PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 10, Pages 3115–3120
S 0002-9939(05)07875-5
Article electronically published on April 20, 2005

LINE ARRANGEMENTS IN $\mathbb{H}^3$

PETER MILLEY

(Communicated by Ronald A. Fintushel)

Dedicated to my wife, Cheryl

ABSTRACT. If $M = \mathbb{H}^3/G$ is a hyperbolic manifold and $\gamma \subset M$ is a simple closed geodesic, then γ lifts to a collection of lines in $\mathbb{H}^3$ acted upon by G. In this paper we show that such a collection of lines cannot contain a particular type of subset (called a *bad triple*) unless G has orientation-reversing elements. This fact allows us to extend certain lower bounds on hyperbolic volume to the non-orientable case.

1. Introduction

Suppose that G is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ acting freely and properly-discontinuously, and that l is a directed line in $\mathbb{H}^3$ such that for all G either $g(l) = l$ or else $g(l)$ and l are disjoint and do not share an endpoint. For example, G might be the fundamental group of a hyperbolic manifold, and l the pre-image of a simple closed geodesic in that manifold. Let $\mathcal{C}$ be the collection of all unordered pairs $\{l_1,\ l_2\}$ where l_1 and l_2 are distinct images of l under the action of G. Clearly G acts on $\mathcal{C}$ and defines an equivalence relation. Define a *bad triple* to be a triple of distinct lines $\{l_1,\ l_2,\ l_3\}$ such that $\{l_1,\ l_2\}$, $\{l_2,\ l_3\}$, and $\{l_3,\ l_1\}$ are equivalent elements of $\mathcal{C}$.

Using the above terminology, the goal of this paper is to prove the following:

Theorem 1.1. *If a bad triple exists, then G contains an orientation-reversing element which fixes l.*

The above theorem has immediate applications in the case where G is the fundamental group of a hyperbolic manifold and l the pre-image of a closed geodesic. In that context, the absence of any bad triples has implications concerning the volume of the manifold; these implications and the connection between Theorem 1.1 and [6] are explored in section 3. The proof of Theorem 1.1 is contained in the following section.

Received by the editors April 15, 2004 and, in revised form, June 3, 2004.
2000 *Mathematics Subject Classification.* Primary 57M60, 51M09; Secondary 57M50.
Key words and phrases. Hyperbolic geometry, non-orientable manifolds.
The author was supported in part by NSF Grants DMS-9505253 and DMS-0071852.
The author would like to thank David Gabai for his comments and support, and the reviewer for his comments and corrections.

2. Proof of Theorem 1.1

Lemma 2.1. *If $\{l_1,\ l_2\}$ and $\{l_3,\ l_4\}$ are equivalent elements of $\mathcal{C}$, then there is a* unique *element of G which sends $\{l_1,\ l_2\}$ to $\{l_3,\ l_4\}$.*

Proof. Clearly it suffices to show that if g sends the pair $\{l_1,\ l_2\} \in \mathcal{C}$ to itself, then g must be the identity. By assumption l_1 and l_2 are distinct lines without common endpoints; hence there is a unique shortest line segment γ joining l_1 to l_2. If g fixes l_1 and l_2, then g fixes γ point-wise; hence g is the identity since G acts freely. If g swaps l_1 with l_2, then g fixes γ but reverses its orientation; hence g fixes some point on γ, a contradiction since g is non-trivial and G acts freely. □

The above lemma allows us to make a few additional definitions.

First, we say that two equivalent pairs have the same *orientation* if and only if the group element which sends one pair to the other is orientation-preserving.

Second, if $\{l_1,\ l_2\}, \{l_3,\ l_4\} \in \mathcal{C}$ are equivalent pairs, we say the *ordered* pairs $(l_1,\ l_2)$ and $(l_3,\ l_4)$ are *similarly ordered* if the corresponding group element g sends l_1 to l_3 and l_2 to l_4. If $g(l_1) = l_4$ and $g(l_2) = l_3$, we say the ordered pairs are *differently ordered*.

We now begin proving Theorem 1.1 by cases. Let $\{l_1,\ l_2,\ l_3\}$ be a bad triple corresponding to the group G and the line l.

For the first case, suppose the three pairs do not all have the same orientation. Since there are three pairs and only two possible orientations, without loss of generality we may assume that $\{l_2,\ l_3\}$ and $\{l_3,\ l_1\}$ have the same orientation while $\{l_1,\ l_2\}$ and $\{l_2,\ l_3\}$ have different orientations. Consider the ordered pairs $(l_1,\ l_2)$ and $(l_2,\ l_3)$. If those two ordered pairs are differently ordered, then by definition there is an orientation-reversing $g \in G$ such that $g(l_2) = l_2$, which (after conjugation) proves the theorem. Similarly, if the ordered pairs $(l_1,\ l_2)$ and $(l_3,\ l_1)$ are differently ordered, then there is an orientation-reversing $g \in G$ such that $g(l_1) = l_1$.

So suppose $(l_1,\ l_2)$, $(l_2,\ l_3)$, and $(l_3,\ l_1)$ are all similarly ordered. Then there is an orientation-reversing element $g \in G$ such that $g(l_1) = l_2$ and $g(l_2) = l_3$, and an orientation-preserving $h \in G$ such that $h(l_2) = l_3$ and $h(l_3) = l_1$. Then $gh^{-1}(l_3) = l_3$, and gh^{-1} reverses orientation. This proves the theorem in the first case.

It remains to consider the case where the three pairs all have the same orientation. It is trivial to check that among the ordered pairs $(l_1,\ l_2)$, $(l_2,\ l_3)$, and $(l_3,\ l_1)$, some two pairs must be similarly ordered. Without loss of generality, assume the first two pairs are similarly ordered. Hence there exists an orientation-preserving $g \in G$ such that $g(l_1) = l_2$ and $g(l_2) = l_3$. We claim that g is elliptic of order three, contradicting the assumption that G acts freely and proving Theorem 1.1.

To prove the claim, note first that since G acts freely and l is a directed line in $\mathbb{H}^3$, we may assume that l_1, l_2, and l_3 are all directed lines and that the action of G preserves direction. Furthermore we may assume that l_2 is the line from 0 to ∞ in the upper half-space model of $\mathbb{H}^3$, that the shortest line from l_2 to l_1 is the line from -1 to $+1$, and that l_1 separates l_2 from $+1$ on this line. Then l_1 must be the line from a^{-1} to a for some non-zero $a \in \mathbb{C}$.

Let T be the following element of $\mathrm{PGL}(2, \mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$:

$$T = \begin{pmatrix} a & -1 \\ 1 & -a \end{pmatrix}.$$

Note that $-a^2+1 \neq 0$ since a and a^{-1} must be distinct. Also, as a fractional linear transformation T sends a to ∞, ∞ to a, a^{-1} to 0, and 0 to a^{-1}. Hence T swaps the directed lines l_1 and l_2. Thus $g \circ T$ fixes l_2 and must be equivalent to a diagonal matrix in $\mathrm{PGL}(2,\mathbb{C})$ with diagonal entries ρ, ρ^{-1} for some non-zero $\rho \in \mathbb{C}$. Furthermore $g \circ T$ sends l_1 to l_3. Hence l_3 must be the line from $\rho^2 a^{-1}$ to $\rho^2 a$; in particular, $\rho^2 \neq 1$.

Now we use the fact that the pair $\{l_1,\ l_3\}$ is equivalent to the pair $\{l_1,\ l_2\}$ and has the same orientation. For any four distinct points $z_1,\ z_2,\ w_1,\ w_2 \in \mathbb{C} \cup \infty$, the *cross-ratio* $R(z_1,\ z_2,\ w_1,\ w_2)$ defined by

$$\frac{(w_2-z_1)(w_1-z_2)}{(w_2-z_2)(w_1-z_1)}$$

is fixed by orientation-preserving isometries and remains constant if the w_i's are exchanged with the z_i's. (See for example [7].) In particular, the cross-ratio $R(0,\ \infty,\ a^{-1},\ a)$ must equal $R(a^{-1},\ a,\ \rho^2 a^{-1},\ \rho^2 a)$. Hence

$$\begin{aligned} a^2 &= \frac{(\rho^2 a - a^{-1})(\rho^2 a^{-1} - a)}{(\rho^2 a - a)(\rho^2 a^{-1} - a^{-1})} \\ &= \frac{\rho^4 - (a^2 + a^{-2})\rho^2 + 1}{(\rho^2-1)^2} \\ \Rightarrow\ a^2 - 1 &= \frac{-(a^2 - 2 + a^{-2})\rho^2}{(\rho^2-1)^2}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1-a^2}{(a-a^{-1})^2} &= \frac{\rho^2}{(\rho^2-1)^2} \\ \Rightarrow\ \frac{a}{a^{-1}-a} &= \frac{1}{(\rho-\rho^{-1})^2}. \end{aligned}$$

Additionally we know that $g = PT^{-1}$, where P is the diagonal matrix with diagonal entries ρ, ρ^{-1}. Hence

$$g = \frac{1}{1-a^2}\begin{pmatrix} -\rho a & \rho \\ -\rho^{-1} & \rho^{-1} a \end{pmatrix}.$$

Hence

$$\begin{aligned} \frac{\mathrm{tr}(g)^2}{\det(g)} &= \frac{a^2(\rho^{-1}-\rho)^2}{1-a^2} \\ &= \frac{a^2}{1-a^2}\frac{a^{-1}-a}{a} \\ &= 1. \end{aligned}$$

But this implies that g is elliptic of order three, as desired. This completes the proof of Theorem 1.1.

3. Applications to hyperbolic 3-manifolds

In [6], it was proved that the volume of a complete orientable hyperbolic three-manifold was greater than or equal to 0.943 if either the shortest closed geodesic in the manifold had length $\leq$ 0.0717 or if the shortest closed geodesic had an embedded tubular neighborhood of radius ≥ 1.464. A natural question is whether or not these results can be extended to non-orientable manifolds; Theorem 1.1 answers this question in the affirmative, as we shall see.

In [6], the following terminology is used. Let M be a complete (orientable) hyperbolic three-manifold with a shortest closed geodesic γ, and let $G = \pi_1(M)$. Let $\{\gamma_i\}$ be the collection of lines in $\mathbb{H}^3$ which projects to γ, and let γ_0 denote a fixed lift. Then two lines γ_i, γ_j with $i, j \neq 0$ are in the same *ortholine class* if (in the language of the present paper) the pairs $\{\gamma_0,\ \gamma_i\}$ and $\{\gamma_0,\ \gamma_j\}$ are equivalent with respect to the action of G. Associated to each ortholine class is the distance from each element of the class to γ_0, which is clearly well-defined. Let $\mathcal{O}(1)$, $\mathcal{O}(2)$, $\ldots$ denote the ortholine classes ordered so that if $O(i)$ denotes the corresponding distance, then $O(1) \leq O(2) \leq \cdots$. Let $r = O(1)/2$, which corresponds to the radius of the maximal embedded tubular neighborhood of γ in M. Let d denote the hyperbolic distance function.

Then, given the above language, lemma 4.2 of [6] can be restated as follows (the formulation given in [6] is slightly different, but equivalent):

Lemma 3.1. *Suppose M is an orientable manifold, γ_i, $\gamma_j \in \mathcal{O}(1)$, $\gamma_i \neq \gamma_j$, and $O(2) > 2r$. Then $d(\gamma_i, \gamma_j) \geq O(2)$.*

The above result is proved in [6] using direct calculation in $\mathrm{PSL}(2, \mathbb{C})$; hence, the proof applies only in the orientable case. It is, however, the only result in [6] which does require orientability. Furthermore, the reader will note that the lemma is a corollary of Theorem 1.1: if M is orientable, then the group G has no orientation-reversing elements and hence no bad triples. Thus the pair $\{\gamma_i,\ \gamma_j\}$ is not equivalent to the pair $\{\gamma_0,\ \gamma_i\}$. But since $O(2) > O(1)$ we must have $O(n) > O(1)$ for all $n > 1$, and furthermore by symmetry $d(\gamma_i, \gamma_j) = O(1)$ if and only if $\{\gamma_i,\ \gamma_j\}$ is equivalent to $\{\gamma_0,\ \gamma_i\}$. Hence $d(\gamma_i, \gamma_j)$ must be at least as large as $O(2)$, as desired.

From the above argument, it is clear how to extend Lemma 3.1 to the non-orientable case. The following lemma follows directly from Theorem 1.1:

Lemma 3.2. *Suppose M is a non-orientable manifold, γ_i, $\gamma_j \in \mathcal{O}(1)$, $\gamma_i \neq \gamma_j$, and $O(2) > 2r$. Then either $d(\gamma_i, \gamma_j) \geq O(2)$ for all choices of γ_i and γ_j, or else the closed geodesic γ has orientation-reversing holonomy.*

So if M is non-orientable and the holonomy of γ preserves orientation, then the arguments of [6] still hold provided we substitute Lemma 3.2 above for Lemma 4.2 of that paper. Suppose instead the holonomy of γ reverses orientation.

In the upper half-space model of $\mathbb{H}^3$, any isometry corresponding to such a geodesic is an anti-Möbius transformation of the form $z \mapsto (a\bar{z} + b)/(c\bar{z} + d)$ for some complex numbers a, b, c, and d. It is easy to verify that the square of such a transformation is a Möbius transformation and that the trace of the corresponding element of $\mathrm{PGL}(2, \mathbb{C})$ is $a\bar{a} + b\bar{c} + c\bar{b} + d\bar{d}$, which is real. Therefore if N is the orientable double cover of M, then γ lifts to a single closed geodesic δ in N which has zero torsion; that is, the corresponding isometry is a pure translation. (More specifically, the isometry corresponding to γ must be a hyperbolic glide-reflection. For further information about the classification of orientation-reversing isometries of $\mathbb{H}^3$, see for example [7], chapter IV.)

Let l be the length of γ, so that $2l$ is the length of δ, and note that δ has an embedded tubular neighborhood of radius r just like γ. Let δ_0 be a particular line in $\mathbb{H}^3$ which projects to δ and let δ_1 be another lift of δ such that $d(\delta_0, \delta_1) = 2r$. Let p_1, p_2 be the points on δ_1, $g(\delta_1)$ respectively which are closest to δ_0, where g

generates the subgroup of $\pi_1(N)$ which fixes δ_0, and let $s = d(p_1, p_2)$. Then by symmetry we must have $s \geq 2r$. But since δ has zero holonomy, p_1 and p_2 are two vertices of a plane hyperbolic quadrilateral with sides of length $2l$, $2r$, s, and $2r$, and in which the two angles adjacent to the side of length $2l$ are right angles. By elementary hyperbolic trigonometry (see for example [7], Chapter VI), we have

$$\cosh(s) = \cosh^2(2r)\cosh(2l) - \sinh^2(2r).$$

Since $s \geq 2r$ this implies that

$$l \geq \frac{1}{2}\cosh^{-1}\left(\frac{\sinh^2(2r) + \cosh(2r)}{\cosh^2(2r)}\right). \tag{3.1}$$

We now proceed as in [6], using equation (3.1) above in place of equation (4.1) of [6] in the case where M is non-orientable and γ has orientation-reversing holonomy. Specifically, note that substituting (3.1) into $\pi l \sinh^2(r)$ (the volume of the maximal embedded tubular neighborhood of γ) results in an increasing function of r, and that when $r \geq 0.846$, then $\pi l \sinh^2(r)$ is at least 0.943, which is greater than the volume of the Weeks manifold. This is clearly at least as good as the result of [6] in the case where M is orientable or γ has orientation-preserving holonomy, which is that $\pi l \sinh^2(r) \geq 0.943$ whenever $r \geq 1.464$.

We should note at this point that the results of [6] have already been improved upon in the orientable case by other authors, and that Theorem 1.1 can be used in some cases to extend these results to the non-orientable case as well. For example, in the proof of Theorem 4.9 of [8] it is shown that $\pi l \sinh^2(r) \geq 0.94274$ whenever M is orientable and $r \geq 1.332$. An examination of [8] shows that the proof of Theorem 4.9 of that paper doesn't use orientability except for the fact that it uses the results of [6]. Hence by combining the proof of that theorem with the above argument we can conclude that $\pi l \sinh^2(r) \geq 0.94274$ whenever $r \geq 1.332$, in any complete hyperbolic 3-manifold.

Furthermore, while the right-hand side of (3.1) is not a monotonic function of r, by combining (3.1) with a previous estimate by Cao-Gehring-Martin [2] (which appears in [6] as equation 3.3 and which is monotonic) we can obtain a lower bound on r in terms of l whenever $l \leq 0.113$. Furthermore this combined estimate implies that whenever $l \leq 0.113$, then $\pi l \sinh^2(r) \geq 6.54$! This is a vast improvement on known results in the case where M is orientable or γ has orientation-preserving holonomy. In particular the above argument extends the result of Theorem 4.9 of [8] (that the tube volume exceeds 0.94274 whenever $l \leq 0.09$) to the non-orientable case.

Thus we obtain the following result, which can be thought of as a corollary of Theorem 1.1, Theorem 4.9 of [8], and the results of [6]:

Corollary 3.3. *If W is a maximal tube of radius r about a geodesic γ in the complete hyperbolic 3-manifold M and either* length$(\gamma) \leq 0.09$ *or $r \geq 1.332$, then* volume$(W) \geq 0.943$.

4. Remarks

To the best of my knowledge, the results in Corollary 3.3 are the best known for non-orientable manifolds; however, they are not the best results known in the orientable case. Ian Agol has been able to show in [1] that the volume is at least 0.943 if length$(\gamma) \leq 0.1036$, while Hodgson and Kerckhoff have shown in [4] that the volume is at least 1.701 if the length$(\gamma) \leq 0.162$. However, it is not immediately clear in either case if the techniques used can be extended to non-orientable manifolds. This is clearly an area deserving of future investigation.

It is also unknown at this time whether or not there exist a group G and a line l which admit a bad triple at all. There is, however, a kind of "analogue-example" in the world of cusped hyperbolic manifolds. We can define a *bad triple of horoballs* in a manner analogous to the previously defined bad triple of lines, in the obvious fashion. Then the fundamental group of the Geiseking manifold, together with the pre-image of the Geiseking manifold's rigid cusp, provides an example of a bad triple of horoballs. The fact that the cusp in the Geiseking manifold is rigid (i.e., is a Klein bottle cusp) is a necessary condition; the analogue of Theorem 1.1 for bad triples of horoballs is also true and has a very similar proof to the one given here for lines.

This suggests that the lower bound for the volume of cusped orientable hyperbolic 3-manifolds found in [3] might be extended to non-orientable cusped manifolds, provided that the cusps in the manifolds are all orientable (i.e., no Klein bottle cusps). Indeed, a quick search of the SnapPea census ([9]) shows that the smallest known non-orientable cusped hyperbolic manifold with no Klein bottle cusps is the manifold m131. This manifold has the same volume as a regular ideal hyperbolic octahedron, approximately 3.664, which is more than half again as large as the smallest orientable cusped manifolds, and more than three times as large as the Geiseking manifold. Further investigation in this area will hopefully shed some light on the minimum-volume problem for non-orientable manifolds.

References

1. I. Agol, *Volume Change Under Drilling*, Geom. Topol. **6** (2002) 905–916 (electronic). MR1943385 (2004e:57021)
2. C. Cao, F.W. Gehring, G.J. Martin, *Lattice Constants and a Lemma of Zagier*, Lipa's Legacy, Contemp. Math. **211** 107–120 Amer. Math. Soc., Providence, R.I., 1997. MR1476983 (99a:30040)
3. C. Cao and R. Meyerhoff, *The orientable cusped hyperbolic 3-manifolds of minimum volume*, Invent. Math. **146** (2001) 451–478. MR1869847 (2002i:57016)
4. C. Hodgson and S. Kerckhoff, *Universal Bounds for Hyperbolic Dehn Surgery*, Preprint.
5. T. Marshall and G. Martin, *Volumes of Hyperbolic 3-Manifolds*, Conform. Geom. Dyn. **7** (2003) 34–48 (electronic). MR1992036 (2004e:57022)
6. D. Gabai, R. Meyerhoff, and P. Milley, *Volumes of Tubes in Hyperbolic 3-Manifolds*, J. Diff. Geometry **57** (2001) 23–46. MR1871490 (2002i:57017)
7. W. Fenchel, *Elementary Geometry in Hyperbolic Space*, de Gruyter Stud. in Math. Vol. 11, 1989. MR1004006 (91a:51009)
8. A. Przeworski, *Cones in Hyperbolic 3-manifolds*, Preprint.
9. J. Weeks, *SnapPea*, http://www.northnet.org/weeks/.

Department of Mathematics, Princeton University, Princeton, New Jersey 08544-1000

E-mail address: `milley@math.princeton.edu`

Current address: Department of Mathematics, University of California–Riverside, Riverside, California 92521-0135

E-mail address: `milley@math.ucr.edu`