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ON THE ABSENCE OF UNIFORM DENOMINATORS IN HILBERT'S 17TH PROBLEM

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ABSTRACT. Hilbert showed that for most (n, m) there exist positive semidefinite forms $p(x_1, \dots, x_n)$ of degree m which cannot be written as a sum of squares of forms. His 17th problem asked whether, in this case, there exists a form h so that h^2p is a sum of squares of forms; that is, p is a sum of squares of rational functions with denominator h. We show that, for every such (n, m) there does not exist a single form h which serves in this way as a denominator for *every* positive semidefinite $p(x_1, \dots, x_n)$ of degree m.

1. Introduction

Let $H_d(\mathbb{R}^n)$ denote the set of real homogeneous forms of degree d in n variables ("n-ary d-ics"). By identifying $p \in H_d(\mathbb{R}^n)$ with the $N = \binom{n+d-1}{n-1}$-tuple of its coefficients, we see that $H_d(\mathbb{R}^n) \approx \mathbb{R}^N$. Suppose m is an even integer. A form $p \in H_m(\mathbb{R}^n)$ is called *positive semidefinite* or *psd* if $p(x_1, \dots, x_n) \ge 0$ for all $(x_1, \dots, x_n) \in \mathbb{R}^n$. Following [1], we denote the set of psd forms in $H_m(\mathbb{R}^n)$ by $P_{n,m}$. Since $P_{n,m}$ is closed under addition and closed under multiplication by positive scalars, it is a convex cone. In fact, $P_{n,m}$ is a *closed* convex cone: if $p_n \to p$ coefficient-wise, and each p_n is psd, then so is p. A psd form is called *positive definite* or *pd* if $p(x_1, \dots, x_n) = 0$ implies $x_j = 0$ for $1 \le j \le n$. The pd n-ary m-ics are the interior of the cone $P_{n,m}$.

A form $p \in H_m(\mathbb{R}^n)$ is called *a sum of squares* or *sos* if it can be written as a sum of squares of polynomials; that is, $p = \sum_k h_k^2$. It is easy to show in this case that each $h_k \in H_{m/2}(\mathbb{R}^n)$. Again following [1], we denote the set of sos forms in $H_m(\mathbb{R}^n)$ by $\Sigma_{n,m}$. Clearly, $\Sigma_{n,m}$ is a convex cone; less obviously, it is a closed cone, a result proved in general by R. M. Robinson [22], although shown for $\Sigma_{3,6}$ by Hilbert [9].

In light of the inclusion $\Sigma_{n,m} \subseteq P_{n,m}$, let $\Delta_{n,m} = P_{n,m} \setminus \Sigma_{n,m}$. It was well known by the late 19th century that $P_{n,m} = \Sigma_{n,m}$ when $m = 2$ or $n = 2$. In 1888, Hilbert proved [9] that $\Sigma_{3,4} = P_{3,4}$; more specifically, every $p \in P_{3,4}$ can be written as the sum of three squares of quadratic forms. (An elementary proof, with "five"

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squares is in [2, pp. 16-17]; for modern expositions of Hilbert's proof, see [26] and [23].) Hilbert also proved in [9] that the preceding are the *only* cases for which $\Delta_{n,m} = \emptyset$. That is, if $n \geq 3$ and $m \geq 6$ or $n \geq 4$ and $m \geq 4$, then there exist psd n-ary m-ics that are not sos.

In 1893, Hilbert [10] generalized his three-square result for $P_{3,4}$ to ternary forms of higher degree. Suppose $p \in P_{3,m}$ with $m \geq 6$. Then there exist $p_1 \in P_{3,m-4}$ and $h_{1k} \in H_{m-2}(\mathbb{R}^3)$, $1 \leq k \leq 3$, so that

$$p_1 p = h_{11}^2 + h_{12}^2 + h_{13}^2.$$

(Hilbert's proof seems to be non-constructive and lacks a modern exposition. In the very recent paper [11], de Klerk and Pasechnik discuss the implementation of an algorithm to find p_1 so that $p_1 p$ is sos, though not necessarily as a sum of *three* squares. This paper uses Hilbert's result without giving an independent proof.)

If $m = 6$ or 8, then p_1 is a sum of three squares of forms, and hence (as Landau later noted [12]), the four-square identity implies that $p_1^2 p = p_1(p_1 p)$ is the sum of four squares of forms. If $m \geq 10$, then the argument can be applied to p_1: there exists $p_2 \in P_{3,m-8}$ with $p_2 p_1 = h_{21}^2 + h_{22}^2 + h_{23}^2$. Thus, if $m = 10$ or 12 (so that $P_{3,m-8} = \Sigma_{3,m-8}$), then $(p_1 p_2)^2 p = p_2(p_2 p_1)(p_1 p)$ is the sum of four squares of forms. An easy induction shows that there exists $q \in H_t(\mathbb{R}^3)$ with $t = \lfloor \frac{(m-2)^2}{8} \rfloor$ so that $q^2 p$ is the sum of four squares of forms.

Hilbert's 17th Problem asked whether this generalizes to $n > 3$ variables; that is, if $p \in P_{n,m}$, must there exist some form q so that $q^2 p$ is sos? Artin proved that there must be, in a way that gives no information about q. Much more on the history of this subject can be found in the survey paper [20].

This discussion leads to two closely related questions. Suppose $p \in P_{n,m}$. Can we *find* a form h such that hp is sos? Can we *find* a form q so that $q^2 p$ is sos? If we've answered the second, we've answered the first. Conversely, if $p \neq 0$ is psd and hp is sos, then h is psd. But it needn't be sos; indeed, a trivial answer to the first question is to take $h = p$. Stengle proved [25] that if $p(x, y, z) = x^3 z^3 + (y^2 z - x^3 - z^2 x)^2$, then $p^{2s+1} \in \Delta_{3,6(2s+1)}$ for every integer s. That is, $p^{2s-1} \cdot p$ is sos, but $p^{2s} \cdot p$ is not. Choi and Lam showed [1] that for $S \in \Delta_{3,6}$ (see (3) below), the product $S(x, y, z)S(x, z, y)$ is actually sos.

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2. What is known about the denominator

The first concrete result about a denominator in Hilbert's 17th Problem was found by Pólya [17]. He showed that if $f \in H_d(\mathbb{R}^n)$ is positive on the unit simplex $\{(x_1, \dots, x_n) \mid x_j \geq 0, \sum x_j = 1\}$, then for sufficiently large N, $(\sum_j x_j)^N f$ has positive coefficients. Replacing each x_j by x_j^2, we see that if $p \in H_{2d}(\mathbb{R}^n)$ is an even positive definite form, then $(\sum_j x_j^2)^N p$ is a sum of even monomials with positive coefficients, and so, as it stands, is a sum of squares of monomials. Taking even N, we see that $q = (\sum_j x_j^2)^{N/2}$ is a denominator for p. Habicht [7] generalized Pólya's proof to give an alternate solution to Hilbert's 17th Problem for pd forms; however, h is not readily constructible and in general is no longer a power of $\sum x_j^2$. Except for one example, Pólya did not attempt to determine an explicit value of N. A good exposition of the theorems of Pólya and Habicht can be found in [8].

For positive definite $p \in P_{n,m}$, let

$$\epsilon(p) := \frac{\inf\{p(u) : u \in S^{n-1}\}}{\sup\{p(u) : u \in S^{n-1}\}}$$

measure how "close" p is to having a zero. The author [19] showed that if

$$N \geq \frac{nm(m-1)}{(4\log 2)\epsilon(p)} - \frac{n+m}{2},$$

then $(\sum_j x_j^2)^N p$ is a sum of $(m+2N)$-th powers of linear forms, and so is sos. A similar lower bound has been shown to apply in Pólya's Theorem; the bound goes to infinity as p approaches the boundary of $P_{n,m}$. (See papers by de Loera and Santos [13] and by Powers and the author [18].)

The restriction to positive definite forms is necessary. There exist psd forms p in $n \geq 4$ variables so that, if $h^2 p$ is sos, then h must have a specified zero. The existence of these unavoidable singularities, or so-called "bad points", insures that $(\sum x_j^2)^r p$ can never be a sum of squares of forms for *any* r. Habicht's Theorem implies that no positive definite form can have a bad point. Bad points were first noted by Straus and have been extensively studied by Delzell; see, e.g. [5, 6].

Little specific is known about the degree of the denominator in more than 3 variables. A. Robinson proved [21, p. 268] that there exists $d(n,m)$ so that $p \in P_{n,m}$ implies that there exists $q \in H_{d(n,m)}(\mathbb{R}^n)$ so that $q^2 p$ is sos. Moreover, $d(n,m)$ is a general recursive function of n and m. Various improvements have been made in the description of d, but no "practical" bounds are known. See [4, §§5.4–5.6, 5.11–5.13, 9.1–9.7] for a detailed survey. The existence of $d(n,m)$ is also a special case of a quantitative version of the Positivstellensatz constructed by Lombardi and Roy [14].

3. Recent results and a new theorem

Scheiderer has shown in very recent work [24] that for $p \in P_{3,m}$, there exists $N = N(p)$ so that $(x^2+y^2+z^2)^N p(x,y,z)$ is sos; indeed, $x^2+y^2+z^2$ can be replaced by any positive definite form. This is a strong refutation to the existence of bad points for ternary forms.

Suppose (n,m) is such that $\Delta_{n,m} \neq \emptyset$. Theorem 1 below states that there is no *single* form h so that, if $p \in P_{n,m}$, then hp is sos. Corollary 2 says that there is not even a *finite* set of forms $\mathcal{H}$ so that, if $p \in P_{n,m}$, then there exists $h \in \mathcal{H}$ so that hp is sos. In particular, there does not exist a finite set of denominators which apply to all of $P_{n,m}$. This result implies that $N(p)$ in Scheiderer's theorem is not bounded as p ranges over $P_{3,m}$. It also implies that the denominators in the Lombardi-Roy theorem cannot be chosen from a finite, predetermined set.

The proof of the theorem is elementary and relies on a few simple observations. If $p \neq 0$ is psd and hp is sos, then h is psd. As previously noted, $\Sigma_{n,m}$ is a closed cone for all (n,m). This cone is invariant under the action of taking invertible linear changes of variable. Thus, if h' is derived from h by such a linear change, and if hp is sos for every $p \in P_{n,m}$, then so is $h'p$. Suppose ℓ is a linear form, $p = \sum_k g_k^2$ is sos, and $\ell \mid p$. Then $\ell^2 \mid p$ and $\ell \mid g_k$ for each k, and by induction, $\ell^{2s} \mid p \implies \ell^s \mid g_k$. Thus, we can "peel off" squares of linear factors from any sos form; this is a common practice, dating back at least to [22, p. 267]. We use this observation in the contrapositive: if $p \in \Delta_{n,m}$, then $\ell^{2s} p \in \Delta_{n,m+2s}$.

Theorem 1. *Suppose $\Delta_{n,m} \neq \emptyset$. Then there does not exist a non-zero form h so that if $p \in P_{n,m}$, then hp is sos.*

Proof. Suppose to the contrary that such a form h exists. Since $h \neq 0$, there exists a point $a \in \mathbb{R}^n$ so that $h(a) \neq 0$. By making an invertible linear change of variables, we can take $a = (1, 0, \dots, 0)$. Thus, we may assume without loss of generality that $h(x_1, 0, \dots, 0) = \alpha x_1^d$, where $\alpha > 0$ and d is even. In the sequel, we distinguish x_1 from the other variables.

Choose $p \in P_{n,m} \setminus \Sigma_{n,m}$. Then

$$h(x_1, x_2, \dots, x_n) p(x_1, r x_2, \dots, r x_n)$$

is sos for every $r \in \mathbb{N}$. By making the change of variables $x_i \to x_i/r$ for $i \geq 2$, we see that

$$h(x_1, r^{-1}x_2, \dots, r^{-1}x_n) p(x_1, x_2, \dots, x_n)$$

is also sos. Since

$$\lim_{r\to\infty} h(x_1, r^{-1}x_2, \dots, r^{-1}x_n) = h(x_1, 0, \dots, 0) = \alpha x_1^d,$$

and since $\Sigma_{n,m+d}$ is closed, it follows that

$$\lim_{r\to\infty} h(x_1, r^{-1}x_2, \dots, r^{-1}x_n) p(x_1, x_2, \dots, x_n) = \alpha x_1^d p(x_1, \dots, x_n)$$

is sos. Thus p is sos, a contradiction. □

The following elegant proof is due to Claus Scheiderer and is included with his permission; it supersedes the proof in an earlier version of this manuscript.

Corollary 2. *Suppose $\Delta_{n,m} \neq \emptyset$. Then there does not exist a finite set of non-zero forms $\mathcal{H} = \{h_1, ..., h_N\}$ with the property that, if $p \in P_{n,m}$, then $h_k p$ is sos for some $h_k \in \mathcal{H}$.*

Proof. Suppose $\mathcal{H}$ exists. For each k, there exists non-zero $p \in \Delta_{n,m}$ so that $h_k p$ is sos. (Otherwise, we may delete h_k harmlessly from $\mathcal{H}$.) Thus, each h_k is psd, and there exists a form q_k so that $q_k^2 h_k$ is sos. Define $h = \prod_k q_k^2 h_k$. We now show that for every $p \in P_{n,m}$, hp is sos: this contradicts Theorem 1 and proves the corollary. By hypothesis, there exists $h_j \in \mathcal{H}$ so that $h_j p$ is sos. Thus,

$$hp = \left(\prod_{k \neq j} q_k^2 h_k \right) \cdot q_j^2 \cdot h_j p$$

is a product of sos factors, and so is sos. □

Finally, we know by Hilbert's theorem that for $p \in P_{3,6}$, there exists quadratic h so that $hp \in \Sigma_{3,8}$. The three simplest forms in $\Delta_{3,6}$ are

(1) $$M(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2, \quad \text{due to Motzkin [15];}$$

R. M. Robinson's [22] simplification of Hilbert's construction

(2) $$R(x,y,z) = x^6 + y^6 + z^6 - (x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + y^4z^2 + y^2z^4) + 3x^2y^2z^2;$$

and

(3) $$S(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2, \quad \text{due to Choi and Lam [1, 2].}$$

It is not too difficult to consider qM, qR, qS for $q(x,y,z) = a^2x^2 + b^2y^2 + c^2z^2$ and determine whether these are sos using the algorithm of [3] directly or its implementation in, e.g., [16].

Interestingly enough, these conditions are the same in each case: the forms are sos if and only if

$$2(a^2b^2 + a^2c^2 + b^2c^2) \geq a^4 + b^4 + c^4.$$

This expression factors rather neatly into

$$(a + b + c)(a + b - c)(b + c - a)(c + a - b) \geq 0,$$

so if $a \geq b \geq c \geq 0$ without loss of generality, the only non-trivial condition is that $b + c \geq a$; that is, there is a (possibly degenerate) triangle with sides a, b, c. (Robinson [22, p. 273] has a superficially similar condition, but note that his multiplier is $ax^2 + by^2 + cz^2$.)

If we scale variables as in the proof of Theorem 1, it follows from this computation that the three forms

$$(x^2 + y^2 + z^2)M(x, \lambda y, \lambda z),\ (x^2 + y^2 + z^2)R(x, \lambda y, \lambda z),\ (x^2 + y^2 + z^2)S(x, \lambda y, \lambda z)$$

are sos if and only if $0 \leq |\lambda| \leq 2$.

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