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CAN A LARGE CARDINAL BE FORCED FROM A CONDITION IMPLYING ITS NEGATION?

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Abstract. In this note, we provide an affirmative answer to the title question by giving two examples of cardinals satisfying conditions implying they are non-Rowbottom which can be turned into Rowbottom cardinals via forcing. In our second example, our cardinal is also non-Jonsson.

A well-known phenomenon is that under certain circumstances, it is possible to force over a given model V of ZFC containing a cardinal κ satisfying a large cardinal property $\varphi(\kappa)$ to create a universe $\overline{V}$ in which $\varphi(\kappa)$ no longer holds, yet over which $\varphi(\kappa)$ can be resurrected via forcing. A folklore example of this is provided by supposing that $V \vDash$ "ZFC + GCH + κ is measurable". If $\mathbb{P}$ is defined as the reverse Easton iteration of length κ which adds a Cohen subset to each inaccessible cardinal below κ, then in $V^{\mathbb{P}}$, κ is no longer measurable, by, e.g., a simpler version of the argument given in Lemma 2.4 of [1]. However, if one forces over $V^{\mathbb{P}}$ by adding a Cohen subset to κ, then the standard reverse Easton arguments show that κ's measurability has been resurrected, since the entire forcing can now be viewed as the length $\kappa + 1$ reverse Easton iteration over V that adds a Cohen subset to each inaccessible cardinal less than or equal to κ.

In this note, we consider a different, stronger phenomenon. Specifically, we examine the following.

Question. Suppose φ is a given large cardinal property. Is it possible to find a formula ψ in one free variable in the language of set theory and a cardinal κ such that $\psi(\kappa)$ holds, ZFC $\vdash$ "$\forall\lambda[\psi(\lambda) \implies \neg\varphi(\lambda)]$", yet there is a partial ordering $\mathbb{P}$ such that $\Vdash_{\mathbb{P}} \varphi(\kappa)$?

We show that the answer to our Question is yes for φ the large cardinal property of Rowbottomness by proving the following theorem.

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Theorem. (1) *There is a formula* $\psi_1(x)$ *in one free variable such that* ZFC $\vdash$ "$\psi_1(\kappa) \implies [\kappa$ *is not Rowbottom, yet for some partial ordering* $\mathbb{P}$, $\Vdash_{\mathbb{P}}$ "κ *is a Rowbottom cardinal carrying a Rowbottom filter*"] *for all cardinals* κ".

(2) *There is a formula* $\psi_2(x)$ *in one free variable such that* ZFC $\vdash$ "$\psi_2(\kappa) \implies [\kappa$ *is not Jonsson, yet for some partial ordering* $\mathbb{P}$, $\Vdash_{\mathbb{P}}$ "κ *is a Rowbottom cardinal carrying a Rowbottom filter*"] *for all cardinals* κ".

Note that $\mathcal{F}$ is a Rowbottom filter for κ if for any $\lambda < \kappa$ and $f : [\kappa]^{<\omega} \to \lambda$, there is some $A \in \mathcal{F}$ such that $|f''[A]^{<\omega}| \leq \omega$ (where as usual, $[\kappa]^{<\omega}$ is the set of both empty and non-empty finite sequences of elements of κ, where each non-empty finite sequence is written in increasing order).

In other words, there is a cardinal κ_1 which satisfies a condition making it automatically non-Rowbottom, yet which can be turned into a Rowbottom cardinal carrying a Rowbottom filter via forcing. Further, there is another cardinal κ_2 which satisfies a condition making it automatically non-Jonsson, yet which can be turned into a Rowbottom cardinal carrying a Rowbottom filter via forcing. This is in sharp contrast to the example given in the first paragraph of this note and the examples to be given in the next to last paragraph of this note. There, one simply has a model V, a large cardinal property φ, a cardinal κ, and a partial ordering $\mathbb{P}$ such that $V \vDash \neg\varphi(\kappa)$ and $V^{\mathbb{P}} \vDash \varphi(\kappa)$. In the Theorem, however, one has that for *every* model V, for the appropriate φ and ψ, if $V \vDash \psi(\kappa)$, then both $V \vDash \neg\varphi(\kappa)$ and there is a partial ordering $\mathbb{P} \in V$ with $V^{\mathbb{P}} \vDash \varphi(\kappa)$. This provides a strong positive answer to our Question.

Before beginning the proof of the Theorem, we briefly mention that we assume throughout a basic understanding of large cardinals and forcing. We refer readers to [5] for anything left unexplained.

We turn now to the proof of clause (1) of the Theorem.

Proof. To prove clause (1) of the Theorem, let κ_1 be a singular limit of δ measurable cardinals, where $\delta < \kappa_1$ is a regular uncountable cardinal. Take $\psi_1(x)$ as the formula in one free variable asserting this fact. Let $\mathbb{P}$ be the partial ordering which collapses δ to ω. After forcing with $\mathbb{P}$, by the Lévy-Solovay results [9], a final segment of measurable cardinals below κ_1 remains measurable. Further, κ_1 can now be written as a limit of ω measurable cardinals. Therefore, by a theorem of Prikry (see [5], Theorem 8.7, page 90), κ_1 has become a Rowbottom cardinal carrying a Rowbottom filter.[1] However, it is a theorem of ZFC (see [5], Exercise 8.6, page 90) that originally, since κ_1 had uncountable cofinality, κ_1 was not a Rowbottom cardinal. Thus, we have turned a cardinal satisfying a condition making it non-Rowbottom via forcing into a Rowbottom cardinal, thereby proving clause (1) of the Theorem. □

We note that since the partial ordering $\mathbb{P}$ defined above has cardinality less than κ_1, our proof of clause (1) of the Theorem illustrates the interesting occurrence that it is possible for small forcing to turn a provably non-Rowbottom cardinal into a Rowbottom cardinal.

We turn now to the proof of clause (2) of the Theorem.

[1]Although Theorem 8.7 of [5] does not explicitly state that a limit of ω measurable cardinals carries a Rowbottom filter, this is clear from the proof presented in [5].

Proof. To prove clause (2) of the Theorem, let κ_2 be the least cardinal which is both regular and a limit of strongly compact cardinals. Take $\psi_2(x)$ as the formula in one free variable asserting this fact. We immediately have the following lemma.

Lemma 1. *κ_2 is not a Jonsson cardinal.*

Proof. If κ_2 were a Jonsson cardinal, then since κ_2 is strongly inaccessible, it is a theorem of Shelah (see [10], Chapters 3 and 4) that κ_2 must also be a Mahlo cardinal. Since the set C of strongly compact cardinals below κ_2 is unbounded in κ_2, by the Mahloness of κ_2, the collection C' of limit points of C must contain a strongly inaccessible cardinal. This means that κ_2 is not the least cardinal which is both regular and a limit of strongly compact cardinals, a contradiction which completes the proof of Lemma 1. □

Now that we know that κ_2 is not a Jonsson cardinal, we define the partial ordering $\mathbb{P}$ which turns κ_2 into a Rowbottom cardinal carrying a Rowbottom filter. $\mathbb{P}$ is a version of the "modified Prikry forcing" given in [4] and [2] but defined using more than one ultrafilter. More specifically, since κ_2 is both regular and a limit of strongly compact cardinals, let $\langle \kappa^*_\alpha : \alpha < \kappa_2 \rangle$ be a sequence of strongly compact cardinals whose limit is κ_2, and let $\langle \mathcal{U}_\alpha : \alpha < \kappa_2 \rangle$ be a sequence of ultrafilters such that each $\mathcal{U}_\alpha$ is a κ^*_α-additive uniform ultrafilter over κ_2. $\mathbb{P}$ may now be defined as the set of all finite sequences of the form $\langle \alpha_1, \ldots, \alpha_n, f \rangle$ satisfying the following properties.

(1) $\langle \alpha_1, \ldots, \alpha_n \rangle \in [\kappa_2]^{<\omega}$.
(2) f is a function having domain

$$T_{\alpha_1,\ldots,\alpha_n} = \{\langle \beta_1, \ldots, \beta_m \rangle \in [\kappa_2]^{<\omega} : \langle \alpha_1, \ldots, \alpha_n \rangle \text{ is an initial segment of } \langle \beta_1, \ldots, \beta_m \rangle\}$$

such that $f(\langle \beta_1, \ldots, \beta_m \rangle) \in \mathcal{U}_{\beta_m}$.

The ordering on $\mathbb{P}$ is given by

$$\langle \beta_1, \ldots, \beta_m, g \rangle \geq \langle \alpha_1, \ldots, \alpha_n, f \rangle$$

($\langle \beta_1, \ldots, \beta_m, g \rangle$ is stronger than $\langle \alpha_1, \ldots, \alpha_n, f \rangle$) iff the following criteria are met.

(1) $\langle \alpha_1, \ldots, \alpha_n \rangle$ is an initial segment of $\langle \beta_1, \ldots, \beta_m \rangle$.
(2) For $i = n+1, \ldots, m$, $\beta_i \in f(\langle \alpha_1, \ldots, \alpha_n, \ldots, \beta_{i-1} \rangle)$.
(3) For every $s \in \mathrm{dom}(g)$ (which must be a subset of $\mathrm{dom}(f)$), $g(s) \subseteq f(s)$.

Lemma 2. *Given any formula φ in the forcing language with respect to $\mathbb{P}$ and any condition $\langle \alpha_1, \ldots, \alpha_n, f \rangle \in \mathbb{P}$, there is a condition*

$$\langle \alpha_1, \ldots, \alpha_n, f' \rangle \geq \langle \alpha_1, \ldots, \alpha_n, f \rangle$$

deciding φ.

Proof. The proof of Lemma 2 is essentially the same as the proof of Lemma 4.1 of [4] or Lemma 1.1 of [2], taking into account that different ultrafilters are used in the definition of $\mathbb{P}$. We follow the proofs of these lemmas almost verbatim, making the necessary minor changes where warranted. Specifically, let $s = \langle \alpha_1, \ldots, \alpha_n \rangle$. For any $t \in T_s$, call t sufficient if, for some g, $\langle t, g \rangle \parallel \varphi$. For t sufficient, let g_t be a witness, with $g_t(r) = \kappa_2$ for all $r \in \mathrm{dom}(g_t)$ if t is not sufficient. If s is sufficient, then we are done. If not, then for any $t \in T_s$, sufficient or otherwise, one of the sets

$$X_t = \{\alpha < \kappa_2 : \exists g[\langle t^\frown \alpha, g\rangle \Vdash \varphi]\},$$
$$Y_t = \{\alpha < \kappa_2 : \exists g[\langle t^\frown \alpha, g\rangle \Vdash \neg\varphi]\}, \text{ or}$$
$$Z_t = \{\alpha < \kappa_2 : \forall g[\langle t^\frown \alpha, g\rangle \text{ does not decide } \varphi]\}$$

is an element of $\mathcal{U}_{\max(t)}$. Let A_t be that set, and for $i \leq \text{length}(t)$, let $t \restriction i$ be the first i members of t. For $t \in T_s$, define f' by

$$f'(t) = f(t) \cap \bigcap_{n \leq i \leq \text{length}(t)} g_{t\restriction i}(t) \cap A_t.$$

Note that by the definition of $\mathbb{P}$, $f'(t) \in \mathcal{U}_{\max(t)}$, which means that $\langle s, f'\rangle$ is a well-defined member of $\mathbb{P}$ extending $\langle s, f\rangle$.

Now, let t be sufficient and of minimal length $m+1 > n$, with $\langle t, f''\rangle \geq \langle s, f'\rangle$ and $f'' = f' \restriction T_t$. Let t' be the sequence t without its last element. It then follows that $A_{t'}$ must be either $X_{t'}$ or $Y_{t'}$, so we suppose without loss of generality that $A_{t'} = X_{t'}$. It must be the case that $\langle t', f' \restriction T_{t'}\rangle \Vdash \varphi$, since if some extension $\langle t'', g'\rangle \Vdash \neg\varphi$, such a condition must add elements to t', since t' isn't sufficient. The first element added to t', α, must come from $X_{t'}$, yielding a condition $\langle t'^\frown\{\alpha\}^\frown u, g'\rangle \Vdash \neg\varphi$. However, by construction,

$$\langle t'^\frown\{\alpha\}^\frown u, g'\rangle \geq \langle t'^\frown\{\alpha\}, f' \restriction T_{t'^\frown\{\alpha\}}\rangle \geq \langle t'^\frown\{\alpha\}, g_{t'^\frown\{\alpha\}}\rangle \Vdash \varphi,$$

which is a contradiction. Thus, $\langle t', f' \restriction T_{t'}\rangle \parallel \varphi$, which contradicts the minimality of the length of t for sufficiency. This completes the proof of Lemma 2. □

Lemma 3. *Forcing with $\mathbb{P}$ adds no new subsets to any $\delta < \kappa_2$.*

Proof. Given $\delta < \kappa_2$, suppose that $p = \langle \alpha_1, \ldots, \alpha_n, f\rangle \Vdash$ "$\tau \subseteq \delta$". Without loss of generality, by extending p if necessary, we also assume that $\kappa^*_{\alpha_n} > \delta$. Further, by Lemma 2, for each $\beta < \tau$, we let $\langle \alpha_1, \ldots, \alpha_n, f_\beta\rangle$ be such that $\langle \alpha_1, \ldots, \alpha_n, f_\beta\rangle \parallel$ "$\beta \in \tau$".

Note that the domains of all of the f_β's for $\beta < \delta$ and f are the same, namely $T_{\alpha_1,\ldots,\alpha_n}$. Therefore, by the choice of p and the definition of $\mathbb{P}$, for each $s \in T_{\alpha_1,\ldots,\alpha_n}$, $f_\beta(s)$ and $f(s)$ lie in an ultrafilter $\mathcal{U}_{\max(s)}$ that is $\kappa^*_{\alpha_n}$-additive. This means that $g(s) = \bigcap_{\beta<\delta} f_\beta(s) \cap f(s)$ is such that $g(s) \in \mathcal{U}_{\max(s)}$, and $q = \langle \alpha_1, \ldots, \alpha_n, g\rangle$ is a well-defined element of $\mathbb{P}$ such that $q \geq p$ and q decides the statement "$\beta \in \tau$" for every $\beta < \delta$. Hence, forcing with $\mathbb{P}$ adds no new subsets to δ. This completes the proof of Lemma 3. □

The proof of clause (2) of the Theorem now easily follows. The usual density arguments show that after forcing with $\mathbb{P}$, $\text{cof}(\kappa_2) = \omega$. Further, by Lemma 3, since forcing with $\mathbb{P}$ adds no new bounded subsets to κ_2, after forcing with $\mathbb{P}$, κ_2 remains a limit of measurable cardinals. Therefore, once again by Prikry's theorem used in the proof of clause (1) of the Theorem, κ_2 is after forcing with $\mathbb{P}$ a Rowbottom cardinal carrying a Rowbottom filter. This completes the proof of clause (2) of the Theorem, and hence also completes the proof of the Theorem. □

It is a theorem of ZFC (see [5], Theorem 8.7, page 90) that a κ_1 as in clause (1) of the Theorem is Jonsson (or more specifically, is δ^+ Rowbottom). Thus, in clause (1) of the Theorem, we actually have that a cardinal which is both Jonsson and partially Rowbottom is turned into a fully Rowbottom cardinal carrying a

Rowbottom filter via small forcing. This contrasts with what occurs in clause (2) of the Theorem, where a cardinal satisfying a condition making it automatically non-Jonsson is transformed into a Rowbottom cardinal carrying a Rowbottom filter. Further, as readers may verify for themselves, our methods of proof yield that the least cardinal λ which is both regular and a limit of cardinals which are λ strongly compact suffices as the κ_2 of clause (2) of the Theorem.

We take this opportunity to make some remarks concerning relative consistency results related to the Theorem. First, as Devlin has shown in [3], it is consistent, relative to the existence of a Ramsey cardinal, for the least Jonsson cardinal κ not to be Rowbottom. Since Kleinberg has shown (see [6], [7], or [5], Propositions 8.15(c) and 10.18, pages 95 - 96 and 128 - 129 respectively) that under these circumstances, κ is δ Rowbottom for some $\delta < \kappa$ and can be transformed into a Rowbottom cardinal via a forcing having size less than κ, we have that the existence of a non-Rowbottom Jonsson cardinal which can be changed into a Rowbottom cardinal via small forcing is consistent relative to a Ramsey cardinal. Further, Woodin has shown (see page 60 of [8]) that the stationary tower forcing can be used, assuming either a proper class of measurable cardinals (for class forcing), or an inaccessible limit of measurable cardinals with a Woodin cardinal above it (for set forcing), to change the cofinality of the least inaccessible limit κ of measurable cardinals to ω (or indeed, to any regular cardinal $\delta < \kappa$) without adding bounded subsets to κ. This thereby gives that the existence of a non-Jonsson cardinal which can be turned into a Rowbottom cardinal carrying a Rowbottom filter via forcing is consistent relative to assumptions much weaker than even the existence of a cardinal λ which is λ^+ strongly compact. However, we emphasize again that these results, like the one given in the first paragraph of this note, are only relative consistency results. This is in sharp contrast to the Theorem, which is an outright ZFC theorem, and thus provides a much stronger phenomenon.

In conclusion, we ask if there are other large cardinal or combinatorial properties that provide answers to our main Question for a particular cardinal κ. If so, can these properties be added by a forcing which doesn't add bounded subsets to κ and doesn't collapse cardinals, or by a forcing which is small relative to κ? Is there such a property implying that κ is a regular cardinal?

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