

## A CHARACTERIZATION OF THE PROJECTIVE LINE

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ABSTRACT. Let  $X$  be a set (with at least three different points) and let  $G$  be a group of bijections of  $X$ . If the action of  $G$  on  $X$  satisfies three natural conditions, then  $X$  admits a canonical structure of a projective line over a commutative field, such that  $G$  is the group of all projective transformations of  $X$ .

### 1. INTRODUCTION

Given a commutative field  $K$ , let us recall that the projective line  $\mathbb{P}_1$  over  $K$  is defined to be the set of all 1-dimensional subspaces of some 2-dimensional vector space  $V$  over  $K$ . A projective transformation is a bijection of  $\mathbb{P}_1$  induced by some linear automorphism of  $V$ .

The purpose of this note is to prove the following group-theoretic characterization of the projective line.

**Theorem 1.1.** *Let  $G \times X \rightarrow X$  be an action of a group  $G$  on a set  $X$  (with at least three different points) satisfying the following conditions:*

- A<sub>1</sub> *Given three different points  $x_1, x_2, x_3 \in X$  and another three different points  $y_1, y_2, y_3 \in X$ , there exists a unique transformation  $g \in G$  such that  $g(x_i) = y_i$  for any  $i = 1, 2, 3$ .*
- A<sub>2</sub> *Given three different points  $p, x, y \in X$ , there exists a unique transformation  $g \in G$  such that  $g(x) = y$  and  $p$  is the unique fixed point of  $g$ .*
- B *Let  $g \in G$ . If there exist two different points  $x, y \in X$  such that  $g(x) = y$  and  $g(y) = x$ , then  $g$  is an involution:  $g^2 = Id$ .*

*Then  $X$  admits a natural structure of a projective line over a commutative field  $K$ , such that  $G$  is the group of all projective transformations of  $X$ .*

Axiom A<sub>1</sub> implies that any transformation  $g \in G$  with three fixed points is the identity map. We may think of axiom A<sub>2</sub> as a degenerate case of A<sub>1</sub> imagining  $p$  as a double-counted point.

As a preliminary step in the proof of Theorem 1.1, we give a similar characterization of the affine line.

In the case of dimension  $n > 1$ , there exist group-theoretic definitions of the affine and projective geometries in terms of the group of dilatations (see [2] and [3]).

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## 2. CHARACTERIZATION OF THE AFFINE LINE

As is well known, there exist two elementary definitions for the affine space: the synthetic definition and the algebraic one. For dimension 1, only the algebraic definition makes sense. Let us recall it.

Let  $K$  be a commutative field. An *affine space* is a set  $\mathbb{A}$  endowed with an action  $\mu: V \times \mathbb{A} \rightarrow \mathbb{A}$  of a  $K$ -vector space  $V$  on  $\mathbb{A}$ . It is assumed that the action satisfies the following property: Given  $p, q \in \mathbb{A}$ , there exists a unique vector  $v \in V$  such that  $q = p + v$  (we denote  $p + v = \mu(v, p)$ ). The *dimension* of  $\mathbb{A}$  is, by definition, the dimension of the vector space  $V$ .

A bijection  $\varphi: \mathbb{A} \rightarrow \mathbb{A}$  is said to be an *affine transformation* if there exists a (necessarily unique)  $K$ -linear automorphism  $\phi: V \rightarrow V$  such that

$$\varphi(p + v) = \varphi(p) + \phi(v)$$

for any  $p \in \mathbb{A}$  and  $v \in V$ .

Of course, an *affine line*  $\mathbb{A}$  is an affine space of dimension 1. Once we fix a point  $p_0$  (the *origin*) in the affine line  $\mathbb{A}$  and a vector  $v \neq 0$  in  $V$ , we may write any point  $p \in \mathbb{A}$  in the form  $p = p_0 + xv$  for some scalar  $x \in K$ ; the scalar  $x$  is said to be the affine coordinate of  $p$ . In terms of affine coordinates, any affine transformation  $\varphi: \mathbb{A} \rightarrow \mathbb{A}$  may be written in the form

$$\varphi(x) = ax + b \quad \text{for some } a, b \in K.$$

**Theorem 2.1.** *Let  $G \times X \rightarrow X$  be an action of a group  $G$  on a set  $X$  satisfying the following conditions:*

- $C_1$  *Given two different points  $x_1, x_2 \in X$  and another two different points  $y_1, y_2 \in X$ , there exists a unique transformation  $g \in G$  such that  $g(x_i) = y_i$  for any  $i = 1, 2$ .*
- $C_2$  *Given two different points  $x, y \in X$ , there exists a unique transformation  $g \in G$ , without fixed points, such that  $g(x) = y$ .*
- $D$  *For each point  $x \in X$ , the isotropy subgroup  $H_x$  is commutative.*

*Then  $X$  admits a natural structure of an affine line over a commutative field  $K$  such that  $G$  is the group of all affine transformations of  $X$ .*

The proof of this theorem is a modification of the introduction of coordinates in the affine space used by Artin [1].

Under the hypotheses of the above theorem, let us introduce some terminology. A transformation  $\tau \in G$  is said to be a translation if it has no fixed points or it is the identity. Given points  $x, y \in X$ , axiom  $C_2$  states that there exists a unique translation  $\tau \in G$  such that  $\tau(x) = y$ .

A transformation  $\sigma \in G$  with a fixed point  $p_0$  is said to be a homothety with centre  $p_0$ . Axiom  $D$  states that the subgroup of all homotheties with centre  $p_0$  is commutative.

**Lemma 2.2.** *The set  $T$  of all translations is a normal subgroup of  $G$ .*

*Proof.*  $T$  is a subgroup: Since the identity is a translation and the inverse of a translation is clearly a translation, we only need to show that the composition of two translations  $\tau_1\tau_2$  is a translation, i.e.,  $\tau_1\tau_2$  has no fixed points or it is the identity. If  $x$  is a fixed point of  $\tau_1\tau_2$ , then  $\tau_2(x) = \tau_1^{-1}(x)$ . By the uniqueness condition of  $C_2$ , we obtain that  $\tau_2 = \tau_1^{-1}$ ; hence  $\tau_1\tau_2 = Id$ .

It is immediate to check that  $T$  is normal in  $G$ . □

**Lemma 2.3.** *Let  $\tau_1, \tau_2$  be translations different from  $Id$ . Given a point  $p_0 \in X$ , there exists a unique homothety  $\sigma$  with centre  $p_0$  such that  $\sigma\tau_1\sigma^{-1} = \tau_2$ .*

*Proof.* Let  $p_1 = \tau_1(p_0)$  and  $p_2 = \tau_2(p_0)$ . By  $C_1$ , there exists a unique homothety  $\sigma$  with centre  $p_0$  such that  $\sigma(p_1) = p_2$ . Both translations  $\sigma\tau_1\sigma^{-1}$  and  $\tau_2$  map  $p_0$  to  $p_2$ ; hence they coincide:  $\sigma\tau_1\sigma^{-1} = \tau_2$ . Conversely, if  $\sigma$  is a homothety with centre  $p_0$  such that  $\sigma\tau_1\sigma^{-1} = \tau_2$ , then  $(\sigma\tau_1\sigma^{-1})(p_0) = \tau_2(p_0)$ ; hence  $\sigma(p_1) = p_2$ .  $\square$

**Proposition 2.4.** *The group  $T$  of all translations is commutative.*

*Proof.* Let  $\tau_1, \tau_2$  be translations different from  $Id$ . By Lemma 2.3, there exists a homothety  $\sigma$  such that

$$\sigma(\tau_2\tau_1\tau_2^{-1})\sigma^{-1} = \tau_1.$$

If  $\sigma\tau_2$  has some fixed point (i.e., it is a homothety), then the uniqueness part of Lemma 2.3 used in the equality

$$(\sigma\tau_2)\tau_1(\tau_2^{-1}\sigma^{-1}) = \tau_1$$

implies that  $\sigma\tau_2 = Id$ ; hence  $\sigma = \tau_2^{-1}$  has no fixed point, contradicting that  $\sigma$  is a homothety. Therefore,  $\sigma\tau_2$  has no fixed point (i.e., it is a translation); hence  $\sigma$  is a translation. Since  $\sigma$  is a homothety, we conclude that  $\sigma = Id$  and then

$$\tau_2\tau_1\tau_2^{-1} = \tau_1$$

or, equivalently,  $\tau_2\tau_1 = \tau_1\tau_2$ .  $\square$

Since  $T$  is a normal subgroup of  $G$ , we may consider the action of  $G$  on  $T$  by conjugation. Let us consider the following ring with unity:

$$K := \text{End}_G(T).$$

Each element  $\lambda \in K$  is an endomorphism of  $T$  which commutes with the action of  $G$ . The product operation in  $K$  is the composition of endomorphisms. Let us show that  $K$  is a field.

**Proposition 2.5.** *Let  $\lambda: T \rightarrow T$  be an endomorphism which commutes with the action of  $G$ . Then  $\lambda$  is the zero map or it is a bijection.*

*As a consequence, it follows that  $K$  is a field.*

*Proof.* By Lemma 2.3,  $G$  acts transitively on  $T - \{0\}$  (additive notation: 0 represents the identity translation). Since  $\lambda$  commutes with the action of  $G$ , we conclude that  $\ker \lambda = 0$  or  $\ker \lambda = T$ , and  $\text{Im } \lambda = T$  or  $\text{Im } \lambda = 0$ .  $\square$

Now we shall show that the product in  $K$  is commutative.

**Lemma 2.6.** *Fix a point  $p_0 \in X$ . Given a transformation  $g \in G$ , there exists a unique translation  $\tau$  and a unique homothety  $\sigma$  with centre  $p_0$  such that  $g = \sigma\tau$ .*

*Proof.* Let  $p_1 \in X$  such that  $g(p_1) = p_0$  and let  $\tau$  be the unique translation such that  $\tau(p_1) = p_0$ . Then  $\sigma := g\tau^{-1}$  is the desired homothety with centre  $p_0$ .  $\square$

Let  $H$  be the subgroup of  $G$  of all homotheties with centre  $p_0$ .  $H$  is a commutative subgroup by axiom D. The above lemma implies that the natural morphism

$$H \longrightarrow G/T$$

is an isomorphism. Since the action by conjugation  $G \rightarrow \text{End}(T)$  factors through the quotient  $G/T = H$  and  $H$  is commutative, we conclude that any element  $g \in G$  defines, acting by conjugation on  $T$ , a scalar  $\lambda_g \in K = \text{End}_G(T)$ .

**Proposition 2.7.** *The field  $K = \text{End}_G(T)$  is commutative, and  $T$  is a  $K$ -vector space of dimension 1.*

*Proof.* Given two non-zero translations  $\tau_1, \tau_2 \in T$ , there exists a homothety  $\sigma \in H$  such that  $\sigma\tau_1\sigma^{-1} = \tau_2$  (see Lemma 2.3); hence  $\lambda_\sigma \cdot \tau_1 = \tau_2$ . This fact implies that  $T$  has dimension 1 and that the morphism  $H \rightarrow K^*$ ,  $\sigma \mapsto \lambda_\sigma$ , is surjective. Since  $H$  is commutative (axiom D), we conclude that  $K$  is a commutative field.  $\square$

*Proof of Theorem 2.1.* Since  $T$  is a  $K$ -vector space of dimension 1,  $X$  inherits the structure of an affine line via the action of  $T$  on  $X$ .

Given  $\sigma \in G$ , for any translation  $\tau \in T$  we have  $\sigma\tau\sigma^{-1} = \lambda_\sigma \cdot \tau$ ; hence  $\sigma\tau = (\lambda_\sigma \cdot \tau)\sigma$ . Rewriting this last equality using the additive notation  $\tau(p) = p + v$ , we obtain that

$$\sigma(p + v) = \sigma(p) + \lambda_\sigma v;$$

hence  $\sigma$  is an affine transformation of  $X$ . Conversely, using axiom C<sub>1</sub>, it is immediate that any affine transformation of  $X$  coincides with some element of  $G$ .  $\square$

### 3. PROOF OF THE MAIN RESULT

With the hypotheses of Theorem 1.1, let us fix a point  $p_\infty \in X$ . Let

$$X' = X - \{p_\infty\}, \quad G' = \{g \in G : g(p_\infty) = p_\infty\}.$$

**Lemma 3.1.** *The action of  $G'$  on  $X'$  satisfies all the conditions of Theorem 2.1.*

*Proof.* It is clear that axioms A<sub>1</sub> and A<sub>2</sub> imply C<sub>1</sub> and C<sub>2</sub>.

To prove condition D, given any point  $p_0 \in X'$ , let us consider a transformation  $g \in G$  such that  $g(p_0) = p_\infty$  and  $g(p_\infty) = p_0$ . By axiom B, we have that  $g^2 = Id$ . Let

$$H_{p_0} = \{\sigma \in G' : \sigma(p_0) = p_0\} = \{\sigma \in G : \sigma(p_0) = p_0 \text{ and } \sigma(p_\infty) = p_\infty\}$$

be the isotropy subgroup (in  $G'$ ) of  $p_0$ . By conjugation with  $g$ , we obtain an automorphism of groups

$$\varphi: H_{p_0} \longrightarrow H_{p_0}, \quad \sigma \mapsto g\sigma g.$$

For any  $\sigma \in H_{p_0}$ , we have that  $g(\sigma(p_0)) = p_\infty$  and  $g(\sigma(p_\infty)) = p_0$ ; hence  $g\sigma g = Id$  by axiom B. This implies that  $\varphi(\sigma) = \sigma^{-1}$  and, since  $\varphi$  is an automorphism of groups, we conclude that the isotropy group  $H_{p_0}$  is commutative.  $\square$

*Proof of Theorem 1.1.* Applying Theorem 2.1 to the action  $G' \times X' \rightarrow X'$ , we obtain that  $X'$  is an affine line over a commutative field  $K$ , and  $G'$  is the group of all affine transformations of  $X'$ . Taking an affine coordinate on  $X'$ , we may write  $X = K \cup \{\infty\}$ , i.e.,  $X$  is a projective line over the field  $K$ .

Now we have to prove that  $G$  is the group of all projective transformations of  $X$ .

Let  $g \in G$  be the unique transformation such that  $g(0) = \infty$ ,  $g(1) = 1$  and  $g(\infty) = 0$  (note that  $g$  is an involution by axiom B). Now we shall prove that  $g(x) = x^{-1}$  for any  $x \in X$ . Let  $\sigma_\lambda \in G'$  be the homothety with centre 0 and ratio  $\lambda$ , i.e.,  $\sigma_\lambda(x) = \lambda x$ . The same argument used in the proof of Lemma 3.1 gives that  $g\sigma_\lambda g\sigma_\lambda = Id$ ; hence  $g\sigma_\lambda g = \sigma_{\lambda^{-1}}$ . On the other hand,  $g\sigma_\lambda g = \sigma_{g(\lambda)}$  because  $g\sigma_\lambda g$  is a homothety with centre 0 and  $(g\sigma_\lambda g)(1) = g(\lambda)$ . In conclusion, it follows that  $g(\lambda) = \lambda^{-1}$ .

Since  $G$  contains the transformation  $x \mapsto ax + b$  (because  $G$  contains the group  $G'$  of the affine transformations) and it also contains the transformation  $g : x \mapsto x^{-1}$ , we conclude that  $G$  contains all the projective transformations  $x \mapsto \frac{ax+b}{cx+d}$ . Conversely, using the uniqueness part of axiom  $A_1$ , it is immediate that any element of  $G$  is a projective transformation.  $\square$

#### FURTHER COMMENTS

The projective line  $\mathbb{P}_1$  over a commutative field  $K$  is usually defined as the set of all 1-dimensional subspaces of some 2-dimensional vector space  $V$  over  $K$ . This definition is not satisfactory since it presents the projective line as a set without any structure. This makes obscure the notion of an automorphism of  $\mathbb{P}_1$ . An automorphism of  $\mathbb{P}_1$  should be a bijection preserving the “structure”, but which structure? If  $\mathbb{P}_1$  were a mere set, then its automorphisms should be bijections of this set; then we ought to define the notion of projective transformation as an arbitrary bijection!

Of course, the question about the structure of  $\mathbb{P}_1$  is illuminated by modern Algebraic Geometry:  $\mathbb{P}_1$  has the structure of a scheme. Scheme automorphisms of  $\mathbb{P}_1$  are just projective transformations in the sense of Staudt (= bijections of  $\mathbb{P}_1$  induced by semilinear automorphisms of  $V$ ), while ordinary projective transformations are automorphisms of  $\mathbb{P}_1$  as a  $K$ -scheme, i.e., scheme automorphisms of  $\mathbb{P}_1$  which define the identity on  $H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}) = K$ . A conceptually rigorous foundation of Projective Geometry may be considered as a first great result of Grothendieck’s theory of schemes.

Our characterization allows us to give an alternative “structural” definition of the projective line, in the spirit of Klein’s Erlangen Program: A projective line is a pair  $(X, G)$  where  $X$  is a set and  $G$  is a subgroup of  $\text{Bij}(X)$  satisfying the conditions of Theorem 1.1. The automorphisms of this structure are the bijections  $\varphi : X \rightarrow X$  preserving the subgroup  $G$ , i.e.,  $\varphi G \varphi^{-1} = G$ . These automorphisms correspond to Staudt transformations in the conventional definition of  $\mathbb{P}_1$ , while the elements of  $G$  correspond to ordinary projective transformations.

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