HYPERBOLIC GROUPS
HAVE FINITE ASYMPTOTIC DIMENSION

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Abstract. We detail a proof of a result of Gromov, that hyperbolic groups
(and metric spaces) have finite asymptotic dimension. This fact has become
important in recent work on the Novikov conjecture.

1. Introduction

Let $X$ be a metric space, with basepoint $x_0$. We use the notation $|x|$ to denote
distance $d(x,x_0)$. If $x,y \in X$, then the Gromov product $(x|y)$ is the positive real number
\[ \frac{1}{2}(|x| + |y| - d(x,y)) \]
By definition \cite{3}, $X$ is hyperbolic if there is $\delta > 0$ such that
\[ (x|z) \geq \min\{(x|y), (y|z)\} - \delta \]
for all $x, y, z \in X$.

Let $\mathcal{U}$ be a family of subsets of $X$. We say that $\mathcal{U}$ is $d$-disconnected if the minimum
distance between any two distinct sets of the family $\mathcal{U}$ is at least $d$. We say that $\mathcal{U}$ is $r$-bounded if each set in the family has diameter $\leq r$. One says that $X$ has finite asymptotic dimension if there is a number $N$ such that for each $d > 0$
there is an $r > 0$ such that $X$ can be covered by at most $N + 1$ $d$-disconnected, $r$-bounded families. The least such $N$ is the asymptotic dimension of $X$.

This definition is due to Gromov \cite{4} page 29 and was crucial to the work of Yu \cite{5} on the Novikov conjecture. On page 31 of \cite{4}, Gromov remarks that word hyperbolic groups have finite asymptotic dimension. Below we present a short proof of (a slight generalization of) this result. Our proof is related to those of the finite-dimensionality of the Gromov boundary of a hyperbolic group given in \cite{2} and \cite{1}.

2. The Proof

Let $X$ be a geodesic metric space. Say that $X$ has bounded growth if for each $s > 0$ there is a number $N_s$ such that each ball of radius $S + s$ in $X$ can be covered by at most $N_s$ balls of radius $S$.

Since $X$ is geodesic, one may take for $N_s$ the supremum (if finite) of the cardinalities of $s$-separated subsets in balls of radius $2s$. This observation shows that a
space of bounded geometry has bounded growth. In particular, the Cayley graph of a finitely generated group has bounded growth.

**Theorem 2.1.** Let $X$ be a hyperbolic geodesic metric space with bounded growth. Then $X$ has finite asymptotic dimension.

**Proof.** Fix a basepoint $x_0 \in X$, and let $d > 0$ be given. Suppose that $X$ is $\delta$-hyperbolic. Let $A_k$ denote the annulus $\{x \in X : kd \leq |x| \leq (k+1)d\}$ in $X$. It will suffice to show that there is a number $N$, independent of $d$, such that each annulus $A_k$ can be covered by a family of sets $\{U_i\}$, each having diameter no more than $4d + 4\delta$, and such that no more than $N$ of the sets $U_i$ have nonempty intersection, in $A_k$, with any set of diameter $d$.

Let $\{x_i\}$ be a maximal $d$-separated subset of the sphere $\{x : |x| = kd\}$ of radius $kd$ and define $U_i = \{x \in A_k : (x|x_i|) \geq (k - \frac{1}{2})d - \delta\}$. If $x \in A_k$ let $x'$ denote the point where a geodesic from $x_0$ to $x$ intersects the sphere of radius $kd$. Then $|x'| = (x|x_i|) = kd$. By maximality there is some $i$ for which $d(x', x_i) \leq d$ and therefore $(x'|x_i|) \geq (k - \frac{1}{2})d$. By hyperbolicity $(x|x_i|) \geq \min\{(x|x'|), (x'|x_i)\} - \delta \geq (k - \frac{1}{2})d - \delta$ and so $x \in U_i$. Thus the $U_i$ cover $A_k$ as asserted.

Suppose $x \in U_i$. Then $d(x, x_i) = |x| + |x_i| - 2|x|x_i|) \leq 2d + 2\delta$. Thus the $U_i$ have uniformly bounded diameter.

Suppose that $U_i$ meets the ball of radius $d$ around some $x \in A_k$; let $y$ be a point in the intersection. Let $x''$ be the point where a geodesic ray from $x_0$ to $x$ intersects the sphere of radius $(k - \frac{1}{2})d$, so that $|x''| = (x|x''|) = (k - \frac{1}{2})d$. We also have $(x|y) \geq (k - \frac{1}{2})d$, and $(x_i|y) \geq (k - \frac{1}{2})d - \delta$, so $(x|x''\) \geq (k - \frac{1}{2})d - 3\delta$. It follows that $d(x_i, x'') = |x_i| + |x''| - 2|x|x''|) \leq \frac{1}{2}d + 6\delta$. The maximum number of $U_i$ that meet the ball of radius $d$ around $x$ is therefore bounded by the maximum cardinality of a $d$-separated subset in a ball of radius $\frac{1}{2}d + 6\delta$. But this cardinality is bounded by the number $N_{d\delta}$ arising from the definition of bounded growth. The proof is complete. 

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**References**


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