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ORDERINGS AND MAXIMAL IDEALS OF RINGS OF ANALYTIC FUNCTIONS

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Dedicated to Eberhard Becker on the occasion of his 60th birthday

Abstract. We prove that there is a natural injective correspondence between the maximal ideals of the ring of analytic functions on a real analytic set X and those of its subring of bounded analytic functions. By describing the maximal ideals in terms of ultrafilters we see that this correspondence is surjective if and only if X is compact. This approach is also useful for studying the orderings of the field of meromorphic functions on X.

Introduction

Given a global analytic set X of $\mathbb{R}^N$ we consider the ring of analytic functions, $\mathcal{O}(X)$, and the ring of bounded analytic functions, $\mathcal{O}_b(X)$. In this note we investigate the relations between the maximal ideals of these two rings describing them by ultrafilters.

In [AB90] it is shown that in the case of a global analytic set of dimension 1 there is a natural injective correspondence between $\operatorname{Specmax}\mathcal{O}(X)$ and $\operatorname{Specmax}\mathcal{O}_b(X)$, and it is asked whether or not this correspondence is surjective. In section 1 we see that this natural correspondence can be extended to an arbitrary dimension (cf. Proposition 1.2), and that it is surjective if and only if X is compact (cf. Corollary 1.4).

In section 2 we attach to each ordering of the field of meromorphic functions on X a maximal ideal of $\mathcal{O}_b(X)$ and then an ultrafilter. This attachment has proved to be very useful; cf. [AB90], [ABR96], chapter VIII, [Cas91], [Cas94a], [DA01] and [Jaw91]. However, to the best of our knowledge, it was not known whether or not there are orderings with attached ultrafilter of dimension greater than zero. We prove that, in fact, such orderings exist at least if the part of maximal dimension of X is unbounded; cf. Theorem 2.1.

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1. Ultrafilters and maximal ideals

Let X be a global analytic subset of $\mathbb{R}^N$, that is, X is the zero set of a finite number of analytic functions on $\mathbb{R}^N$. The ideal of analytic functions vanishing on X is $I(X) := \{f \in \mathcal{O}(\mathbb{R}^N) \,|\, f = 0 \text{ on } X\}$ and generates the coherent sheaf of ideals I_X whose stalk at x is $I_{X,x} = I(X)\mathcal{O}(\mathbb{R}^N_x)$. The analytic functions on X are the global sections of the coherent sheaf $\mathcal{O}_X := \dfrac{\mathcal{O}_{\mathbb{R}^N}}{I_X}$, and they form the ring $\mathcal{O}(X) = \dfrac{\mathcal{O}(\mathbb{R}^N)}{I(X)}$.

For describing maximal ideals of $\mathcal{O}(X)$ in terms of ultrafilters we will follow [GJ60]. The lattice on X whose elements are zero sets of functions of $\mathcal{O}(X)$ will be denoted by Δ. Given an ideal $I \subset \mathcal{O}(X)$ the family $\mathcal{Z}(I) := \{Z(f) \,|\, f \in I\}$ is a Δ-filter called the filter of zeros of I. Conversely, given a Δ-filter $\mathcal{F}$ its ideal is defined as $\mathcal{I}(\mathcal{F}) := \{f \in \mathcal{O}(X) \,|\, Z(f) \in \mathcal{F}\}$ and is a proper ideal of $\mathcal{O}(X)$.

It can be checked that $\mathcal{I}(\mathcal{Z}(I)) \supset I$ and $\mathcal{Z}(\mathcal{I}(\mathcal{F})) = \mathcal{F}$ and both maps are order-preserving for the inclusion relation. This implies that the map $\mathcal{Z} : \operatorname{Specmax} \mathcal{O}(X) \to \tilde{\Delta}$, where $\tilde{\Delta}$ is the set of all Δ-ultrafilters, is a bijection whose inverse is $\mathcal{I}$.

The Δ-ultrafilter $\mathcal{F}$ and its corresponding maximal ideal $\mathcal{I}(\mathcal{F})$ are called *free* if $\bigcap_{V \in \mathcal{F}} V = \emptyset$ and *fixed* otherwise. Note that a fixed ultrafilter is the principal Δ-ultrafilter attached to the unique limit point of $\mathcal{F}$. An important result is the following; cf. [Cas94b], Chapter 2.

Theorem 1.1. *Let $\mathcal{F}$ be a Δ-ultrafilter. Then $\mathcal{F}$ contains some discrete subset $D \subset X$.*

Proof. Let Y be a non-discrete element of $\mathcal{F}$ and let $Y = \bigcup Y_i$ be its decomposition into irreducible components. We will see that there is an element $Y' \in \mathcal{F}$ such that $\dim Y' < \dim Y$, whence the result follows.

We pick one point $x_i \in Y_i \setminus \bigcup_{j \neq i} Y_i$ in each component and put $D := \bigcup \{x_i\}$. Let $\mathcal{Z}(f) \in \mathcal{F}$. If $\dim \mathcal{Z}(f) \cap Y < \dim Y$, then we take $Y' = \mathcal{Z}(f) \cap Y \in \mathcal{F}$.

Suppose now that $\dim \mathcal{Z}(f) \cap Y = \dim Y$ for all $\mathcal{Z}(f) \in \mathcal{F}$. Then for every such f at least one component Y_k of Y verifies $\dim Y_k = \dim \mathcal{Z}(f) \cap Y_k$, which implies $Y_k \subset \mathcal{Z}(f)$ since Y_k is irreducible. In particular, $x_k \in \mathcal{Z}(f)$. Thus $\mathcal{Z}(f) \cap D \neq \emptyset$ for all $\mathcal{Z}(f) \in \mathcal{F}$. But $\mathcal{F}$ is an ultrafilter; hence $D \in \mathcal{F}$. $\square$

We will relate the maximal ideals of the subring $\mathcal{O}_b(X)$ of bounded analytic functions on X to another kind of ultrafilters. To be precise, the global semianalytic subsets of X are those which can be written as

$$\bigcup_{i=1}^{p} \{x \in X | f_i(x) = 0, g_{i1}(x) > 0, \ldots, g_{ij_i}(x) > 0\}$$

where $f_i, g_{ij} \in \mathcal{O}(X)$. We will denote by $\mathcal{C}$ the family of global semianalytic subsets of X which are closed in the usual topology. This family is closed under finite unions and intersections. So it makes sense to consider filters of subsets of $\mathcal{C}$.

For any maximal ideal $m \subset \mathcal{O}_b(X)$ the family

$$\mathcal{U}_m := \{Y \in \mathcal{C} \,|\, Y \cap f^{-1}([-\delta, \delta]) \neq \emptyset, \forall f \in m \text{ and } \forall \delta > 0\}$$

is a $\mathcal{C}$-ultrafilter; cf. [Cas94a]. Conversely, given a $\mathcal{C}$-ultrafilter $\mathcal{U}$ the subset of $\mathcal{O}_b(X)$ defined as

$$\mathfrak{M}_{\mathcal{U}} := \{f \in \mathcal{O}_b(X) \mid f^{-1}([-\delta,\delta]) \in \mathcal{U},\ \forall \delta > 0\}$$

is a maximal ideal of $\mathcal{O}_b(X)$: let $f \in \mathcal{O}_b(X)$ and denote by $f(\mathcal{U})$ the collection $\{f(A)\}_{A\in\mathcal{U}}$. Then $f(\mathcal{U})$ is a filter basis of $\mathbb{R}$ with a unique limit point which will be denoted as $\lambda_{\mathcal{U}}(f)$. In fact, $\lambda_{\mathcal{U}} : \mathcal{O}_b(X) \to \mathbb{R}$ defines a surjective homomorphism whose kernel is precisely $\mathfrak{M}_{\mathcal{U}}$. Therefore, $\mathfrak{M}_{\mathcal{U}}$ is a maximal ideal of $\mathcal{O}_b(X)$.

Given $m \in \operatorname{Specmax}\mathcal{O}_b(X)$ we have $m = \mathfrak{M}_{\mathcal{U}_m}$, since $f \in m$ implies that $f^{-1}([-\delta,\delta]) \cap Y \neq \emptyset$, $\forall\delta > 0$, $\forall Y \in \mathcal{U}_m$. In particular, $f^{-1}([-\delta,\delta]) \in \mathcal{U}_m$, $\forall\delta > 0$, so $f \in \mathfrak{M}_{\mathcal{U}_m}$. In a similar way it can be proved that $\mathcal{U} = \mathcal{U}_{\mathfrak{M}_{\mathcal{U}}}$. Therefore these two correspondences are inverses of each other. So they define a bijection between $\tilde{C}$, the set of all $\mathcal{C}$-ultrafilters of X, and $\operatorname{Specmax}\mathcal{O}_b(X)$.

There is a natural correspondence between Δ and $\mathcal{C}$-ultrafilters. Namely, if $\mathcal{F}$ is a Δ-ultrafilter we define $\mathcal{F}^* := \{S \in \mathcal{C} \mid S \supset Z \text{ for some } Z \in \mathcal{F}\}$. As can be checked $\mathcal{F}^*$ is a $\mathcal{C}$-ultrafilter: let $T \in \mathcal{C}$ be such that $T \cap S \neq \emptyset$ for all $S \in \mathcal{F}^*$. In particular, if $D \in \mathcal{F}$ is a discrete set, which exists by Theorem 1.1, then $T \cap D$ is also a discrete set and for all $W \in \mathcal{F}$ we have $(T \cap D) \cap W = T \cap (D \cap W) \neq \emptyset$ so that $T \cap D \in \mathcal{F}$ and then $T \in \mathcal{F}^*$. We will now consider the relation between the maximal ideals of the rings $\mathcal{O}(X)$ and $\mathcal{O}_b(X)$.

Proposition 1.2. *a) Let m be a maximal ideal of $\mathcal{O}(X)$. Then there is a unique maximal ideal m^* of $\mathcal{O}_b(X)$ such that $m \cap \mathcal{O}_b(X) \subset m^*$. Moreover, the equality $m \cap \mathcal{O}_b(X) = m^*$ holds if and only if m is fixed.*

b) The correspondence $\operatorname{Specmax}\mathcal{O}(X) \to \operatorname{Specmax}\mathcal{O}_b(X) : m \to m^*$ *is injective.*

Proof. a) The Δ-ultrafilter $\mathcal{Z}(m)$ can be extended to a discrete $\mathcal{C}$-ultrafilter which we will call $\mathcal{U}$. We put $m^* = \mathfrak{M}_{\mathcal{U}}$. It is clear that $m \cap \mathcal{O}_b(X) \subset m^*$ since $f \in m \cap \mathcal{O}_b(X)$ implies $\mathcal{Z}(f) \in \mathcal{Z}(m)$ and then $\mathcal{Z}(f) \in \mathcal{U}$. Thus $f^{-1}([-\delta,\delta]) \in \mathcal{U}$, $\forall\delta > 0$, so $f \in m^*$.

To prove uniqueness suppose there is an $m' \in \operatorname{Specmax}\mathcal{O}_b(X)$ such that $m \cap \mathcal{O}_b(X) \subset m'$ and $m^* \neq m'$. Then $\mathcal{U}_{m'} \neq \mathcal{U}_{m^*}$, that is, there are $C_1 \in \mathcal{U}_{m'}$ and $C_2 \in \mathcal{U}_{m^*}$ such that $C_1 \cap C_2 = \emptyset$. By Theorem 1.1 we can suppose C_2 is a discrete set.

Let $f \in \mathcal{O}(X)$ be an analytic function whose zero set is $\mathcal{Z}(f) = C_2$. We can take such a function bounded by 1 and non-negative (if needed, we can replace f by $f^2/(1+f^2)$) so that $f \in m \cap \mathcal{O}_b(X)$. Now, by Tietze's theorem there is a non-negative continuous function G on X whose restriction to C_1 coincides with $1/f$ (which is continuous on C_1). Take an analytic approximation $p \in \mathcal{O}(X)$ of G such that $G(x) < p(x) < G(x) + 1$, $\forall x \in X$ (cf. [Hir76]), and put $h = \dfrac{fp}{1+fp}$. Then $h \in m \cap \mathcal{O}_b(X) \subset m'$ and $C_1 \cap h^{-1}([-1/3, 1/3]) = \emptyset$, as a direct computation shows. But this implies $C_1 \notin \mathcal{U}_{m'}$, a contradiction.

If m is fixed, then it is the ideal of some point $p \in X$. But $f^{-1}([-\delta,\delta]) \in \mathcal{U}$, $\forall\delta > 0$, in this case implies $f(p) = 0$ and hence $m \cap \mathcal{O}_b(X) = m^*$.

If m is not fixed we consider $f = \dfrac{1}{1+\|x\|^2} \in \mathcal{O}_b(X)$. It is clear that $f \notin m$ since it is a unit in $\mathcal{O}(X)$. But $f \in m^*$ as can easily be checked.

b) If $m_1 \neq m_2$ are maximal ideals of $\mathcal{O}(X)$, then there are discrete sets $D_1 \in \mathcal{Z}(m_1)$ and $D_2 \in \mathcal{Z}(m_2)$ such that $D_1 \cap D_2 = \emptyset$. Then $D_1 \in \mathcal{Z}(m_1)^*$, $D_2 \in \mathcal{Z}(m_2)^*$ and so the $\mathcal{C}$-ultrafilters $\mathcal{U}_1 := \mathcal{Z}(m_1)^*$ and $\mathcal{U}_2 := \mathcal{Z}(m_2)^*$ are not equal and then neither are the ideals $m_1^* = \mathfrak{M}_{\mathcal{U}_1}$ and $m_2^* = \mathfrak{M}_{\mathcal{U}_2}$. $\square$

By what has been seen above we have the following commutative square:

$$\begin{array}{ccc} \tilde{\Delta} & \longrightarrow & \tilde{\mathcal{C}} \\ {\scriptstyle \mathcal{Z}}\uparrow\downarrow{\scriptstyle \mathcal{I}} & & {\scriptstyle \mathcal{U}}\uparrow\downarrow{\scriptstyle \mathfrak{M}} \\ \operatorname{Specmax} \mathcal{O}(X) & \longrightarrow & \operatorname{Specmax} \mathcal{O}_b(X) \end{array}$$

The map at the top is $\mathcal{F} \to \mathcal{F}^*$ and at the bottom is $m \to m^*$; cf. Proposition 1.2. These maps are clearly injective. The ideals of $\mathcal{O}_b(X)$ which are the image of some maximal ideal of $\mathcal{O}(X)$ under this map are called *discrete* since their corresponding $\mathcal{C}$-ultrafilters contain a discrete set. The question asked in [AB90] is whether or not these horizontal maps are surjective. We will see that the answer is positive if and only if X is compact.

The dimension of an ultrafilter is defined as the minimum of the dimensions of the sets in that ultrafilter. Thus, Theorem 1.1 says that every Δ-ultrafilter has dimension zero. Therefore the image of the mapping $\mathcal{F} \to \mathcal{F}^*$ is the $\mathcal{C}$-ultrafilters of dimension 0. What we are going to see now is that in case X is non-compact there are $\mathcal{C}$-ultrafilters of dimension strictly greater than zero.

Without loss of generality we will suppose X to be irreducible of dimension $n \geq 1$. We decompose X as $X = X^{(1)} \cup \ldots \cup X^{(n)}$ where $X^{(j)}$ is the closure of the set $\{x \in X \mid \dim X_x = j\}$. At least one of these components, say $X^{(k)}$, is unbounded. We take a non-compact discrete set

$$D \subset \operatorname{Reg} X^{(k)} := \{x \in X^{(k)} \mid \mathcal{O}(X_x) \text{ is a regular ring}\}$$

and for every $p \in D$ we choose a regular system of parameters $x_{1,p}, \ldots, x_{N,p} \in \mathcal{O}(\mathbb{R}^N_p)$ such that $X_p = \{x_{k+1,p} = \ldots = x_{N,p} = 0\}$ and the restrictions of $x_{1,p}, \ldots, x_{k,p}$ to X_p form a regular system of parameters of $\mathcal{O}(X_p)$. By Cartan's Theorem B there are global analytic functions $H_1, \ldots, H_k \in \mathcal{O}(\mathbb{R}^N)$ such that for all $p \in D$ we have $H_{i,p} \equiv x_{i,p} \mod m_p^2$, where m_p is the maximal ideal of $\mathcal{O}(\mathbb{R}^N_p)$. But then denoting by primes the restriction to X_p we also have $H'_{i,p} \equiv x'_{i,p} \mod m'^2_p$. For the sake of convenience, in the following the restriction of H_i to X will be denoted again by H_i instead of H'_i.

Now, for every $p \in D$ we take a neighborhood $U(p, \epsilon_p)$ of arbitrarily small radius ϵ_p and a global analytic function $h_\epsilon \in \mathcal{O}(X)$ such that

$$\bigcup_{p \in D} U(p, \epsilon_p/2) \subset \{h_\epsilon > 0\} \quad \text{and} \quad X \setminus \bigcup_{p \in D} U(p, \epsilon_p) \subset \{h_\epsilon < 0\}.$$

Such an h_ϵ can be constructed by approximating a $C^\infty(X)$ function with the same property; cf. [Hir76], Chapter 2.

Thus $U_\epsilon := \{h_\epsilon > 0\}$ and $U'_\epsilon := \{h_\epsilon \geq 0\}$ are open and closed, respectively, small neighborhoods of the discrete set D. By taking small enough ϵ_p's we can suppose that the germ of $\{H_{i+1} = \ldots = H_k = 0\}$ is a regular analytic set germ of dimension i at every $x \in U'_\epsilon$.

We define Y_i as the Zariski closure of $\{H_{i+1} = \ldots = H_k = 0\} \cap U_\epsilon$. In this way, Y_i is a global analytic set of dimension i whose germ at every $x \in U_\epsilon$ coincides with the germ of $\{H_{i+1} = \ldots = H_k = 0\}$. In particular, $\operatorname{Reg} Y_i$ is not bounded.

Given any closed global semianalytic set $E \subset Y_i$ of dimension $\leq i-1$ we take an open global semianalytic set $U_E \subset Y_i$ containing an open neighborhood (in Y_i) of $E \cap U_\epsilon$ such that

$$(*) \qquad \frac{\mathrm{vol}_i(U_E \cap U_p)}{\mathrm{vol}_i(Y_i \cap U_p)} \overset{\|p\| \to \infty}{\longrightarrow} 0$$

where $U_p := U(p, \epsilon_p) \cap U_\epsilon$ (so that $U_\epsilon = \bigcup_{p \in D} U_p$) and vol_i denotes the i-dimensional volume; see [KR89]. This can be done as follows. Let $D = \bigcup_{m=1}^{\infty} p_m$ and let $H \in \mathcal{O}(X)$ be a positive equation of the Zariski closure of E. As $\mathrm{vol}_i(\{H < \delta\} \cap U_{p_m} \cap Y_i) \overset{\delta \to 0}{\longrightarrow} 0$ we can choose δ_m such that

$$\frac{\mathrm{vol}_i(\{H < \delta_m\} \cap U_{p_m} \cap Y_i)}{\mathrm{vol}_i(U_{p_m} \cap Y_i)} < \frac{1}{m}.$$

Now, take a global non-negative analytic function $f \in \mathcal{O}(X)$ such that $f < \frac{1}{2}\delta_m$ on U_{p_m}. Then $U_E := \{H < f\}$ does the job.

By $(*)$, the family of closed sets $A_E := (U'_\epsilon \setminus U_E) \cap Y_i$ is a filter basis, so they generate a $\mathcal{C}$-filter $\mathcal{F}_i$. We take one ultrafilter refining $\mathcal{F}_i$ and call it $\mathcal{U}_i$. It is easy to see that the dimension of this ultrafilter is i. First of all, by definition of the $\mathcal{C}$-filter $\mathcal{F}_i$ we have $Y_i \in \mathcal{F}_i$ and so also $Y_i \in \mathcal{U}_i$. Suppose now there is some $E \subset Y_i$ such that $\dim E \leq i-1$ and $E \in \mathcal{U}_i$. Then $A_E \cap E = \emptyset$, which would imply $A_E \notin \mathcal{U}_i$, but this is a contradiction. Hence we have proved the following.

Theorem 1.3. *Let $X \subset \mathbb{R}^N$ be a non-compact global analytic set of dimension $n \geq 1$. Then there are $\mathcal{C}$-ultrafilters which are not discrete.*

More precisely, if X has a decomposition $X = X^{(1)} \cup \ldots \cup X^{(n)}$, where $X^{(j)}$ is the closure of the set $\{x \in X \mid \dim X_x = j\}$, and $X^{(k)}$ is unbounded, then there are $\mathcal{C}$-ultrafilters of dimensions $0, 1, \ldots, k$.

As a consequence we have

Corollary 1.4. *If $X \subset \mathbb{R}^N$ is a non-compact global analytic set of dimension $n \geq 1$, then the correspondences*

$$\begin{array}{ccc} \tilde{\Delta} & \longrightarrow & \tilde{\mathcal{C}} \\ \mathcal{F} & \longrightarrow & \mathcal{F}^* \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Specmax}\, \mathcal{O}(X) & \longrightarrow & \mathrm{Specmax}\, \mathcal{O}_b(X) \\ m & \longrightarrow & m^* \end{array}$$

are not surjective.

2. Orderings and ultrafilters

Let $X \subset \mathbb{R}^N$ be an irreducible global analytic set of dimension n. The field of fractions of the domain $\mathcal{O}(X)$ is the field of meromorphic functions on X and is denoted as $\mathcal{M}(X)$. In this section we will be interested in orderings of this field. For background on the real spectrum of a field we refer to [ABR96] and [BCR98].

Given an ordering β of $\mathcal{M}(X)$ the convex hull of $\mathbb{R}$ in $\mathcal{M}(X)$ is the valuation ring (cf. [Jaw91] and [Cas94a])

$$W_\beta = \{f \in \mathcal{M}(X) \mid f^2 <_\beta r^2 \text{ for some } r \in \mathbb{R}\}$$

whose maximal ideal is the set of all infinitely small elements of $\mathcal{M}(X)$:

$$n_\beta = \{f \in \mathcal{M}(X) \mid f^2 <_\beta r \text{ for all } r \in \mathbb{R}\}.$$

All bounded analytic functions belong to W_β and the center of W_β in $\mathcal{O}_b(X)$, defined as $m_\beta = n_\beta \cap \mathcal{O}_b(X)$, is a maximal ideal of $\mathcal{O}_b(X)$. We will attach to each ordering of β of $\mathcal{M}(X)$ the $\mathcal{C}$-ultrafilter $\mathcal{U}_\beta := \mathcal{U}_{m_\beta}$; see section 1. One open question is to determine whether or not there are orderings attached to ultrafilters of dimension greater than zero. The following theorem assures the existence of such orderings if the part of maximal dimension is unbounded.

Theorem 2.1. *Let $X \subset \mathbb{R}^N$ be an irreducible global analytic set of dimension n whose part of maximal dimension $X^{(n)}$ is not compact. Then for every $k = 0, \ldots, n$ there are orderings β_k of $\mathcal{M}(X)$ with attached ultrafilters $\mathcal{U}_{\beta_k}$ of dimension k.*

Proof. The idea of the proof is quite similar to that of Theorem 1.3. We will closely follow the notation in that proof.

We take a non-compact discrete set $D \subset \operatorname{Reg} X^{(n)}$ and for every $p \in D$ we choose a regular system of parameters $x_{1,p}, \ldots, x_{n,p} \in \mathcal{O}(X_p)$ and global analytic functions $H_1, \ldots, H_n \in \mathcal{O}(X)$ such that for all $p \in D$ we have $H_{i,p} \equiv x_{i,p} \mod m_p^2$, where m_p is the maximal ideal of $\mathcal{O}(X_p)$.

The set $U_\epsilon := \{h_\epsilon > 0\}$ is a neighborhood of the discrete set D small enough to suppose that the germ of $\{H_{i+1} = \ldots = H_n = 0\}$ is a regular analytic set germ of dimension i at every $x \in U_\epsilon$.

We define Y_i as the Zariski closure of $\{H_{i+1} = \ldots = H_n = 0\} \cap U_\epsilon$. In this way, Y_i is a global analytic set of dimension i whose germ at every $x \in U_\epsilon$ coincides with the germ of $\{H_{i+1} = \ldots = H_n = 0\}$.

For every Y_k we will define an ordering $\alpha_k \in \operatorname{Spec}_{\mathrm{r}} \mathcal{O}(Y_k)$. Given any closed global semianalytic set $E \subset Y_k$ of dimension $\leq k-1$ we take a closed global semianalytic set $V_E \subset Y_k$ containing an open neighborhood (in Y_k) of $E \cap U_\epsilon$ such that

$$\frac{\operatorname{vol}_k(V_E \cap U_p)}{\operatorname{vol}_k(Y_k \cap U_p)} \overset{\|p\| \to \infty}{\longrightarrow} 0.$$

The family of open sets $A_E := (U_\epsilon \setminus V_E) \cap Y_k$ is a filter basis that generates a filter of open global semianalytic sets $\mathcal{F}_k$. We take one ultrafilter (of open global semianalytic sets) refining $\mathcal{F}_k$ and call it ν_k. This ultrafilter ν_k defines in the usual way an ordering α_k of $\mathcal{O}(Y_k)$, namely, given $f \in \mathcal{O}(Y_k)$ we say that f is positive in the ordering α_k if and only if $\{f > 0\} \in \nu_k$.

The ultrafilter ν_k can be lifted to an ultrafilter ν'_k (of open subsets of X) as follows: an open subset $A \subset X$ belongs to ν'_k iff $\overline{A \cap \{H_{k+1} > 0, \ldots, H_n > 0\}} \cap Y_k$ contains an element of ν_k. The corresponding ordering of $\mathcal{O}(X)$ will be denoted as β_k. It is easy to check that β_k is a total ordering of $\mathcal{O}(X)$ and so can be extended to an ordering of $\mathcal{M}(X)$.

Now, we will see that the dimension of the $\mathcal{C}$-ultrafilter $\mathcal{U}_{\beta_k}$ attached to β_k is k. First of all, we have $Y_k \in \mathcal{U}_{\beta_k}$. Otherwise there is a closed semianalytic set $C_k \in \mathcal{U}_{\beta_k}$ such that $C_k \cap Y_k = \emptyset$. Then there will be $h \in \mathcal{O}(X)$ such that $C_k \subset \{h > 0\}$ and $Y_k \subset \{h < 0\}$ and so, by [Cas94a], Lemma 2.5, $h >_{\beta_k} 0$, but this contradicts the definition of β_k.

Suppose there is some $E \subset Y_k$ such that $\dim E \leq k-1$ and $E \in \mathcal{U}_{\beta_k}$. Then we can find $f \in \mathcal{O}(X)$ such that $E \subset \{f > 0\}$ and $\{f < 0\} \cap Y_k \in \nu_k$, but this is a

contradiction since the first assertion implies that $f >_{\beta_k} 0$ while the second means $f <_{\beta_k} 0$. $\square$

In the case of a compact global analytic set X we have that all orderings are centered in a point; that is, the $\mathcal{C}$-ultrafilters corresponding to orderings of $\mathcal{M}(X)$ are fixed. But if X is non-compact and $X^{(n)}$ is bounded we do not know in general. We conjecture that if $X^{(n)}$ is bounded, then there are only orderings whose attached ultrafilters are fixed and, in particular, they have dimension 0, as happens in the following example.

Example 2.2. Let $X \subset \mathbb{R}^3$ be the analytic surface of equation $x^2(1-z^2) = x^4+y^4$. Its part of maximal dimension is bounded although X is not since it contains the whole z-axis; see Figure 1.

With regard to maximal ideals of $\mathcal{O}_b(X)$, as only $X^{(1)}$ is unbounded they correspond to $\mathcal{C}$-ultrafilters of dimension 0 and 1 but not of dimension 2.

In this case it is possible to prove that all orderings of $\mathcal{M}(X)$ are fixed and, in particular, their attached $\mathcal{C}$-ultrafilters have dimension 0.

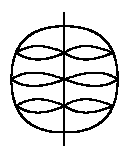

FIGURE 1. $X : x^2(1-z^2) = x^4 + y^4$

Suppose $\beta \in \operatorname{Spec}_{\mathrm{r}} \mathcal{M}(X)$ is not fixed. Then the corresponding $\mathcal{C}$-ultrafilter, $\mathcal{U}_\beta$, has no compact subsets. Thus $V := X \cap (\{z > 2\} \cup \{z < -2\})$ must intersect any subset of $\mathcal{U}_\beta$ and then $V \in \mathcal{U}_\beta$. We define $f := \dfrac{z^2-1}{z^2+1} \in \mathcal{O}_b(X)$ so that $f > 0$ on V. This implies that f is positive in the ordering β; cf. [Cas94a], Lemma 2.5 or [ABR96], Proposition VIII.4.7. But this is a contradiction since $-f = \dfrac{x^4+y^4}{x^2(z^2+1)}$; that is, $-f$ is a sum of squares of meromorphic functions and so $-f$ is positive in every ordering of $\mathcal{M}(X)$.

In fact, it is possible to prove the conjecture if $X \subset \mathbb{R}^N$ is algebraic: suppose X is unbounded of dimension n but $X^{(n)}$ is bounded. For some positive constant A the restriction of the polynomial $f := A - \sum_1^N x_i^2$ to X is positive on $X^{(n)}$. Then by the positive solution of Hilbert's 17th problem in the algebraic case (cf. [BCR98]), f is a sum of squares of rational functions on X and, in particular, is a sum of squares in $\mathcal{M}(X)$. On the other hand, if $\beta \in \operatorname{Spec}_{\mathrm{r}} \mathcal{M}(X)$ is not fixed, then $V := X \cap \{f \le -1\} \in \mathcal{U}_\beta$ and f is strictly negative on V. Therefore, by [Cas94a], Lemma 2.5, f is negative in β, a contradiction since f is a sum of squares.

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