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A NEW CHARACTERIZATION OF GAPS BETWEEN TWO SUBSPACES

HONG-KE DU AND CHUN-YUAN DENG

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Abstract. In this note, a new characterization of gaps between two subspaces of a Hilbert space is established.

Let $\mathcal{M}$ and $\mathcal{N}$ be two closed linear subspaces of a Hilbert space $\mathcal{H}$. The gap between $\mathcal{M}$ and $\mathcal{N}$ (see [2]) is defined by

$$\operatorname{gap}(\mathcal{M},\mathcal{N}) = \max\{\delta(\mathcal{M},\mathcal{N}), \delta(\mathcal{N},\mathcal{M})\}, \tag{1}$$

where

$$\delta(\mathcal{M},\mathcal{N}) = \sup\{\operatorname{dist}(x,\mathcal{N}) : x \in \mathcal{M}, \| x \| \leq 1\},$$

and the conorm between $\mathcal{M}$ and $\mathcal{N}$ is defined by

$$\gamma(\mathcal{M},\mathcal{N}) = \inf\{\| x \| : x \in \mathcal{M}, \operatorname{dist}(x,\mathcal{N}) \geq 1\}. \tag{2}$$

In this note, the specific representations of the gap and the conorm between two subspaces are established.

For two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$, denote $\mathcal{H}_1 = \mathcal{M} \cap \mathcal{N}$, $\mathcal{H}_2 = \mathcal{M} \cap \mathcal{N}^{\perp}$, $\mathcal{H}_3 = \mathcal{M}^{\perp} \cap \mathcal{N}$, $\mathcal{H}_4 = \mathcal{M}^{\perp} \cap \mathcal{N}^{\perp}$ and let $\mathcal{H}_5 = \mathcal{M} \cap (\mathcal{H} \ominus (\bigoplus_{i=1}^4 \mathcal{H}_i))$ and $\mathcal{H}_6 = (\mathcal{H} \ominus (\bigoplus_{i=1}^4 \mathcal{H}_i)) \ominus \mathcal{H}_5$. It is clear that $\mathcal{H}_i \perp \mathcal{H}_j$, $j \neq i$ and $1 \leq i, j \leq 6$. If $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ are orthogonal projections onto $\mathcal{M}$ and $\mathcal{N}$, respectively, we have the following lemma which is useful later.

Lemma 1 (see [1])**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. Then $P_{\mathcal{M}}\mathcal{H}_i \subseteq \mathcal{H}_i$ and $P_{\mathcal{N}}\mathcal{H}_i \subseteq \mathcal{H}_i$, $1 \leq i \leq 4$, and $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the following operator matrices:*

$$P_{\mathcal{M}} = \begin{pmatrix} I &&&&& \\ & I &&&& \\ && 0 &&& \\ &&& 0 && \\ &&&& I & \\ &&&&& 0 \end{pmatrix} P_{\mathcal{N}} = \begin{pmatrix} I &&&&& \\ & 0 &&&& \\ && I &&& \\ &&& 0 && \\ &&&& Q & Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}D \\ &&&& D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}} & D^*(I-Q)D \end{pmatrix}, \tag{3}$$

with respect to the space decomposition $\mathcal{H} = \sum_{i=1}^6 \mathcal{H}_i$, respectively, where Q is a positive contraction on $\mathcal{H}_5$, 0 and 1 are not eigenvalues of Q, and D is a unitary operator from $\mathcal{H}_6$ onto $\mathcal{H}_5$.

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If $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices (3), for $x \in \mathcal{M}$, we have $x = x_1 + x_2 + x_5$ with $x_i \in \mathcal{H}_i$, $i = 1, 2, 5$, and $P_{\mathcal{N}} x = x_1 + Qx_5 + D^* Q^{\frac{1}{2}} (I - Q)^{\frac{1}{2}} x_5$. Observing that $\operatorname{dist}(x, \mathcal{N})^2 = \| x \|^2 - \| P_{\mathcal{N}} x \|^2$, then

$$\begin{aligned} \operatorname{dist}(x, \mathcal{N})^2 &= \| x_1 \|^2 + \| x_2 \|^2 + \| x_5 \|^2 - \| x_1 \|^2 \\ &\quad - \| Qx_5 \|^2 - \| D^* Q^{\frac{1}{2}} (I - Q)^{\frac{1}{2}} x_5 \|^2 \\ &= \| x_2 \|^2 + \| (I - Q)^{\frac{1}{2}} x_5 \|^2 \leq \| x_2 \|^2 + \| x_5 \|^2 \\ &\leq \| x \|^2, \end{aligned}$$

that is,

$$\operatorname{dist}(x, \mathcal{N}) = (\| x_2 \|^2 + \| (I - Q)^{\frac{1}{2}} x_5 \|^2)^{\frac{1}{2}} \leq \| x \| . \tag{4}$$

Proposition 2. *Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$, and let $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices* (3). *If $\mathcal{M} \cap \mathcal{N}^{\perp} \neq \{0\}$, then $\delta(\mathcal{M}, \mathcal{N}) = 1$. If $\mathcal{M} \cap \mathcal{N}^{\perp} = \{0\}$, then $\delta(\mathcal{M}, \mathcal{N}) = \| (I - Q)^{\frac{1}{2}} \|$.*

Proof. By Lemma 1, $0 \leq Q \leq I$. Then $0 \leq I - Q \leq I$. So $I - Q$ is also a positive contraction.

If $\mathcal{M} \cap \mathcal{N}^{\perp} \neq \{0\}$, and for a vector $x \in \mathcal{M}$, x has the decomposition $x = x_1 + x_2 + x_5$ with $x_i \in \mathcal{H}_i$, $i = 1, 2, 5$, then

$$\begin{aligned} \delta(\mathcal{M}, \mathcal{N}) &= \sup\{\operatorname{dist}(x, \mathcal{N}) : x \in \mathcal{M}, \| x \| \leq 1\} \\ &= \sup\{(\| x_2 \|^2 + \| (I - Q)^{\frac{1}{2}} x_5 \|^2)^{\frac{1}{2}} : x_i \in \mathcal{H}_i, i = 1, 2, 5, \\ &\quad \text{and } \textstyle\sum_{i=1,2,5} \| x_i \|^2 \leq 1\} \\ &\leq 1. \end{aligned}$$

To prove the reverse inequality, choose $x_0 = x_2 \in \mathcal{H}_2$ with $\| x_2 \| = 1$, $\operatorname{dist}(x_0, \mathcal{N}) = \| x_0 \| = 1$, i.e., $\delta(\mathcal{M}, \mathcal{N}) \geq 1$. Hence $\delta(\mathcal{M}, \mathcal{N}) = 1$.

If $\mathcal{M} \cap \mathcal{N}^{\perp} = \{0\}$, then $\mathcal{H}_2 = \{0\}$. In this case, for a vector $x \in \mathcal{M}$, x has the decomposition $x = x_1 + x_5$ with $x_1 \in \mathcal{H}_1$ and $x_5 \in \mathcal{H}_5$. We have

$$\begin{aligned} \delta(\mathcal{M}, \mathcal{N}) &= \sup\{\operatorname{dist}(x, \mathcal{N}) : x \in \mathcal{M}, \| x \| \leq 1\} \\ &= \sup\{\| (I - Q)^{\frac{1}{2}} x_5 \| : x_i \in \mathcal{H}_i, i = 1, 5, \text{ and } \textstyle\sum_{i=1,5} \| x_i \|^2 \leq 1\} \\ &= \sup\{\| (I - Q)^{\frac{1}{2}} x_5 \| : x_5 \in \mathcal{H}_5, \| x_5 \| \leq 1\} \\ &= \| (I - Q)^{\frac{1}{2}} \| . \end{aligned}$$

□

Proposition 3. *Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$, and let $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices* (3). *If $\mathcal{M}^{\perp} \cap \mathcal{N} \neq \{0\}$, then $\delta(\mathcal{N}, \mathcal{M}) = 1$. If $\mathcal{M}^{\perp} \cap \mathcal{N} = \{0\}$, then $\delta(\mathcal{N}, \mathcal{M}) = \| (I - Q)^{\frac{1}{2}} \|$.*

Proof. If $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices as the formula (3), since $\mathcal{M} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_5$, $\mathcal{N} = \mathcal{H}_1 \oplus \mathcal{H}_3 \oplus R(P_{\mathcal{N}} |_{\mathcal{H}_5 \oplus \mathcal{H}_6})$ and

$$\begin{aligned} & \begin{pmatrix} Q \\ D^* Q^{\frac{1}{2}} (I - Q)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} Q & Q^{\frac{1}{2}} (I - Q)^{\frac{1}{2}} D \end{pmatrix} \\ = & \begin{pmatrix} Q^2 & Q^{\frac{3}{2}} (I - Q)^{\frac{1}{2}} D \\ D^* Q^{\frac{3}{2}} (I - Q)^{\frac{1}{2}} & D^* Q (I - Q) D \end{pmatrix} \\ = & \begin{pmatrix} Q & Q^{\frac{1}{2}} (I - Q)^{\frac{1}{2}} D \\ D^* Q^{\frac{1}{2}} (I - Q)^{\frac{1}{2}} & D^* (I - Q) D \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & D^* Q D \end{pmatrix}, \end{aligned}$$

we obtain

$$\begin{aligned}\overline{R(\begin{pmatrix} Q \\ D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}} \end{pmatrix})} &= R(\begin{pmatrix} Q & Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}D \\ D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}} & D^*(I-Q)D \end{pmatrix}) \\ &= R(P_{\mathcal{N}} \mid_{\mathcal{H}_5 \oplus \mathcal{H}_6})\end{aligned}$$

and by Lemma 1 that Q is injective and positive, where $R(T)$ denotes the range of an operator T and $\overline{K}$ the closure of a set K. Denoting

$$\mathcal{N}^0 = \mathcal{H}_1 \oplus \mathcal{H}_3 \oplus R(\begin{pmatrix} Q \\ D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}} \end{pmatrix}),$$

it is obvious that $\overline{\mathcal{N}^0} = \mathcal{N}$, so $\delta(\mathcal{N},\mathcal{M}) = \delta(\mathcal{N}^0,\mathcal{M})$. For a vector $y \in \mathcal{N}^0$, there exist vectors $y_1 \in \mathcal{H}_1$, $y_3 \in \mathcal{H}_3$ and $y_5 \in \mathcal{H}_5$ such that $y = y_1 + y_3 + Qy_5 + D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5$ and $P_{\mathcal{M}}y = y_1 + Qy_5$. Observing that $\| y \|^2 = \| y_1 \|^2 + \| y_3 \|^2 + \| Qy_5 \|^2 + \| D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5 \|^2 = \| y_1 \|^2 + \| y_3 \|^2 + \| Q^{\frac{1}{2}}y_5 \|^2$ and $\| P_{\mathcal{M}}y \|^2 = \| y_1 \|^2 + \| Qy_5 \|^2$, we have

$$\begin{aligned}\operatorname{dist}(y,\mathcal{M})^2 &= \| y \|^2 - \| P_{\mathcal{M}}y \|^2 \\ &= \| y_3 \|^2 + \| D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5 \|^2 \\ &= \| y_3 \|^2 + \| Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5 \|^2 .\end{aligned}$$

Hence,

$$\begin{aligned}\delta(\mathcal{N},\mathcal{M}) &= \delta(\mathcal{N}^0,\mathcal{M}) \\ &= \sup\{\operatorname{dist}(y,\mathcal{M}) : y \in \mathcal{N}^0, \| y \| \leq 1\} \\ &= \sup\{(\| y_3 \|^2 + \| Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5 \|^2)^{\frac{1}{2}} : \\ &\qquad \| y_1 \|^2 + \| y_3 \|^2 + \| Q^{\frac{1}{2}}y_5 \|^2 \leq 1, y_i \in \mathcal{H}_i, i = 1,3,5\} \\ &\leq 1.\end{aligned}$$

If $\mathcal{M}^\perp \cap \mathcal{N} \neq \{0\}$, then $\mathcal{H}_3 \neq \{0\}$ and $\mathcal{H}_3 \subset \mathcal{N}^0$; hence $\delta(\mathcal{N}^0,\mathcal{M}) \geq \sup\{\| y_3 \| : y_3 \in \mathcal{H}_3, \| y_3 \| \leq 1\} = 1$. Therefore, $\delta(\mathcal{N},\mathcal{M}) = 1$.

If $\mathcal{M}^\perp \cap \mathcal{N} = \{0\}$, $\mathcal{H}_3 = \{0\}$ and $y_3 = 0$, then

$$\begin{aligned}\delta(\mathcal{N},\mathcal{M}) &= \delta(\mathcal{N}^0,\mathcal{M}) \\ &= \sup\{\| Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5 \| : \| y_1 \|^2 + \| Q^{\frac{1}{2}}y_5 \|^2 \leq 1, y_1 \in \mathcal{H}_1, y_5 \in \mathcal{H}_5\} \\ &= \sup\{\| (I-Q)^{\frac{1}{2}}Q^{\frac{1}{2}}y_5 \| : y_5 \in \mathcal{H}_5, \| Q^{\frac{1}{2}}y_5 \| \leq 1\} \\ &= \| (I-Q)^{\frac{1}{2}} \| .\end{aligned}$$

Here the last equation holds since $\overline{R(Q^{\frac{1}{2}})} = \mathcal{H}_5$. □

Proposition 4. *Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$, and let $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices* (3). *If $\mathcal{M} \cap \mathcal{N}^\perp \neq \{0\}$, then $\gamma(\mathcal{M},\mathcal{N}) = 1$. If $\mathcal{M} \cap \mathcal{N}^\perp = \{0\}$, then $\gamma(\mathcal{M},\mathcal{N}) = \| (I-Q)^{\frac{1}{2}} \|^{-1}$.*

Proof. Let $x \in \mathcal{M}$. Then there exist vectors $x_1 \in \mathcal{H}_1$, $x_2 \in \mathcal{H}_2$ and $x_5 \in \mathcal{H}_5$ such that $x = x_1 + x_2 + x_5$, $\| x \|^2 = \sum_{i=1,2,5} \| x_i \|^2$ and

$$\begin{aligned}\operatorname{dist}(x,\mathcal{N})^2 &= \| x \|^2 - \| P_{\mathcal{N}}x \|^2 \\ &= \| x_2 \|^2 + \| x_5 \|^2 - \| Qx_5 \|^2 - \| D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}x_5 \|^2 \\ &= \| x_2 \|^2 + \| (I-Q)^{\frac{1}{2}}x_5 \|^2 .\end{aligned}$$

Hence,

$$\begin{aligned}\gamma(\mathcal{M},\mathcal{N}) &= \inf\{\| x \|: x\in\mathcal{M}, \operatorname{dist}(x,\mathcal{N})\geq 1\}\\ &= \inf\{\| x \|: x=\textstyle\sum_{i=1,2,5} x_i, x_i\in\mathcal{H}_i, i=1,2,5,\\ &\qquad \| x_2 \|^2+\| (I-Q)^{\frac{1}{2}}x_5 \|^2\geq 1\}.\end{aligned}$$

The operator Q being a positive contraction, $\operatorname{dist}(x,\mathcal{N})\geq 1$ implies that $1\leq\| x_2 \|^2 + \| (I-Q)^{\frac{1}{2}}x_5 \|^2\leq\| x_2 \|^2+\| x_5 \|^2\leq\| x \|^2$. Hence $\gamma(\mathcal{M},\mathcal{N})\geq 1$.

If $\mathcal{M}\cap\mathcal{N}^\perp\neq\{0\}$, then it follows upon choosing $x_0=x_2\in\mathcal{H}_2$ with $\| x_2 \|=1$ that $\operatorname{dist}(x_0,\mathcal{N})=1$, i.e., $\gamma(\mathcal{M},\mathcal{N})=1$.

If, instead, $\mathcal{M}\cap\mathcal{N}^\perp=\{0\}$, then

$$\begin{aligned}\gamma(\mathcal{M},\mathcal{N}) &= \inf\{\| x_1+x_5 \|: x_1\in\mathcal{H}_1, x_5\in\mathcal{H}_5, \| (I-Q)^{\frac{1}{2}}x_5 \|\geq 1\}\\ &= \inf\{\| x_5 \|: x_5\in\mathcal{H}_5, \| (I-Q)^{\frac{1}{2}}x_5 \|\geq 1\}.\end{aligned}$$

Observing $1\leq\| (I-Q)^{\frac{1}{2}}x_5 \|\leq\| (I-Q)^{\frac{1}{2}} \|\| x_5 \|$, it follows that $\| x_5 \|\geq\| (I-Q)^{\frac{1}{2}} \|^{-1}$, and hence that $\| (I-Q)^{\frac{1}{2}} \|^{-1}\leq\gamma(\mathcal{M},\mathcal{N})$. To prove the reverse inequality, we let Q have the spectral representation $Q=\int_{\mu_0}^{\nu_0}\lambda dE_\lambda$, where $\mu_0=\min\{\lambda:\lambda\in\sigma(Q)\}\geq 0$, $\nu_0=\max\{\lambda:\lambda\in\sigma(Q)\}\leq 1$, and $\sigma(Q)$ denotes the spectrum of Q. For $\epsilon>0$ small enough, choose a vector $x_\epsilon\in E([\mu_0,\mu_0+\epsilon])\mathcal{H}_5$ with $1+\epsilon\geq\| (I-Q)^{\frac{1}{2}}x_\epsilon \|\geq 1$. Then $1+\epsilon\geq\| (I-Q)^{\frac{1}{2}}x_\epsilon \|=\| \int_{\mu_0}^{\nu_0}(1-\lambda)^{\frac{1}{2}}dE_\lambda x_\epsilon \|=\| \int_{\mu_0}^{\mu_0+\epsilon}(1-\lambda)^{\frac{1}{2}}dE_\lambda x_\epsilon \|\geq(1-\mu_0-\epsilon)^{\frac{1}{2}}\| x_\epsilon \|$, i.e., $\| x_\epsilon \|\leq(1-\mu_0-\epsilon)^{-\frac{1}{2}}(1+\epsilon)$. Since ϵ is arbitrary and $\| (I-Q)^{\frac{1}{2}} \|^{-1}=(1-\mu_0)^{-\frac{1}{2}}$, $\gamma(\mathcal{M},\mathcal{N})\leq\| (I-Q)^{\frac{1}{2}} \|^{-1}$. Hence $\gamma(\mathcal{M},\mathcal{N})=\| (I-Q)^{\frac{1}{2}} \|^{-1}$. □

Proposition 5. *Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$, and let $P_\mathcal{M}$ and $P_\mathcal{N}$ have the operator matrices* (3)*. If $\mathcal{M}^\perp\cap\mathcal{N}\neq\{0\}$, then $\gamma(\mathcal{N},\mathcal{M})=1$. If $\mathcal{M}^\perp\cap\mathcal{N}=\{0\}$, then $\gamma(\mathcal{N},\mathcal{M})=\| (I-Q)^{\frac{1}{2}} \|^{-1}$.*

Proof. Similar to the proof of Proposition 3, we can show that

$$\gamma(\mathcal{N},\mathcal{M})=\gamma(\mathcal{N}^0,\mathcal{M}).$$

To complete the proof, it is enough to consider the quantum $\gamma(\mathcal{N}^0,\mathcal{M})$. Since

$$\begin{aligned}\gamma(\mathcal{N}^0,\mathcal{M})= &\inf\{\| y \|: y=y_1+y_3+Qy_5+D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5,\\ &y_i\in\mathcal{H}_i, i=1,3,5, \| y_3 \|^2+\| D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5 \|^2\geq 1\},\end{aligned}$$

it is clear that $\gamma(\mathcal{N}^0,\mathcal{M})\geq 1$ in general.

Let $\mathcal{M}^\perp\cap\mathcal{N}\neq\{0\}$, choosing $y_0=y_3\in\mathcal{H}_3$ with $\| y_3 \|=1$, so that $y_0\in\mathcal{N}^0$, $\| y_0 \|=1$ and $\operatorname{dist}(y_0,\mathcal{M})=1$, i.e., $\gamma(\mathcal{N}^0,\mathcal{M})\leq 1$. Hence $\gamma(\mathcal{N}^0,\mathcal{M})=1$.

If $\mathcal{M}^\perp\cap\mathcal{N}=\{0\}$, then $\mathcal{H}_3=\{0\}$, so

$$\begin{aligned}\gamma(\mathcal{N}^0,\mathcal{M})= &\inf\{\| y \|: y=y_1+Qy_5+D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5,\\ &y_i\in\mathcal{H}_i, i=1,5, \| D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5 \|^2\geq 1\}.\end{aligned}$$

Noting that

$$\begin{aligned}\| y \|^2 &=\| y_1 \|^2+\| Qy_5 \|^2+\| D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5 \|^2\\ &=\| y_1 \|^2+\| Q^{\frac{1}{2}}y_5 \|^2\geq\| Q^{\frac{1}{2}}y_5 \|^2\end{aligned}$$

and $\overline{R(Q^{\frac{1}{2}})} = \mathcal{H}_5$, then

$$\begin{aligned}\gamma(\mathcal{N}^0, \mathcal{M}) &= \inf\{\| Q^{\frac{1}{2}} y_5 \| : y_5 \in \mathcal{H}_5, \| Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}} y_5 \| \geq 1\} \\ &= \inf\{\| z \| : z \in \mathcal{H}_5, \| (I-Q)^{\frac{1}{2}} z \| \geq 1\} \\ &= \| (I-Q)^{\frac{1}{2}} \|^{-1} .\end{aligned}$$

So $\gamma(\mathcal{N}^0, \mathcal{M}) = \| (I-Q)^{\frac{1}{2}} \|^{-1}$. □

By Propositions 2−5, the following consequences are clear.

Corollary 6. *If $\mathcal{M}$ and $\mathcal{N}$ are two closed subspaces of $\mathcal{H}$, then $\delta(\mathcal{M},\mathcal{N})\gamma(\mathcal{M},\mathcal{N}) = 1$.*

Combining Proposition 2 with Proposition 3, we get

Corollary 7. *Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. If $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ as projections onto $\mathcal{M}$ and $\mathcal{N}$, respectively, have the formulas* (3) *of operator matrices, then*

$$\operatorname{gap}(\mathcal{M},\mathcal{N}) = \begin{cases} 1, & \mathcal{M} \cap \mathcal{N}^{\perp} \neq \{0\} \text{ or } \mathcal{M}^{\perp} \cap \mathcal{N} \neq \{0\}; \\ \| (I-Q)^{\frac{1}{2}} \|, & \mathcal{M} \cap \mathcal{N}^{\perp} = \{0\} \text{ and } \mathcal{M}^{\perp} \cap \mathcal{N} = \{0\}. \end{cases}$$

Remarks. (1) $\delta(\mathcal{M},\mathcal{N}) = \| P_{\mathcal{M}}(I - P_{\mathcal{N}}) \|$.

Proof. Observing that $\| P_{\mathcal{M}}(I-P_{\mathcal{N}}) \|^2 = \| P_{\mathcal{M}}(I-P_{\mathcal{N}})P_{\mathcal{M}} \|$, from the formula (2), we obtain

$$P_{\mathcal{M}}(I-P_{\mathcal{N}})P_{\mathcal{M}} = \begin{pmatrix} 0 & & & & & \\ & I & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & I-Q & \\ & & & & & 0 \end{pmatrix}.$$

If $\mathcal{H}_2 \neq \{0\}$, then $\| P_{\mathcal{M}}(I-P_{\mathcal{N}}) \| = \max\{1, \| (I-Q)^{\frac{1}{2}} \|\} = 1$ since Q is a positive contraction. If $\mathcal{H}_2 = \{0\}$, then $\| P_{\mathcal{M}}(I-P_{\mathcal{N}}) \| = \| (I-Q)^{\frac{1}{2}} \|$. That is, $\delta(\mathcal{M},\mathcal{N}) = \| P_{\mathcal{M}}(I-P_{\mathcal{N}}) \|$. □

(2) Proposition 3 can be seen as a corollary of Proposition 2. In fact, from Remark (1) and Proposition 2, we have $\delta(\mathcal{N},\mathcal{M}) = \| P_{\mathcal{N}}(I-P_{\mathcal{M}}) \|$. Then

$$\delta(\mathcal{N},\mathcal{M}) = \| P_{\mathcal{N}}(I-P_{\mathcal{M}}) \| = \begin{cases} 1, & \text{if } \mathcal{M}^{\perp} \cap \mathcal{N} \neq \{0\}, \\ \| (I-Q)^{\frac{1}{2}} \|, & \text{if } \mathcal{M}^{\perp} \cap \mathcal{N} = \{0\}. \end{cases}$$

(3) Similarly, we have $\gamma(\mathcal{M},\mathcal{N}) = \| P_{\mathcal{M}}(I-P_{\mathcal{N}}) \|^{-1}$. Proposition 5 can also be derived as a corollary of Proposition 4.

But here we still give the direct proofs of Propositions 3 and 5 because the direct proofs give us more information of the geometrical structures between two subspaces.

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College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, People's Republic of China
E-mail address: `hkdu@snnu.edu.cn`

College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, People's Republic of China
E-mail address: `cy-deng@263.net`