

ON THE PROXIMALITY OF THE UNIT BALL OF PROXIMAL SUBSPACES IN BANACH SPACES: A COUNTEREXAMPLE

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ABSTRACT. A known, and easy to establish, fact in Best Approximation Theory is that, if the unit ball of a subspace G of a Banach space X is proximal in X , then G itself is proximal in X . We are concerned in this article with the reverse implication, as the knowledge of whether the unit ball is proximal or not is useful in obtaining information about other problems. We show, by constructing a counterexample, that the answer is negative in general.

1. INTRODUCTION

Let X be a Banach space with norm $\|\cdot\|$ and distance $d(\cdot, \cdot)$. We need the following definition:

Definition 1. A nonempty subset G of the Banach space X is said to be $\|\cdot\|$ -proximal (or proximal) in X if for every $x \in X$ there exists $g_x \in G$ such that

$$(1.1) \quad \|x - g_x\| = d(x, G) := \inf\{\|x - g\| : g \in G\}.$$

A point $g_x \in G$ that satisfies Eq. (1.1) is called a best approximation of x from G . The set of best approximations from G of a point $x \in X$ is denoted by $\pi_G(x)$ or $\pi(x)$. If every point $x \in X$ admits a unique best approximation from G , then G is said to be Chebyshev in X .

The concept of best approximation is one of the most important in approximation theory. It plays an important role in many areas of science. For this reason, a lot of work has been devoted to the study of proximality of subspaces of Banach spaces. We refer the reader to [2], [5], and the references cited there for a great deal of literature on the theory of best approximation.

In connection with the proximality of subspaces of Banach spaces, one encounters the problem of characterization of the relation, if any, that exists between the proximality of the subspace and the proximality of its unit ball. It is easy to show that if the unit ball B_G of the subspace G of X is proximal in X , then G itself is proximal in X . We are concerned here with the reverse implication. In other words, our aim in this paper is to investigate the following problem.

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Problem 1. *Given a Banach space X and given a subspace G of X , is the unit ball B_G of G proximal in X whenever the subspace G is proximal in X ?*

One may be inclined to think that, if G is a proximal subspace of X , then, for any $x \in X$, the projection of the set $\pi_G(x)$ onto the unit ball B_G of G contains a best approximation of x from B_G , hence the answer to Problem 1 is affirmative. By projection of an element $g \in G$ onto B_G we mean g if $g \in B_G$ and $g/\|g\|$ if $g \notin B_G$. This leads us to the investigation of the following problem.

Problem 2. *Given a proximal subspace G of a Banach space X , is it true that for every $x \in X$ we have*

$$\pi_{B_G}(x) \cap P_{B_G}(\pi_G(x)) \neq \emptyset,$$

where, for every $g \in G$, $P_{B_G}(g) = g$ if $g \in B_G$ and $P_{B_G}(g) = g/\|g\|$ if $g \notin B_G$?

Problem 2 was addressed in [4]. There the author showed that the answer is negative in general, even when B_G is also proximal in X . Moreover, it was shown that if all closed subspaces G of X are Chebyshev in X (equivalently, if X is reflexive and strictly convex, [5, Corollary 3.4, page 111]) and if the answer to Problem 2 was affirmative for all closed subspaces G of a fixed finite dimension n or for all closed subspaces G of a fixed finite codimension $(n-1)$, $2 \leq n < \dim(X)$, then X is isometric to a Hilbert space. But still, there remains the question of whether the unit ball B_G is proximal in X whenever G is proximal in X . We show in this paper, by constructing a counterexample, that the answer to Problem 1 is negative in general.

Finally, we point out that it is important to know whether the unit ball B_G is proximal in X or not, as this information may be used to obtain results about other problems. For instance, in [3] and [1] the authors use the proximality of the unit ball B_G of the subspace G of X to obtain results on the proximality of $L^p(S, G)$ in $L^p(S, X)$, where S is a finite measure space and $L^p(S, X)$ is the usual L^p -space of p -Bochner-integrable functions defined on S with values in X .

2. THE COUNTEREXAMPLE

In any Banach space E with norm denoted by $\|\cdot\|$, the ball centered at $e \in E$ and radius $r \in [0, \infty)$ is denoted by $B_E^{|\cdot|}(e, r)$, or simply by $B_E(e, r)$ when there is no confusion, and is given by

$$B_E^{|\cdot|}(e, r) = B_E(e, r) = \{x \in E : \|x - e\| \leq r\}.$$

The unit ball in E is also denoted by $B_E^{|\cdot|}$ or by B_E ,

$$B_E^{|\cdot|} = B_E = \{x \in E : \|x\| \leq 1\},$$

while the distance in E is denoted by $d_{\|\cdot\|}(\cdot, \cdot)$ or by $d(\cdot, \cdot)$. Finally, the set of real numbers is denoted by \mathbb{R} .

We now start the construction of our counterexample. Let Z be a one-dimensional real Banach space and let G be a real Banach space that contains a nonproximal hyperplane H (note that every nonreflexive Banach space G satisfies this property, [5, page 100]). We denote the norms on G and Z by $\|\cdot\|_G$ and $\|\cdot\|_Z$, respectively.

Let X be the Banach space given by the l^1 -sum of G and Z ,

$$X := G \oplus_1 Z.$$

In other words, X is the direct sum of G with Z , and the norm on X is the l^1 -norm $\|\cdot\|_1$ defined by

$$\|x\|_1 := \|g\|_G + \|z\|_Z,$$

for every $x := g + z$ where $g \in G$ and $z \in Z$. Note that $\|\cdot\|_1 = \|\cdot\|_G$ on G and $\|\cdot\|_1 = \|\cdot\|_Z$ on Z .

Now, let $z_o \in Z$ be fixed, $\|z_o\|_Z = 1$. Then the unit ball in X is the convex hull of the set $\{z_o, -z_o, B_G\}$, i.e.

$$B_X := B_X^{\|\cdot\|_1} = \text{co}\{z_o, -z_o, B_G\}.$$

Note that one can easily check that B_X is indeed closed in X . Since H is nonproximal in G , there exists $\bar{g} \in G$ such that $d(\bar{g}, H) = 1$ and

$$B_G(\bar{g}, 1) \cap H = \emptyset.$$

Let \bar{H} be the manifold given by $\bar{H} := \bar{g} + H$. Then we have

$$(2.1) \quad d(0, \bar{H}) = 1 \text{ and } B_G \cap \bar{H} = \emptyset.$$

Now let A be the subset of \bar{H} defined by $A := B_G(0, 2) \cap \bar{H}$ and let $a \in A$ be fixed. Consider the subset R of H given by

$$R := A - a.$$

Then $0 \in R$ since $a \in A$. Since R is a bounded subset of G , there exists $\gamma \in (0, \infty)$ such that

$$\frac{1}{\gamma}(R \cup (-R)) \subset B_G.$$

Let $D := \frac{1}{\gamma}R - z_o$ and let

$$K := \text{cl}_X(\text{co}\{D, B_G, -D\}),$$

where $\text{cl}_X(M)$ denotes the closure in X of the subset M of X . Since $0 \in R$, it follows that z_o and $-z_o$ are both in K and, consequently, that $B_X \subset K$. This, together with the fact that K is bounded in X , implies that there exists $\rho \in (0, \infty)$ such that

$$B_X \subset K \subset B_X(0, \rho).$$

It follows that the Minkowski functional associated with K defines a norm $\|\cdot\|$ on X that is equivalent to the l^1 -norm $\|\cdot\|_1$.

Note that the unit ball in $(X, \|\cdot\|)$ is given by K . Hence

$$B_X^{\|\cdot\|} = K.$$

To avoid confusion, we substitute the letter K for the letter B when referring to balls in X corresponding to the new norm $\|\cdot\|$. Thus, for instance,

$$K_X := B_X^{\|\cdot\|} = K \text{ and } K_X(x, r) := B_X^{\|\cdot\|}(x, r).$$

Our goal in the remainder of this paper is to show that G is $\|\cdot\|$ -proximal in X while the unit ball K_G of G is not $\|\cdot\|$ -proximal in X .

We start with the following remark which follows directly from the definitions of K and $\|\cdot\|_1$.

Remark 1. The norm on G induced by the new norm $\|\cdot\|$ is the same as the old norm $\|\cdot\|_G$. In other words, we have

$$K_G := K \cap G = B_G.$$

For each $x \in X$, let $\pi(x)$ denote the set of best approximations of x from G with respect to the new norm $\|\cdot\|$,

$$\pi(x) := \{y \in G : \|x - y\| = d_{\|\cdot\|}(x, G)\}.$$

Our next lemma shows that G is $\|\cdot\|$ -proximal in X while explicitly providing the set of best approximations of an arbitrary $x := g + \lambda z_o \in X$, where $g \in G$.

Lemma 1. *The following statements hold true:*

(i) *For every $g \in G$ and every $\lambda \in \mathbb{R}$, we have*

$$(2.2) \quad K(g + \lambda z_o, \tau) \cap G = \begin{cases} \emptyset & \text{if } \tau < \lambda, \\ g + \frac{\lambda}{\gamma} R = g + \lambda(D + z_o) & \text{if } \tau = \lambda. \end{cases}$$

(ii) *For every $g \in G$ and every $\lambda \in \mathbb{R}$, we have*

$$\pi(g + \lambda z_o) = g + \frac{\lambda}{\gamma} R = g + \lambda(D + z_o)$$

and

$$d_{\|\cdot\|}(g + \lambda z_o, G) = \lambda.$$

Hence G is $\|\cdot\|$ -proximal in X .

Proof. First, note that part (ii) follows directly from part (i).

Now, clearly we have, for every $g \in G$ and every $\lambda \in \mathbb{R}$,

$$K(g + \lambda z_o, \tau) \cap G = g + \lambda \left(K\left(z_o, \frac{\tau}{\lambda}\right) \cap G \right).$$

Therefore Eq. (2.2) is equivalent to

$$(2.3) \quad K(z_o, T) \cap G = \begin{cases} \emptyset & \text{if } T < 1, \\ \frac{1}{\gamma} R = D + z_o & \text{if } T = 1. \end{cases}$$

If $T = 0$, then $K(z_o, T) = \{z_o\} \notin G$, hence $K(z_o, T) \cap G = \emptyset$.

Now, suppose that $K(z_o, T) \cap G \neq \emptyset$ for some $T > 0$ and let $h \in K(z_o, T) \cap G$ be fixed. Then

$$\frac{(h - z_o)}{T} \in K := cl_X(co\{D, B_G, -D\}).$$

It follows, since $D = (1/\gamma)R - z_o$, that there exist sequences $\{r_n\}$, $\{r_n^-\}$, and $\{h_n\}$ in $(1/\gamma)R$, $(1/\gamma)(-R)$, and B_G respectively such that

$$\frac{(h - z_o)}{T} = \lim(\alpha_n r_n + \beta_n r_n^- + \delta_n h_n + (\beta_n - \alpha_n) z_o),$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ are sequences in $[0, 1]$ satisfying

$$\alpha_n + \beta_n + \delta_n = 1.$$

This implies, by the definition of $\|\cdot\|_1$, that $\lim(\beta_n - \alpha_n) = -1/T$. Hence $(\alpha_n - \beta_n) \neq 0$ for large n and

$$T = \lim\left(\frac{1}{\alpha_n - \beta_n}\right) \geq 1,$$

where the inequality follows from the fact that α_n and β_n are both in the interval $[0, 1]$. Therefore

$$K(z_o, T) \cap G = \emptyset \text{ if } T < 1.$$

On the other hand, if $T = 1$, then we obtain that $\lim(\alpha_n - \beta_n) = 1$ and, consequently, since α_n and β_n are both in $[0, 1]$ and since $\alpha_n + \beta_n + \delta_n = 1$, that

$$\lim \alpha_n = 1 \text{ and } \lim \beta_n = \lim \delta_n = 0.$$

It follows, since $\{r_n^-\}$ and $\{h_n\}$ are bounded ($r_n^- \in \frac{1}{\gamma}(-R) \subset B_G$ and $h \in B_G$), that $\lim \beta_n r_n^- = \lim \delta_n h_n = 0$ and, consequently, that

$$h = \lim r_n.$$

But $\{r_n\} \subset (1/\gamma)R$, which is closed. Therefore $h \in (1/\gamma)R$ and, consequently,

$$(K(z_o, 1) \cap G) \subset \frac{1}{\gamma}R.$$

The reverse inclusion follows from the fact that $((1/\gamma)R - z_o) \subset K$, hence $(1/\gamma)R \subset K(z_o, 1)$, and the fact that $(1/\gamma)R \subset G$. Therefore

$$K(z_o, 1) \cap G = \frac{1}{\gamma}R.$$

This completes the proof of Eq. (2.3) and, consequently, of Eq. (2.2). The proof of the lemma is now complete. \square

It remains to show that the unit ball K_G of G is not $\|\cdot\|$ -proximal in X . Indeed, we have:

Lemma 2. *The unit ball K_G of G is not $\|\cdot\|$ -proximal in X .*

Proof. First note that, by Remark 1, $K_G = B_G$ and, consequently,

$$(2.4) \quad d_{\|\cdot\|} = d_{\|\cdot\|_1} = d \text{ on } G \times G.$$

By Lemma 1 and the definition of R , we have, for every $g \in G$ and every $\lambda \in \mathbb{R}$,

$$(2.5) \quad \pi(g + \lambda z_o) = g + \frac{\lambda}{\gamma}R = g - \frac{\lambda}{\gamma}a + \frac{\lambda}{\gamma}A.$$

Let \bar{x} be the fixed element of X defined by

$$\bar{x} := a + \gamma z_o.$$

Then we have, by Eq. (2.5) and Lemma 1,

$$(2.6) \quad \pi(\bar{x}) = K(\bar{x}, \gamma) \cap G = A \text{ and } d_{\|\cdot\|}(\bar{x}, G) = \gamma.$$

It follows, by Eqs. (2.1) and (2.4) and by the definition of A , that

$$d(\pi(\bar{x}), B_G) = d(A, B_G) = d(\bar{H}, B_G) = 0$$

and

$$\pi(\bar{x}) \cap B_G = A \cap B_G = \emptyset.$$

This together with Eq. (2.6) imply that $d(\pi(\bar{x}), B_G) = 0$ and, consequently,

$$d_{\|\cdot\|}(\bar{x}, B_G) = d_{\|\cdot\|}(\bar{x}, G) = \gamma \text{ and } K(\bar{x}, \gamma) \cap B_G = \emptyset.$$

This implies that \bar{x} has no best approximation from B_G . It follows, since $K_G = B_G$, that K_G is not $\|\cdot\|$ -proximal in X , which completes the proof of the lemma. \square

In summary, we have proved the following theorem, which represents our main result.

Theorem 1. *Let G be a real Banach space that contains a nonproximal hyperplane, let Z be a one-dimensional real Banach space, and let X be the Banach space given by the l^1 -sum of G and Z , $X := G \oplus_1 Z$. Then there exists a norm $\|\cdot\|$ on X equivalent to the l^1 -norm $\|\cdot\|_1$ such that G is $\|\cdot\|$ -proximal in X , while the unit ball $B_G^{\|\cdot\|}$ of G is not $\|\cdot\|$ -proximal in X .*

Since every nonreflexive Banach space contains a nonproximal hyperplane, [5, page 100], we immediately obtain the following two corollaries:

Corollary 1. *Let G be a nonreflexive real Banach space, let Z be a real Banach space of dimension one, and let X be the Banach space given by the l^1 -sum of G and Z , $X := G \oplus_1 Z$. Then there exists a norm $\|\cdot\|$ on X equivalent to the l^1 -norm $\|\cdot\|_1$ such that G is $\|\cdot\|$ -proximal in X while the unit ball $B_G^{\|\cdot\|}$ of G is not $\|\cdot\|$ -proximal in X .*

Corollary 2. *There exists a Banach space X that contains a proximal subspace G whose unit ball B_G is nonproximal in X .*

We conclude by showing that it was necessary to renorm X in order to obtain a counterexample. Indeed, the unit ball B_G of G is $\|\cdot\|_1$ -proximal in X , as the following proposition shows.

Proposition 1. *Let G be a real Banach space, let Z be a one-dimensional real Banach space, and let X be the Banach space given by the l^1 -sum of G and Z , $X := G \oplus_1 Z$. Then G and its unit ball B_G are both $\|\cdot\|_1$ -proximal in X .*

Proof. Let $x := g_x + z_x \in X$ be fixed, $g_x \in G$ and $z_x \in Z$. Clearly we have, for every $g \in G$,

$$\|x - g\|_1 = \|g_x - g\|_G + \|z_x\|_Z \geq \|z_x\|_Z = \|x - g_x\|_1.$$

Therefore g_x is a best approximation of x from G with respect to $\|\cdot\|_1$ and, consequently,

$$G \text{ is } \|\cdot\|_1\text{-proximal in } X.$$

In fact, g_x is the unique best approximation of x from G with respect to $\|\cdot\|_1$, hence G is Chebyshev in X .

Now, if $g_x \in B_G$, then g_x is also a best approximation of x from B_G with respect to $\|\cdot\|_1$. Therefore, suppose that $g_x \notin B_G$. Then we have

$$\left\| g_x - \frac{g_x}{\|g_x\|} \right\|_G = \inf\{\|g_x - g\|_G : g \in B_G\}.$$

It follows that

$$\begin{aligned} \left\| x - \frac{g_x}{\|g_x\|} \right\|_1 &= \left\| g_x - \frac{g_x}{\|g_x\|} \right\|_G + \|z_x\|_Z \\ &\leq \|g_x - g\|_G + \|z_x\|_Z = \|x - g\|_1, \end{aligned}$$

for all $g \in B_G$. Hence $g_x/\|g_x\|$ is a best approximation of x from B_G with respect to $\|\cdot\|_1$. Therefore in all cases x has a best approximation from B_G with respect to $\|\cdot\|_1$ and, consequently, B_G is $\|\cdot\|_1$ -proximal in X . This completes the proof of the proposition. \square

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