

REGULARIZATION FOR A CLASS OF ILL-POSED CAUCHY PROBLEMS

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ABSTRACT. This paper is concerned with the ill-posed Cauchy problem associated with a densely defined linear operator A in a Banach space. Our main result is that if $-A$ is the generator of an analytic semigroup of angle $\geq \pi/4$, then there exists a family of regularizing operators for such an ill-posed Cauchy problem by using the Gajewski and Zacharias quasi-reversibility method, and semigroups of linear operators.

1. INTRODUCTION

Let A be an unbounded operator in a Banach space X and let $-A$ generate an analytic semigroup $\{S(t)\}$ of angle α , where $\pi/4 \leq \alpha < \pi/2$. In this paper, we consider the following Cauchy problem:

$$(1.1) \quad u'(t) = Au(t) \quad (0 < t \leq T), \quad u(0) = x.$$

A continuous function $u(t)$ ($0 \leq t \leq T$) is said to be a solution of (1.1) if for $0 < t \leq T$, $u(t)$ is continuously differentiable, its values belong to the domain of A , and (1.1) is satisfied.

It is not hard to show that $S(t)u(t) = x$ if $u(t)$ is a solution of (1.1). Since $S(t)$ is invertible for each $t \geq 0$ (cf. [5, p. 69]), we obtain $u(t) = S(t)^{-1}x$ for $0 \leq t \leq T$. However, $S(t)^{-1}$ ($t \geq 0$) is not a family of bounded linear operators. Thus (1.1) is not stable. The problem (1.1) can then lead to the general ill-posed problem, which was introduced by Tikhonov (see [10]).

If there exists a family of regularizing operators for (1.1) (cf. Definition 3.1), then the regularization of the ill-posed problem (1.1) is obtained (cf. [10]). We in this paper obtain the regularization of (1.1) by using the solution of the well-posed Cauchy problem

$$(1.2) \quad u'_\varepsilon(t) + \varepsilon Au'_\varepsilon(t) = Au_\varepsilon(t) \quad (0 < t \leq T), \quad u_\varepsilon(0) = x$$

to approximate the solution of (1.1), where $\varepsilon > 0$.

This method is called the quasi-reversibility method, introduced by Gajewski and Zacharias [2] in Hilbert space at first. Showalter [9] then generalized their results to

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the case that $-A$ generates an analytic contraction semigroup of angle α with $\pi/4 \leq \alpha < \pi/2$ in Hilbert space. In [9], the key point for using the approximate theorem of semigroups is to use the properties of the dissipative operator in Hilbert space. However, it is false in Banach space even if $\{S(t)\}$ is contractive. We overcome this difficulty by complex analysis and the generation theorem of semigroups.

We notice that Mel'nikova [6] (also see [3, 7]) constructed a family of regularizing operators by using the solution of the well-posed Cauchy problem:

$$(1.3) \quad v'_\varepsilon(t) = (A - \varepsilon A^2)v_\varepsilon(t) \quad (0 < t \leq T), \quad v_\varepsilon(0) = x.$$

Such a method is also called the quasi-reversibility method, introduced by Lattes and Lions [4]. Because (1.2) and (1.3) are different, the corresponding semigroup analysis and the construction of the family of regularizing operators in this paper are different from [6]. One advantage is that (1.2) is related to the quasi-parabolic or Sobolev equation if A is a realization of a partially differential operator (cf. [9]).

Throughout the paper, $B(X)$ is the space of all bounded linear operators from X into itself. By $D(B)$, $\sigma(B)$, $\rho(B)$, and $R(\lambda, B)$ ($\lambda \in \rho(B)$) we denote the domain, spectrum set, resolvent set, and resolvent of the operator B , respectively. Set $\Sigma_\alpha = \{\lambda \in \mathbf{C} \setminus \{0\}; |\arg \lambda| < \alpha\}$ for $\alpha \in (0, \pi)$.

This paper is organized as follows. We prove in section 2 that $A(I + \varepsilon A)^{-1} - A$ generates a bounded analytic semigroup $\{E_\varepsilon(t)\}$ if $\{S(t)\}$ is bounded and $0 \in \rho(A)$. Moreover, there exists a positive constant M independent of ε such that $\|E_\varepsilon(t)\| \leq M$ ($t \geq 0$). The main result of this paper is given in section 3, which states that if $-A$ generates an analytic semigroup of angle α with $\pi/4 \leq \alpha < \pi/2$, then there exists a family of regularizing operators for the ill-posed Cauchy problem (1.1). In addition, Remark 3.3 points out that the restriction $\pi/4 \leq \alpha < \pi/2$ cannot be relaxed to $0 < \alpha < \pi/2$ by this paper's method.

2. SEMIGROUP ANALYSIS

Let the operator $-A$ be a generator of a bounded analytic semigroup $S(t)$ of angle α , $\pi/4 \leq \alpha < \pi/2$, and $0 \in \rho(A)$. For $\varepsilon > 0$, set $J_\varepsilon = (I + \varepsilon A)^{-1}$ and $A_\varepsilon = AJ_\varepsilon$. From the identity

$$A_\varepsilon = \varepsilon^{-1}(I - J_\varepsilon)$$

and from J_ε being bounded, it follows that A_ε is bounded. We thus can define a C_0 -group of linear operators by

$$S_\varepsilon(t) = \exp(-tA_\varepsilon), \quad -\infty < t < \infty,$$

where we use the power series to define the exponential function.

By Theorem VII.9.5 in Dunford-Schwartz [1, p. 602],

$$\sigma(A_\varepsilon) = \{\mu(1 + \varepsilon\mu)^{-1}; \mu \in \sigma(A)\} \cup \{1/\varepsilon\}.$$

So, $\Sigma_{2\alpha} \subset \rho(-A_\varepsilon)$. For $\lambda \in \Sigma_{2\alpha}$, we have

$$\begin{aligned} \|R(\lambda, -A_\varepsilon)\| &= \|(\lambda I + (1 + \varepsilon\lambda)A)^{-1} + \frac{\varepsilon}{1 + \varepsilon\lambda} \{I - \lambda(\lambda I + (1 + \varepsilon\lambda)A)^{-1}\}\| \\ &= \|\varepsilon(1 + \varepsilon\lambda)^{-1}I + (1 + \varepsilon\lambda)^{-1}(\lambda I + (1 + \varepsilon\lambda)A)^{-1}\| \\ &\leq \frac{\varepsilon}{|1 + \varepsilon\lambda|} + \frac{M}{|\lambda||1 + \varepsilon\lambda|} \\ &\leq \frac{M}{|\lambda|}, \end{aligned}$$

where the positive constant M is independent of ε . Therefore, there exists a positive constant M_1 independent of ε such that $\|S_\varepsilon(t)\| \leq M_1$ ($t \geq 0$). Also, it is not hard to show that $\overline{(I + A)D(A^2)} = X$. Noticing that A_ε approximates A on $D(A^2)$ for small ε , we have $\lim_{\varepsilon \rightarrow 0} S_\varepsilon(t)x = S(t)x$ for $x \in X$ by the approximation theorem of semigroups.

Theorem 2.1. *For each $\varepsilon > 0$ let $E_\varepsilon(t) = S(t)S_\varepsilon(-t)$ ($t \geq 0$). Then $E_\varepsilon(t)$ is an analytic semigroup on X and $A_\varepsilon - A$ is its generator. In addition, there exists a positive constant M independent of ε such that $\|E_\varepsilon(t)\| \leq M$ ($t \geq 0$).*

Proof. Similar to the proof of Lemma 1 in Showalter [9], $E_\varepsilon(t)$ is a semigroup on X and $A_\varepsilon - A$ is the generator. Since A_ε is bounded and $-A$ generates an analytic semigroup, $E_\varepsilon(t)$ is analytic by the perturbation theorem and the uniqueness of the semigroup.

In order to end the proof we first consider the spectrum of $A_\varepsilon - A$.

For $\lambda \in \mathbf{C}$, we have

$$\begin{aligned} (\lambda I - A_\varepsilon + A)x &= (\lambda I - \varepsilon^{-1}I + \varepsilon^{-1}(I + \varepsilon A)^{-1} + A)x \\ &= [(\lambda I - \varepsilon^{-1}I + A)(I + \varepsilon A) + \varepsilon^{-1}](I + \varepsilon A)^{-1}x \\ &= (\lambda I + \lambda \varepsilon A + \varepsilon A^2)(I + \varepsilon A)^{-1}x, \quad x \in D(A). \end{aligned}$$

Denote $\Lambda = \sqrt{\varepsilon^2\lambda^2 - 4\varepsilon\lambda}$, and take the branch of $\Lambda > 0$ for $\varepsilon\lambda > 4$. The inverse

$$\begin{aligned} &(\lambda I - A_\varepsilon + A)^{-1} \\ &= (I + \varepsilon A)(\lambda I + \lambda \varepsilon A + \varepsilon A^2)^{-1} \\ &= \varepsilon^{-1}(I + \varepsilon A)\left(A + \frac{\varepsilon\lambda - \Lambda}{2\varepsilon}I\right)^{-1}\left(A + \frac{\varepsilon\lambda + \Lambda}{2\varepsilon}I\right)^{-1} \\ &= \left(A + \frac{\varepsilon\lambda + \Lambda}{2\varepsilon}I\right)^{-1} + \left(1 - \frac{\varepsilon\lambda - \Lambda}{2}\right)\left(\varepsilon A + \frac{\varepsilon\lambda - \Lambda}{2}I\right)^{-1}\left(A + \frac{\varepsilon\lambda + \Lambda}{2\varepsilon}I\right)^{-1} \\ &= \left(A - \frac{\varepsilon\lambda - \Lambda}{2\varepsilon}I\right)^{-1} + \left(1 - \frac{\varepsilon\lambda + \Lambda}{2}\right)\left(\varepsilon A + \frac{\varepsilon\lambda + \Lambda}{2}I\right)^{-1}\left(A + \frac{\varepsilon\lambda - \Lambda}{2\varepsilon}I\right)^{-1} \end{aligned}$$

exists (as a bounded operator) if and only if

$$-\frac{\varepsilon\lambda \pm \Lambda}{2\varepsilon} \in \rho(A).$$

This implies (see Figure 1)

$$\sigma(A_\varepsilon - A) \subset \{-\varepsilon\lambda^2(1 + \varepsilon\lambda)^{-1} : \lambda \in \sigma(A)\} := \mathcal{G}.$$

Set $\lambda = re^{\pm i(\pi/2 - \alpha)}$. Then the parametric equations of the curve γ are as follows:

$$\gamma : \begin{cases} x = -\frac{\varepsilon r^2(\cos 2(\pi/2 - \alpha) + \varepsilon r \cos(\pi/2 - \alpha))}{1 + 2r\varepsilon \cos(\pi/2 - \alpha) + \varepsilon^2 r^2} \\ y = \mp \frac{\varepsilon r^2(\sin 2(\pi/2 - \alpha) + \varepsilon r \sin(\pi/2 - \alpha))}{1 + 2r\varepsilon \cos(\pi/2 - \alpha) + \varepsilon^2 r^2}, \end{cases} \quad (r \geq 0),$$

where \mathcal{G} is the left-side region surrounded by the curve γ .

For each $\beta \in [2(\pi/2 - \alpha), \pi/2)$ or $\beta \in (\pi/2, 2\alpha]$, it is not hard to show that the line $y = x \tan \beta$ intersects the curve γ only at the origin. Hence,

$$\Sigma_{2\alpha} \subset \mathbf{C} \setminus \mathcal{G} \subset \rho(A_\varepsilon - A).$$

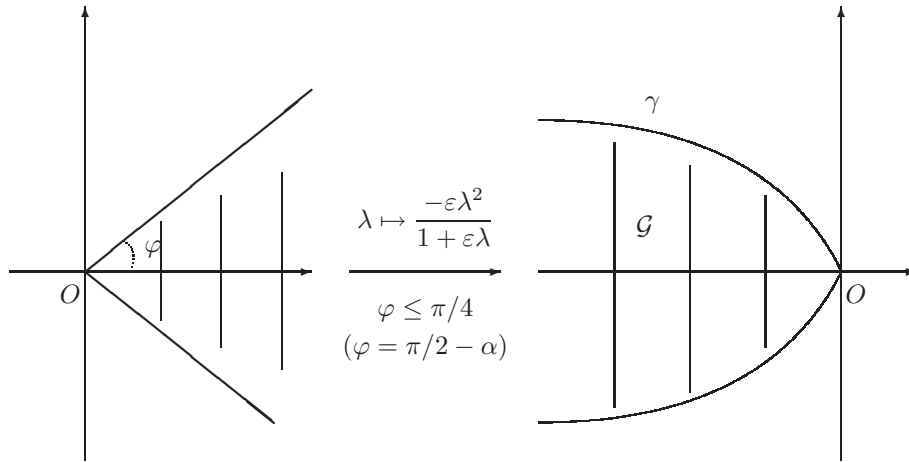


FIGURE 1.

Now we estimate the resolvent. For $\lambda \in \Sigma_{2\alpha} \subset \rho(A_\epsilon - A)$, we have

$$\|R(\lambda, A_\epsilon - A)\| \leq \begin{cases} \left| \frac{2\epsilon}{\epsilon\lambda + \Lambda} \right| + \left| 1 - \frac{\epsilon\lambda - \Lambda}{2} \right| \left| \frac{2}{\epsilon\lambda - \Lambda} \right| \left| \frac{2\epsilon}{\epsilon\lambda + \Lambda} \right|, & \text{Im } \lambda \geq 0, \\ \left| \frac{2\epsilon}{\epsilon\lambda - \Lambda} \right| + \left| 1 - \frac{\epsilon\lambda + \Lambda}{2} \right| \left| \frac{2}{\epsilon\lambda + \Lambda} \right| \left| \frac{2\epsilon}{\epsilon\lambda - \Lambda} \right|, & \text{Im } \lambda < 0. \end{cases}$$

Denote $\epsilon\lambda$ by z , and define

$$f(z) = \frac{z}{z + \sqrt{z^2 - 4z}},$$

for $z \in G_+ := \{z \in \Sigma_{2\alpha}; \text{Im } z > 0\}$ (see Figure 2). Then $f(z)$ is continuous in G_+ .

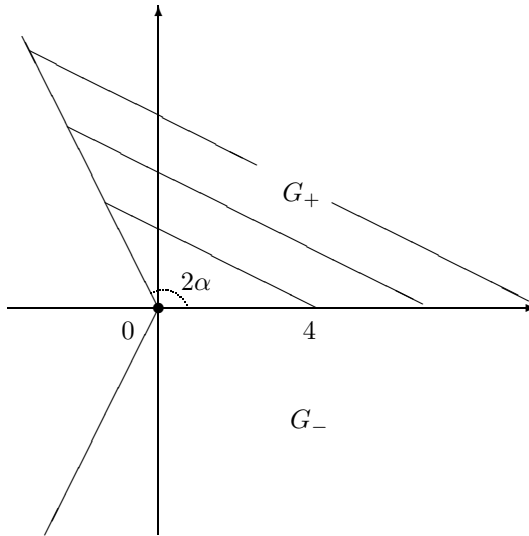


FIGURE 2.

Moreover, $f(z)$ can be continuous to the upper segment of $[0, 4]$. This means $f(z)$ is continuous on $\overline{G_+}$ if we define $f(z) = 0$ (since $f(z) \rightarrow 0$ as $|z| \rightarrow 0, \text{Im } z > 0$), where the upper segment of $[0, 4]$ is a part of ∂G_+ , the boundary of G_+ .

Set $z = re^{i\theta}, 0 \leq \theta \leq 2\alpha$. Then

$$\frac{\sqrt{z^2 - 4z}}{z} = \frac{\sqrt{r^2 e^{i2\theta} - 4r e^{i\theta}}}{r e^{i\theta}} \rightarrow 1 \quad (r \rightarrow \infty).$$

Consequently,

$$\lim_{|z| \rightarrow \infty, z \in G_+} f(z) = \lim_{|z| \rightarrow \infty, z \in G_+} \left(1 + \frac{\sqrt{z^2 - 4z}}{z}\right)^{-1} = 1/2.$$

Thus $f(z)$ is bounded on $\overline{G_+}$. Also, we have by l'Hospital's rule,

$$\begin{aligned} & \lim_{|z| \rightarrow \infty, z \in G_+} \frac{2}{z - \sqrt{z^2 - 4z}} \left(1 - \frac{z - \sqrt{z^2 - 4z}}{2}\right) \\ &= \lim_{|z| \rightarrow \infty, z \in G_+} \frac{2 - z + \sqrt{z^2 - 4z}}{z - \sqrt{z^2 - 4z}} = -1. \end{aligned}$$

Moreover,

$$\begin{aligned} & \lim_{|z| \rightarrow 0, z \in G_+} \frac{2}{z - \sqrt{z^2 - 4z}} \cdot \frac{2z}{z + \sqrt{z^2 - 4z}} \left(1 - \frac{z - \sqrt{z^2 - 4z}}{2}\right) \\ &= \lim_{|z| \rightarrow 0, z \in G_+} \left(1 - \frac{z - \sqrt{z^2 - 4z}}{2}\right) \\ &= 1. \end{aligned}$$

Thus, the function

$$g(z) = \left(1 - \frac{z - \sqrt{z^2 - 4z}}{2}\right) \cdot \frac{2z}{z - \sqrt{z^2 - 4z}} \cdot \frac{2}{z + \sqrt{z^2 - 4z}}$$

is bounded on $\overline{G_+}$, too. Consequently, there exists a positive constant M such that

$$|f(z)| + |g(z)| \leq M, \quad \text{Im } z \geq 0.$$

Substituting z for $\varepsilon\lambda$, it follows that

$$(2.1) \quad \|R(\lambda, A_\varepsilon - A)\| \leq M/|\lambda|, \quad \text{Im } \lambda \geq 0,$$

for $\lambda \in \Sigma_{2\alpha}$. Analogously, we have the same final estimate in G_- . In this case,

$$\frac{\sqrt{\varepsilon^2 \lambda^2 - 4\varepsilon\lambda}}{\varepsilon\lambda} = \frac{\sqrt{\varepsilon^2 r^2 e^{i2\theta} - 4\varepsilon r e^{i\theta}}}{\varepsilon r e^{i\theta}} \rightarrow -1 \quad (r \rightarrow \infty)$$

for $\lambda = re^{i\theta}, -2\alpha < \theta < 0$. This implies that $\sqrt{e^{i2\theta}} = \sqrt{e^{i(2\theta+2\pi)}} = -e^{i\theta}$.

Summing up,

$$\|R(\lambda, A_\varepsilon - A)\| \leq M/|\lambda|, \quad \lambda \in \Sigma_{2\alpha}.$$

By the generation of semigroups, there exists a positive constant M_2 independent of ε such that $\|E_\varepsilon(t)\| \leq M_2$ ($t \geq 0$). □

3. REGULARIZATION FOR (1.1) AND A REMARK

We start with the definition of a family of regularizing operators (cf. [10]).

Definition 3.1. A family $\{R_{\varepsilon,t}; \varepsilon > 0, t \in [0, T]\} \subset B(X)$ is called a family of regularizing operators for the problem (1.1) if for each solution $u(t)$ ($0 \leq t \leq T$) of (1.1) with initial element x , and for any $\delta > 0$, there exists $\varepsilon(\delta) > 0$, such that

- (a) $\varepsilon(\delta) \rightarrow 0$ ($\delta \rightarrow 0$),
- (b) $\|R_{\varepsilon(\delta),t}x_\delta - u(t)\| \rightarrow 0$ ($\delta \rightarrow 0$) for each $t \in [0, T]$ on the condition that x_δ satisfies $\|x_\delta - x\| \leq \delta$.

Define $R_{\varepsilon,t} = S_\varepsilon(-t)$ for $\varepsilon > 0$ and $0 \leq t \leq T$, where $S_\varepsilon(t)$ is the semigroup generated by $-A_\varepsilon$. Then $\{R_{\varepsilon,t}; \varepsilon > 0, t \in [0, T]\} \subset B(X)$.

Theorem 3.2. Assume that $-A$ is the generator of an analytic semigroup of angle α , $\pi/4 \leq \alpha < \pi/2$. Then the operator family $\{R_{\varepsilon,t}\}$ defined above is a family of regularizing operators for (1.1).

Proof. We first consider the case that $-A$ is the generator of a bounded analytic semigroup of angle α and $0 \in \rho(A)$, where $\pi/4 \leq \alpha < \pi/2$.

Let $u(t)$ ($0 \leq t \leq T$) be a solution of (1.1) with initial element x , and let $\|x_\delta - x\| \leq \delta$. By $\|A_\varepsilon\| \leq M\varepsilon^{-1}$, we have

$$\|R_{\varepsilon,t}\| = \|S_\varepsilon(-t)\| \leq e^{Mt/\varepsilon} \leq e^{MT/\varepsilon}, \quad t \in [0, T].$$

Choose $\varepsilon = -2MT(\ln \delta)^{-1}$. Then $\varepsilon \rightarrow 0$ ($\delta \rightarrow 0$) and

$$(3.1) \quad \|R_{\varepsilon,t}x_\delta - R_{\varepsilon,t}x\| \leq \delta e^{MT/\varepsilon} = \sqrt{\delta} \rightarrow 0 \quad (\delta \rightarrow 0).$$

The approximation theorem of semigroups with Theorem 2.1 and (2.1) yields that

$$\lim_{\varepsilon \rightarrow 0} E_\varepsilon(t)x = x, \quad x \in X, \quad t \geq 0.$$

It is easy to show that $x = S(t)u(t)$. It follows that

$$(3.2) \quad \|R_{\varepsilon,t}x - u(t)\| = \|S_\varepsilon(-t)S(t)u(t) - u(t)\| = \|E_\varepsilon(t)u(t) - u(t)\| \rightarrow 0 \quad (\delta \rightarrow 0).$$

Combining (3.1) with (3.2) yields that $\{R_{\varepsilon,t}\}$ is a family of regularizing operators for (1.1) by Definition 3.1.

Next, we return to the general case that $-A$ is the generator of an analytic semigroup. There exists a constant $\omega \in \mathbf{R}$ such that $\omega - A$ is the generator of a bounded analytic semigroup and $0 \in \rho(A - \omega)$. As seen above, there exists a family of regularizing operators $\{R_{\varepsilon,t}\}$ for the problem

$$(3.3) \quad v'(t) = (A - \omega)v(t) \quad (0 < t \leq T), \quad v(0) = x.$$

Let $u(t)$ ($0 \leq t \leq T$) be a solution of (1.1) with initial element x . Then $v(t) := e^{-\omega t}u(t)$ ($0 \leq t \leq T$) is a solution of (3.3) with initial element x . Hence for each $t \in [0, T]$,

$$\|e^{\omega t}R_{\varepsilon(\delta),t}x_\delta - u(t)\| = e^{\omega t}\|R_{\varepsilon(\delta),t}x_\delta - v(t)\| \rightarrow 0 \quad (\delta \rightarrow 0),$$

i.e. $\{e^{\omega t}R_{\varepsilon,t}\}$ is a family of regularizing operators for (1.1). \square

Remark 3.3. The method above is false for $0 < \alpha < \pi/4$.

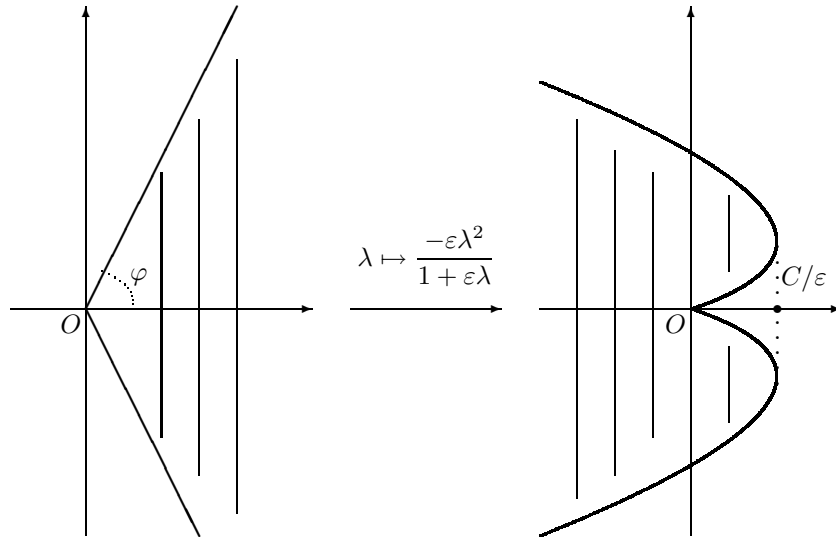


FIGURE 3.

Set $\varphi = \pi/2 - \alpha$. Consider the image of two rays $r \exp(\pm i\varphi)$ ($r \geq 0$) under the map $\lambda \mapsto -\varepsilon\lambda^2(1 + \varepsilon\lambda)^{-1}$. We consider only the case $\lambda = r \exp(i\varphi)$ by the symmetry.

It is easy to obtain that

$$\frac{-\varepsilon(r \exp(i\varphi))^2}{1 + \varepsilon r \exp(i\varphi)} = -\frac{\varepsilon r^2(\cos 2\varphi + \varepsilon r \cos \varphi)}{1 + 2\varepsilon r \cos \varphi + \varepsilon^2 r^2} - i\frac{\varepsilon r^2(\sin 2\varphi + \varepsilon r \sin \varphi)}{1 + 2\varepsilon r \cos \varphi + \varepsilon^2 r^2}.$$

Denote the real part by $f(r)$, that is

$$f(r) = -\frac{\varepsilon r^2(\cos 2\varphi + \varepsilon r \cos \varphi)}{1 + 2\varepsilon r \cos \varphi + \varepsilon^2 r^2}, \quad r \geq 0.$$

It is obvious that $f(r) \leq 0$ for $0 < \varphi \leq \pi/4$.

When $\pi/4 < \varphi < \pi/2$, $f'(r) = 0$ yields the following equation:

$$r^3 \varepsilon^3 \cos \varphi + 4r^2 \varepsilon^2 \cos^2 \varphi + r\varepsilon(3 + 2 \cos 2\varphi) \cos \varphi + 2 \cos 2\varphi = 0.$$

Its unique real solution is

$$r_0 = \frac{1}{\varepsilon} \left(\frac{G^{1/3}}{3} - \frac{1 - \frac{2}{3} \cos^2 \varphi}{G^{1/3}} - \frac{4}{3} \cos \varphi \right),$$

where

$$G = (8 \cos^4 \varphi - 36 \cos^2 \varphi + 27 + 3H) / \cos \varphi,$$

$$H = \sqrt{-48 \cos^6 \varphi + 180 \cos^4 \varphi - 213 \cos^2 \varphi + 81}.$$

This suggests that $f(r_0) = C/\varepsilon = \max_{r \geq 0} f(r)$, where $C = f(r_0)|_{\varepsilon=1} > 0$ for $\pi/4 < \varphi < \pi/2$. We thus understand the range of the spectrum $\sigma(A_\varepsilon - A)$ (see the shadow part of the right side in Figure 3). In view of the above-mentioned facts, it is not necessary to apply the approximation theorem of semigroups to $\lim_{\varepsilon \rightarrow 0} E_\varepsilon(t)$. In fact, although $A_\varepsilon - A$ generates an analytic semigroup, its growth order ω is dependent on ε (indeed, $\omega = M/\varepsilon$ for some positive constant M). This indicates

that it may be impossible to relax the restriction $\pi/4 \leq \alpha < \pi/2$ to $0 < \alpha < \pi/2$ by this paper's method.

REFERENCES

- [1] N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Interscience, New York, 1985. MR1009162 (90g:47001a)
- [2] H. Gajewski and K. Zacharias, Zur Regularisierung einer Klasse nichtkorrekter Probleme bei Evolutionsgleichungen, *J. Math. Anal. Appl.* **38**(1972), 784-789. MR0308625 (46:7739)
- [3] V. K. Ivanov, I. V. Mel'nikova and A.I. Filinkov, *Differential Operator Equations and Ill-posed Problems*, Nauka, Moscow, 1995. (In Russian) MR1415388 (97j:47092)
- [4] R. Lattes and J.-L. Lions, *The Method of Quasi-reversibility Applications to Partial Differential Equations*, Amer. Elsevier Publ. Co., New York, 1969. MR0243746 (39:5067)
- [5] R. deLaubenfels, *Existence Families, Functional Calculi and Evolution Equations*, Springer-Verlag, Berlin, 1994. MR1290783 (96b:47047)
- [6] I. V. Mel'nikova, General theory of the ill-posed Cauchy problem, *J. Inverse and Ill-posed Problems* **3**(1995), 149-171. MR1341036 (97a:34156)
- [7] I. V. Mel'nikova and A.I. Filinkov, *Abstract Cauchy Problems: Three Approaches*, Chapman and Hall, London, 2001. MR1823612 (2002h:34110)
- [8] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin, 1983. MR0710486 (85g:47061)
- [9] R. E. Showalter, The final problem for evolution equations, *J. Math. Anal. Appl.* **47**(1974), 563-572. MR0352644 (50:5131)
- [10] A. N. Tikhonov and V. Y. Arsenin, *Solution of Ill-posed Problems*, Winston and Sons, Washington D.C., 1977. MR0455365 (56:13604)

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