}{1/\sqrt{Q_\lambda} - A_\lambda}\right), \end{aligned}$$

where

$$A_\lambda := \sqrt{\frac{\|u_\lambda\|_\infty - u_\lambda(r_0)}{P_\lambda + Q_\lambda(\|u_\lambda\|_\infty - u_\lambda(r_0))}}.$$

This implies that

$$\sqrt{\frac{(\|u_\lambda\|_\infty - u_\lambda(r_0))}{P_\lambda + Q_\lambda(\|u_\lambda\|_\infty - u_\lambda(r_0))}} \ge \frac{1}{\sqrt{Q_\lambda}} \frac{e^{\sqrt{Q_\lambda} r_0} - 1}{e^{\sqrt{Q_\lambda} r_0} + 1}.$$

By this and (P.2), for $\lambda \gg 1$, we obtain

$$\frac{2}{N}\xi_\lambda = P_\lambda \le \frac{4Q_\lambda \lambda e^{\sqrt{Q_\lambda} r_0}(\|u_\lambda\|_\infty - u_\lambda(r_0))}{(e^{\sqrt{Q_\lambda} r_0} - 1)^2} = o\left(\lambda e^{-\sqrt{2\lambda(1+o(1))/(N+1)} r_0}\right).$$

Thus the proof is complete. □

Proof of Theorem 1 *for* $\Omega = B_R$ *and* $x = 0$. We put $\zeta_1(\lambda) := \pi - \|u_\lambda\|_\infty$. Then $\zeta_1(\lambda) \to 0$ as $\lambda \to \infty$ by (P.2). Then by Lemma 2.5, we obtain

$$\begin{aligned} \lambda \sin \|u_\lambda\|_\infty &= \lambda \sin(\pi - \zeta_1(\lambda)) = \lambda \sin \zeta_1(\lambda) \\ &= \lambda(1+o(1))\zeta_1(\lambda) \\ &= g(\|u_\lambda\|_\infty) + o(\lambda e^{-\sqrt{2\lambda(1+o(1)/(N+1)} r_0}) \\ &= g(\pi)(1+o(1)). \end{aligned} \tag{2.23}$$

This implies that

$$\zeta_1(\lambda) = \frac{g(\pi)}{\lambda} + o\left(\frac{1}{\lambda}\right). \tag{2.24}$$

Therefore, we obtain the second term in (1.5). Then (1.5) is proved by completely the same argument as that in [7, Proof of Theorem 2], which is proceeded by the mathematical induction. Thus the proof of Theorem 1 for the case $\Omega = B_R$ and $x = 0$ is complete. □

3. Proof of Theorem 1

Now we consider (1.1)–(1.3). Let $x \in \Omega$ be fixed. Let $\delta_1 := \mathrm{dist}\{x, \partial\Omega\} > 0$ and $L_1 := \sup\{|y_1 - y_2| : y_1, y_2 \in \Omega\}$. Since (1.1)–(1.3) is autonomous, by translation of the coordinate system, we may assume that $x = 0$. Let $B_1 = B_{\delta_1/2}$ and $B_2 := B_{2L_1}$. Furthermore, let $u_{\lambda,1}$ and $u_{\lambda,2}$ be the solutions of (1.7)–(1.9) for $R = \delta_1/2$ and $R = 2L_1$, respectively. It is clear that u_λ is a super-solution of (1.1)–(1.3) for B_1.

Further, for $0 < \epsilon \ll 1$, we see from (A.2) that $v_\epsilon(|x|) = \epsilon\varphi_1(|x|)$ is a sub-solution of (1.1)–(1.3) for B_1, where φ_1 is the first positive eigenfunction of $-\Delta$ in B_1 with the Dirichlet zero boundary condition. Since $0 < \epsilon \ll 1$, we see that $\epsilon\varphi_1 < u_\lambda$. Then since $u_{\lambda,1}$ is a unique solution of (1.1)–(1.3) for $\Omega = B_1$, we see from [7, p. 24] that for $x \in B_1$

$$\epsilon\varphi_1(x) \le u_{\lambda,1}(x) \le u_\lambda(x). \tag{3.1}$$

By the same argument as above, for $x \in \Omega$, we have

$$u_\lambda(x) \le u_{\lambda,2}(x). \tag{3.2}$$

In particular, by putting $x = 0$, we obtain

$$\|u_{\lambda,1}\|_\infty \le u_\lambda(0) \le \|u_{\lambda,2}\|_\infty. \tag{3.3}$$

Since the formula (1.5) holds for $\|u_{\lambda,1}\|_\infty$ and $\|u_{\lambda,2}\|_\infty$, by (3.2) and (3.3), we immediately obtain (1.5). Thus the proof is complete.

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