LINE ARRANGEMENTS IN $\mathbb{H}^3$

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(Communicated by Ronald A. Fintushel)

Dedicated to my wife, Cheryl

Abstract. If $M = \mathbb{H}^3/G$ is a hyperbolic manifold and $\gamma \subset M$ is a simple closed geodesic, then $\gamma$ lifts to a collection of lines in $\mathbb{H}^3$ acted upon by $G$. In this paper we show that such a collection of lines cannot contain a particular type of subset (called a bad triple) unless $G$ has orientation-reversing elements. This fact allows us to extend certain lower bounds on hyperbolic volume to the non-orientable case.

1. Introduction

Suppose that $G$ is a discrete subgroup of $\text{Isom}(\mathbb{H}^3)$ acting freely and properly-discontinuously, and that $l$ is a directed line in $\mathbb{H}^3$ such that for all $G$ either $g(l) = l$ or else $g(l)$ and $l$ are disjoint and do not share an endpoint. For example, $G$ might be the fundamental group of a hyperbolic manifold, and $l$ the pre-image of a simple closed geodesic in that manifold. Let $C$ be the collection of all unordered pairs $\{l_1, l_2\}$ where $l_1$ and $l_2$ are distinct images of $l$ under the action of $G$. Clearly $G$ acts on $C$ and defines an equivalence relation. Define a bad triple to be a triple of distinct lines $\{l_1, l_2, l_3\}$ such that $\{l_1, l_2\}$, $\{l_2, l_3\}$, and $\{l_3, l_1\}$ are equivalent elements of $C$.

Using the above terminology, the goal of this paper is to prove the following:

Theorem 1.1. If a bad triple exists, then $G$ contains an orientation-reversing element which fixes $l$.

The theorem has immediate applications in the case where $G$ is the fundamental group of a hyperbolic manifold and $l$ the pre-image of a closed geodesic. In that context, the absence of any bad triples has implications concerning the volume of the manifold; these implications and the connection between Theorem 1.1 and \cite{6} are explored in section 3. The proof of Theorem 1.1 is contained in the following section.

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2. Proof of Theorem

Lemma 2.1. If \( \{l_1, l_2\} \) and \( \{l_3, l_4\} \) are equivalent elements of \( \mathcal{C} \), then there is a unique element of \( G \) which sends \( \{l_1, l_2\} \) to \( \{l_3, l_4\} \).

Proof. Clearly it suffices to show that if \( g \) sends the pair \( \{l_1, l_2\} \in \mathcal{C} \) to itself, then \( g \) must be the identity. By assumption \( l_1 \) and \( l_2 \) are distinct lines without common endpoints; hence there is a unique shortest line segment \( \gamma \) joining \( l_1 \) to \( l_2 \). If \( g \) fixes \( l_1 \) and \( l_2 \), then \( g \) fixes \( \gamma \) point-wise; hence \( g \) is the identity since \( G \) acts freely. If \( g \) swaps \( l_1 \) with \( l_2 \), then \( g \) fixes \( \gamma \) but reverses its orientation; hence \( g \) fixes some point on \( \gamma \), a contradiction since \( g \) is non-trivial and \( G \) acts freely.

The above lemma allows us to make a few additional definitions.

First, we say that two equivalent pairs have the same orientation if and only if the group element which sends one pair to the other is orientation-preserving.

Second, if \( \{l_1, l_2\}, \{l_3, l_4\} \in \mathcal{C} \) are equivalent pairs, we say the ordered pairs \((l_1, l_2)\) and \((l_3, l_4)\) are similarly ordered if the corresponding group element \( g \) sends \( l_1 \) to \( l_3 \) and \( l_2 \) to \( l_4 \). If \( g(l_1) = l_4 \) and \( g(l_2) = l_3 \), we say the ordered pairs are differently ordered.

We now begin proving Theorem 1.1 by cases. Let \( \{l_1, l_2, l_3\} \) be a bad triple corresponding to the group \( G \) and the line \( l \).

For the first case, suppose the three pairs do not all have the same orientation. Since there are three pairs and only two possible orientations, without loss of generality we may assume that \( \{l_2, l_3\} \) and \( \{l_3, l_1\} \) have the same orientation while \( \{l_1, l_2\} \) and \( \{l_2, l_3\} \) have different orientations. Consider the ordered pairs \((l_1, l_2)\) and \((l_2, l_3)\). If those two ordered pairs are differently ordered, then by definition there is an orientation-reversing \( g \in G \) such that \( g(l_2) = l_2 \), which (after conjugation) proves the theorem. Similarly, if the ordered pairs \((l_1, l_2)\) and \((l_3, l_1)\) are differently ordered, then there is an orientation-reversing \( g \in G \) such that \( g(l_1) = l_1 \).

So suppose \( \{l_1, l_2\}, \{l_2, l_3\}, \) and \( \{l_3, l_1\} \) all have the same orientation. Then there is an orientation-preserving element \( g \in G \) such that \( g(l_1) = l_2 \) and \( g(l_2) = l_3 \), and an orientation-preserving \( h \in G \) such that \( h(l_2) = l_3 \) and \( h(l_3) = l_1 \). Then \( gh^{-1}(l_3) = l_3 \), and \( gh^{-1} \) reverses orientation. This proves the theorem in the first case.

It remains to consider the case where the three pairs all have the same orientation. It is trivial to check that among the ordered pairs \((l_1, l_2), (l_2, l_3), \) and \((l_3, l_1), \) some two pairs must be similarly ordered. Without loss of generality, assume the first two pairs are similarly ordered. Hence there exists an orientation-preserving \( g \in G \) such that \( g(l_1) = l_2 \) and \( g(l_2) = l_3 \). We claim that \( g \) is elliptic of order three, contradicting the assumption that \( G \) acts freely and proving Theorem 1.1.

To prove the claim, note first that since \( G \) acts freely and \( l \) is a directed line in \( \mathbb{H}^3 \), we may assume that \( l_1, l_2, \) and \( l_3 \) are all directed lines and that the action of \( G \) preserves direction. Furthermore we may assume that \( l_2 \) is the line from 0 to \( \infty \) in the upper half-space model of \( \mathbb{H}^3 \), that the shortest line from \( l_2 \) to \( l_1 \) is the line from \(-1\) to \(+1\), and that \( l_1 \) separates \( l_2 \) from \(+1\) on this line. Then \( l_1 \) must be the line from \( a^{-1} \) to \( a \) for some non-zero \( a \in \mathbb{C} \).

Let \( T \) be the following element of \( \text{PGL}(2, \mathbb{C}) = \text{Isom}^+(\mathbb{H}^3) \):

\[
T = \begin{pmatrix} a & -1 \\ 1 & -a \end{pmatrix}.
\]
Note that $-a^2 + 1 \neq 0$ since $a$ and $a^{-1}$ must be distinct. Also, as a fractional linear transformation $T$ sends $a$ to $\infty$, $\infty$ to $a$, $a^{-1}$ to $0$, and $0$ to $a^{-1}$. Hence $T$ swaps the directed lines $l_1$ and $l_2$. Thus $g \circ T$ fixes $l_2$ and must be equivalent to a diagonal matrix in $\text{PGL}(2, \mathbb{C})$ with diagonal entries $\rho, \rho^{-1}$ for some non-zero $\rho \in \mathbb{C}$. Furthermore $g \circ T$ sends $l_1$ to $l_3$. Hence $l_3$ must be the line from $\rho^2 a^{-1}$ to $\rho^2 a$; in particular, $\rho^2 \neq 1$.

Now we use the fact that the pair $\{l_1, l_3\}$ is equivalent to the pair $\{l_1, l_2\}$ and has the same orientation. For any four distinct points $z_1, z_2, w_1, w_2 \in \mathbb{C} \cup \infty$, the \textit{cross-ratio} $R(z_1, z_2, w_1, w_2)$ defined by

$$\frac{(w_2 - z_1)(w_1 - z_2)}{(w_2 - z_2)(w_1 - z_1)}$$

is fixed by orientation-preserving isometries and remains constant if the $w_i$’s are exchanged with the $z_i$’s. (See for example [7].) In particular, the cross-ratio $R(0, \infty, a^{-1}, a)$ must equal $R(a^{-1}, a, \rho^2 a^{-1}, \rho^2 a)$. Hence

$$a^2 = \frac{(\rho^2 a - a^{-1})(\rho^2 a^{-1} - a)}{\rho^2 a - a)(\rho^2 a^{-1} - a^{-1})} = \frac{\rho^2 - (a^2 + a^{-2})\rho + 1}{(\rho^2 - 1)^2}$$

$$\Rightarrow a^2 - 1 = \frac{-(a^2 - 2 + a^{-2})\rho^2}{(\rho^2 - 1)^2}.$$ 

Hence

$$\frac{1 - a^2}{(a - a^{-1})^2} = \frac{\rho^2}{(\rho^2 - 1)^2}$$

$$\Rightarrow \frac{a}{a^{-1} - a} = \frac{1}{(\rho - \rho^{-1})^2}.$$ 

Additionally we know that $g = PT^{-1}$, where $P$ is the diagonal matrix with diagonal entries $\rho, \rho^{-1}$. Hence

$$g = \frac{1}{1 - a^2} \begin{pmatrix} -\rho a & \rho \\ -\rho^{-1} & \rho^{-1} a \end{pmatrix}.$$ 

Hence

$$\frac{\text{tr}(g)^2}{\text{det}(g)} = \frac{a^2(\rho^{-1} - \rho)^2}{1 - a^2}$$

$$= \frac{a^2}{1 - a^2} \frac{a^{-1} - a}{a}$$

$$= 1.$$ 

But this implies that $g$ is elliptic of order three, as desired. This completes the proof of Theorem [11].

3. \textsc{Applications to hyperbolic 3-manifolds}

In [6], it was proved that the volume of a complete orientable hyperbolic three-manifold was greater than or equal to 0.943 if either the shortest closed geodesic in the manifold had length $\leq 0.0717$ or if the shortest closed geodesic had an embedded tubular neighborhood of radius $\geq 1.464$. A natural question is whether
or not these results can be extended to non-orientable manifolds; Theorem 1.1 answers this question in the affirmative, as we shall see.

In [6], the following terminology is used. Let \( M \) be a complete (orientable) hyperbolic three-manifold with a shortest closed geodesic \( \gamma \), and let \( G = \pi_1(M) \). Let \( \{\gamma_i\} \) be the collection of lines in \( \mathbb{H}^3 \) which projects to \( \gamma \), and let \( \gamma_0 \) denote a fixed lift. Then two lines \( \gamma_i, \gamma_j \) with \( i, j \neq 0 \) are in the same ortholine class if (in the language of the present paper) the pairs \( \{\gamma_0, \gamma_i\} \) and \( \{\gamma_0, \gamma_j\} \) are equivalent with respect to the action of \( G \). Associated to each ortholine class is the distance from each element of the class to \( \gamma_0 \), which is clearly well-defined. Let \( O(1), O(2), \ldots \) denote the ortholine classes ordered so that if \( O(i) \) denotes the corresponding distance, then \( O(1) \leq O(2) \leq \cdots \). Let \( r = O(1)/2 \), which corresponds to the radius of the maximal embedded tubular neighborhood of \( \gamma \) in \( M \). Let \( d \) denote the hyperbolic distance function.

Then, given the above language, lemma 4.2 of [6] can be restated as follows (the formulation given in [6] is slightly different, but equivalent):

**Lemma 3.1.** Suppose \( M \) is an orientable manifold, \( \gamma_i, \gamma_j \in O(1), \gamma_i \neq \gamma_j \), and \( O(2) > 2r \). Then \( d(\gamma_i, \gamma_j) \geq O(2) \).

The above result is proved in [6] using direct calculation in \( \text{PSL}(2, \mathbb{C}) \); hence, the proof applies only in the orientable case. It is, however, the only result in [6] which does require orientability. Furthermore, the reader will note that the lemma is a corollary of Theorem 1.1 if \( M \) is orientable, then the group \( G \) has no orientation-reversing elements and hence no bad triples. Thus the pair \( \{\gamma_i, \gamma_j\} \) is not equivalent to the pair \( \{\gamma_0, \gamma_i\} \). But since \( O(2) > O(1) \) we must have \( O(n) > O(1) \) for all \( n > 1 \), and furthermore by symmetry \( d(\gamma_i, \gamma_j) = O(1) \) if and only if \( \{\gamma_i, \gamma_j\} \) is equivalent to \( \{\gamma_0, \gamma_i\} \). Hence \( d(\gamma_i, \gamma_j) \) must be at least as large as \( O(2) \), as desired.

From the above argument, it is clear how to extend Lemma 3.1 to the non-orientable case. The following lemma follows directly from Theorem 1.1.

**Lemma 3.2.** Suppose \( M \) is a non-orientable manifold, \( \gamma_i, \gamma_j \in O(1), \gamma_i \neq \gamma_j \), and \( O(2) > 2r \). Then either \( d(\gamma_i, \gamma_j) \geq O(2) \) for all choices of \( \gamma_i \) and \( \gamma_j \), or else the closed geodesic \( \gamma \) has orientation-reversing holonomy.

So if \( M \) is non-orientable and the holonomy of \( \gamma \) preserves orientation, then the arguments of [6] still hold provided we substitute Lemma 3.2 above for Lemma 4.2 of that paper. Suppose instead the holonomy of \( \gamma \) reverses orientation.

In the upper half-space model of \( \mathbb{H}^3 \), any isometry corresponding to such a geodesic is an anti-Möbius transformation of the form \( z \mapsto (az + b)/(c\overline{z} + d) \) for some complex numbers \( a, b, c, \) and \( d \). It is easy to verify that the square of such a transformation is a Möbius transformation and that the trace of the corresponding element of \( \text{PGL}(2, \mathbb{C}) \) is \( a\overline{a} + b\overline{c} + \overline{c}b + d\overline{d} \), which is real. Therefore if \( N \) is the orientable double cover of \( M \), then \( \gamma \) lifts to a single closed geodesic \( \delta \) in \( N \) which has zero torsion; that is, the corresponding isometry is a pure translation. (More specifically, the isometry corresponding to \( \gamma \) must be a hyperbolic glide-reflection. For further information about the classification of orientation-reversing isometries of \( \mathbb{H}^3 \), see for example [7], chapter IV.)

Let \( l \) be the length of \( \gamma \), so that \( 2l \) is the length of \( \delta \), and note that \( \delta \) has an embedded tubular neighborhood of radius \( r \) just like \( \gamma \). Let \( \delta_0 \) be a particular line in \( \mathbb{H}^3 \) which projects to \( \delta \) and let \( \delta_1 \) be another lift of \( \delta \) such that \( d(\delta_0, \delta_1) = 2r \).

Let \( p_1, p_2 \) be the points on \( \delta_1 \), \( g(\delta_1) \) respectively which are closest to \( \delta_0 \), where \( g \)
generates the subgroup of $\pi_1(N)$ which fixes $\delta_0$, and let $s = d(p_1, p_2)$. Then by
symmetry we must have $s \geq 2r$. But since $\delta$ has zero holonomy, $p_1$ and $p_2$ are two
vertices of a plane hyperbolic quadrilateral with sides of length $2l$, $2r$, $s$, and $2r$,
and in which the two angles adjacent to the side of length $2l$ are right angles. By
elementary hyperbolic trigonometry (see for example [7], Chapter VI), we have
\[
\cosh(s) = \cosh^2(2r) \cosh(2l) - \sinh^2(2r).
\]
Since $s \geq 2r$ this implies that
\[
(3.1) \quad l \geq \frac{1}{2} \cosh^{-1} \left( \frac{\sinh^2(2r) + \cosh(2l)}{\cosh^2(2r)} \right).
\]

We now proceed as in [6], using equation (3.1) above in place of equation (4.1) of
[6] in the case where $M$ is non-orientable and $\gamma$ has orientation-reversing holonomy.
Specifically, note that substituting (3.1) into $\pi l \sinh^2(r)$ (the volume of the maximal
embedded tubular neighborhood of $\gamma$) results in an increasing function of $r$, and
that when $r \geq 0.846$, then $\pi l \sinh^2(r)$ is at least 0.943, which is greater than the
volume of the Weeks manifold. This is clearly at least as good as the result of [6]
in the case where $M$ is orientable or $\gamma$ has orientation-preserving holonomy, which
is that $\pi l \sinh^2(r) \geq 0.943$ whenever $r \geq 1.464$.

We should note at this point that the results of [6] have already been improved
upon in the orientable case by other authors, and that Theorem 1.1 can be used in
some cases to extend these results to the non-orientable case as well. For example,
in the proof of Theorem 4.9 of [8] it is shown that $\pi l \sinh^2(r)$ (the volume of the maximal
embedded tubular neighborhood of $\gamma$) results in an increasing function of $r$, and
that when $r \geq 0.846$, then $\pi l \sinh^2(r)$ is at least 0.943, which is greater than the
volume of the Weeks manifold. This is clearly at least as good as the result of [6]
in the case where $M$ is orientable or $\gamma$ has orientation-preserving holonomy, which
is that $\pi l \sinh^2(r) \geq 0.943$ whenever $r \geq 1.464$.

Furthermore, while the right-hand side of (3.1) is not a monotonic function of $r$, by combining (3.1) with a previous estimate by Cao-Gehring-Martin [2] (which
appears in [6] as equation 3.3 and which is monotonic) we can obtain a lower bound
on $r$ in terms of $l$ whenever $l \leq 0.113$. Furthermore this combined estimate implies
that whenever $l \leq 0.113$, then $\pi l \sinh^2(r) \geq 6.54^1$! This is a vast improvement on
known results in the case where $M$ is orientable or $\gamma$ has orientation-preserving
holonomy. In particular the above argument extends the result of Theorem 4.9 of
[8] (that the tube volume exceeds 0.94274 whenever $l \leq 0.09$) to the non-orientable
case.

Thus we obtain the following result, which can be thought of as a corollary of
Theorem 1.1, Theorem 4.9 of [8], and the results of [6]:

**Corollary 3.3.** If $W$ is a maximal tube of radius $r$ about a geodesic $\gamma$ in the
complete hyperbolic 3-manifold $M$ and either length($\gamma$) $\leq 0.09$ or $r \geq 1.332$, then
volume($W$) $\geq 0.943$.

### 4. Remarks

To the best of my knowledge, the results in Corollary 3.3 are the best known
for non-orientable manifolds; however, they are not the best results known in the
orientable case. Ian Agol has been able to show in [1] that the volume is at least
0.943 if length($\gamma$) $\leq 0.1036$, while Hodgson and Kerckhoff have shown in [4] that
the volume is at least 1.701 if the length(\(\gamma\)) \(\leq 0.162\). However, it is not immediately clear in either case if the techniques used can be extended to non-orientable manifolds. This is clearly an area deserving of future investigation.

It is also unknown at this time whether or not there exist a group \(G\) and a line \(l\) which admit a bad triple at all. There is, however, a kind of “analogue-example” in the world of cusped hyperbolic manifolds. We can define a bad triple of horoballs in a manner analogous to the previously defined bad triple of lines, in the obvious fashion. Then the fundamental group of the Geiseking manifold, together with the pre-image of the Geiseking manifold’s rigid cusp, provides an example of a bad triple of horoballs. The fact that the cusp in the Geiseking manifold is rigid (i.e., is a Klein bottle cusp) is a necessary condition; the analogue of Theorem 1.1 for bad triples of horoballs is also true and has a very similar proof to the one given here for lines.

This suggests that the lower bound for the volume of cusped orientable hyperbolic 3-manifolds found in \(\mathbb{R}\) might be extended to non-orientable cusped manifolds, provided that the cusps in the manifolds are all orientable (i.e., no Klein bottle cusps). Indeed, a quick search of the SnapPea census (\([9]\)) shows that the smallest known non-orientable cusped hyperbolic manifold with no Klein bottle cusps is the manifold m131. This manifold has the same volume as a regular ideal hyperbolic octahedron, approximately 3.664, which is more than half again as large as the smallest orientable cusped manifolds, and more than three times as large as the Geiseking manifold. Further investigation in this area will hopefully shed some light on the minimum-volume problem for non-orientable manifolds.

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