REAL \( k \)-FLATS TANGENT TO QUADRICS IN \( \mathbb{R}^n \)

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Abstract. Let \( d_{k,n} \) and \( \#_{k,n} \) denote the dimension and the degree of the Grassmannian \( G_{k,n} \), respectively. For each \( 1 \leq k \leq n-2 \) there are \( 2^{d_{k,n}} \cdot \#_{k,n} \) (a priori complex) \( k \)-planes in \( \mathbb{P}^n \) tangent to \( d_{k,n} \) general quadratic hypersurfaces in \( \mathbb{P}^n \). We show that this class of enumerative problems is fully real, i.e., for \( 1 \leq k \leq n-2 \) there exists a configuration of \( d_{k,n} \) real quadrics in (affine) real space \( \mathbb{R}^n \) so that all the mutually tangent \( k \)-flats are real.

Introduction

Understanding the real solutions of a system of polynomial equations is a fundamental problem in mathematics (see, e.g., [13] for some recent lines of research and applications). However, as pointed out in [3, p. 55], even for problem classes with a finite number of complex solutions (enumerative problems), the question of how many solutions can be real is still widely open. A class of enumerative problems is called fully real if there are general real instances for which all the (a priori complex) solutions are real.

One of us (Sottile) began a systematic study of this question in the special Schubert calculus [9, 10], a class of enumerative problems from classical algebraic geometry. This special Schubert calculus asks for linear subspaces of a fixed dimension meeting some given (general) linear subspaces (whose dimensions and number ensure a finite number of solutions) in \( n \)-dimensional complex projective space \( \mathbb{P}^n \). For any given dimensions of the subspaces, this problem is fully real, i.e., there exist real linear subspaces for which each of the a priori complex solutions is real. In particular, for \( 1 \leq k \leq n-2 \) there are \( d_{k,n} := (k+1)(n-k) \) real \( (n-k-1) \)-planes \( U_1, \ldots, U_{d_{k,n}} \) in \( \mathbb{P}^n \) with

\[
\#_{k,n} := \frac{1! \cdots k!((k+1)(n-k))!}{(n-k)!((n-k+1)! \cdots n!)}
\]

real \( k \)-planes meeting \( U_1, \ldots, U_{d_{k,n}} \). Here, \( d_{k,n} \) and \( \#_{k,n} \) are the dimension and the degree of the Grassmannian \( G_{k,n} \), respectively (see [9, 10]). These were the first results showing that a large class of non-trivial enumerative problems is fully real.
Recently, Vakil \cite{14} has shown that any Schubert problem on a Grassmannian is fully real.

We continue this line of research by considering $k$-flats tangent to quadratic hypersurfaces (hereafter \emph{quadrics}). This is also motivated by recent investigations in computational geometry (see \cite{6, 11, 12}). It was shown in \cite{12} that $2n-2$ general spheres in affine real space $\mathbb{R}^n$ have at most $3 \cdot 2^{n-1}$ common tangent lines in $\mathbb{C}^n$, and that there exist spheres for which all the a priori complex tangent lines are real.

The present paper addresses the following question: What is the maximum number of real $k$-flats simultaneously tangent to $d_{k,n}$ general quadrics in $\mathbb{R}^n$ (respectively in $\mathbb{P}^n_{\mathbb{R}}$)? As this problem may be formulated as the complete intersection of $d_{k,n}$ quadratic equations on the Grassmannian of $k$-planes in $\mathbb{P}^n$, the expected number of complex solutions is the product of the degrees of the equations with the degree of the Grassmannian, i.e., $2^{d_{k,n}} \cdot \#_{k,n}$. We show that the problem is fully real:

\textbf{Theorem 1.} Let $1 \leq k \leq n-2$. Given $d_{k,n}$ general quadrics in $\mathbb{P}^n$ there are $2^{d_{k,n}} \cdot \#_{k,n}$ complex $k$-planes that are simultaneously tangent to all $d_{k,n}$ quadrics. Furthermore, there is a choice of quadrics in $\mathbb{R}^n$ for which all the $k$-flats are real, distinct, and lie in the affine space $\mathbb{R}^n$.

When $k=1$, we have $d_{1,n} = 2(n-1)$ and $\#_{1,n}$ is the Catalan number $\#_{1,n} = \frac{1}{n} \binom{2n-2}{n-1}$. Table 1 exhibits the large discrepancy between the number of lines tangent to spheres and the number of lines tangent to general quadrics. When $n=3$ this discrepancy was accounted for by Aluffi and Fulton \cite{1}.

<table>
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<th>5</th>
<th>6</th>
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<tr>
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In Section \textbf{1} we review some facts on Plücker coordinates of $k$-planes in projective space. In Section \textbf{2} we combine recent results in the real Schubert calculus with classical perturbation arguments adapted to the real numbers to prove Theorem \textbf{1}. Since the proof for general $(k,n)$ is non-constructive, we give a symbolic, constructive proof for the case $(k,n) = (1,3)$ in Section \textbf{3}.

\textbf{1. Preliminaries}

We review the well-known \textit{Plücker coordinates} of $k$-dimensional linear subspaces (hereafter $k$-planes) in complex projective space $\mathbb{P}^n$ (see, e.g., \cite{4}). Let $U$ be a $k$-plane in $\mathbb{P}^n$ which is spanned by the columns of an $(n+1) \times (k+1)$-matrix $L$. For every subset $I \subset \{0, \ldots, n\}$ of size $k+1$ let $p_I$ be the $(k+1) \times (k+1)$-subdeterminant of $L$ given by the rows in $I$ and let $N := \binom{n+1}{k+1} - 1$. Then $p := (p_I)_{I \subset \{0, \ldots, n\}, |I|=k+1} \in \mathbb{P}^{N}$ is the Plücker coordinate of $U$. The set of all $k$-planes in $\mathbb{P}^n$ is called the \textit{Grassmannian} of $k$-planes in $\mathbb{P}^n$ and is denoted by $\mathbb{G}_{k,n}$. If the indices are written as ordered tuples, then the Plücker coordinates are skew-symmetric in the indices. $\mathbb{G}_{k,n}$ is in 1-1-correspondence with the set of vectors in...
\[ P^N \text{ satisfying the Plücker relations, i.e.,} \]
\[ \sum_{l=1}^{k+1} (-1)^l p_{i_1 \ldots i_l i_{k+1}} = 0 \]
for every \( I = \{i_1, \ldots, i_{k+1}\}, J = \{j_1, \ldots, j_k\} \subset \{0, \ldots, n\} \) of strictly ordered index sets (where ‘\(^\ast\)’ over an index means that it is omitted). See, e.g., [4, Theorem VII.6].

By Schubert’s results [7], the dimension of \( G_{k,n} \) is \( d_{k,n} = (k+1)(n-k) \) and its degree is \#_{k,n}.

If an \((n-k-1)\)-plane \( V \) is given as the intersection of the \( k + 1 \) hyperplanes \( \sum_{i=0}^{n} v_i^{(0)} x_i = 0, \ldots, \sum_{i=0}^{n} v_i^{(k)} x_i = 0 \), then the dual Plücker coordinate \( q = (q_l)_{l \subset \{0,\ldots,n\}, |l|=k+1} \in \mathbb{P}^N \) of \( V \) is defined by the \((k+1) \times (k+1)\)-subdeterminants of the matrix with columns \( v^{(0)}, \ldots, v^{(k)} \).

A \( k \)-plane \( U \) intersects an \((n-k-1)\)-plane \( V \) in \( \mathbb{P}^n \) if and only if the dot product of the Plücker coordinate \( p \) of \( U \) and the dual Plücker coordinate \( q \) of \( V \) vanishes, i.e., if and only if
\[ \sum_{l \subset \{0,\ldots,n\}, |l|=k+1} p_r q_l = 0 \]
(see, e.g., [4, Theorem VII.5.1]).

We use Plücker coordinates to characterize the \( k \)-planes tangent to a given quadric in \( \mathbb{P}^n \) (see [11]). We identify a quadric \( x^T Q x = 0 \) in \( \mathbb{P}^n \) with its symmetric \((n+1) \times (n+1)\)-representation matrix \( Q \). Further, for \( r \in \mathbb{N} \) let \( \wedge^r \) denote the \( r \)-th exterior power of matrices
\[ \wedge^r : \mathbb{C}^{m \times n} \rightarrow \mathbb{C}^{(\binom{m}{r}) \times (\binom{n}{r})} \]
(see [11]). The row and column indices of the resulting matrix are subsets of cardinality \( r \) of \( \{1, \ldots, m\} \) and \( \{1, \ldots, n\} \), respectively. For \( I \subset \{1, \ldots, m\} \) with \( |I| = r \) and \( J \subset \{1, \ldots, n\} \) with \( |J| = r \), \((\wedge^r A)_{I,J} \) is the subdeterminant of \( A \) whose rows are indexed by \( I \) and whose columns are indexed by \( J \). If a \( k \)-plane \( U \subset \mathbb{P}^n \) is spanned by the columns of an \((n+1) \times (k+1)\)-matrix \( L \), then the \((\binom{n+1}{k+1}) \times 1\)-matrix \( \wedge^{k+1} L \), considered as a vector in \( \mathbb{P}^N \), is the Plücker coordinate of \( U \).

Recall the following algebraic characterization of tangency: A \( k \)-plane \( U \) is tangent to a quadric \( Q \) if the restriction of the quadratic form to \( U \) is singular (which includes the case \( U \subset Q \)). When the quadric is smooth, this implies that \( U \) is tangent to the quadric in the usual geometric sense.

**Proposition 2** (Proposition 5.5.3 of [11]). A \( k \)-plane \( U \subset \mathbb{P}^n \) is tangent to a quadric \( Q \) if and only if the Plücker coordinate \( p_U \) of \( U \) satisfies
\[ p_U^T (\wedge^{k+1} Q) p_U = 0 \]

A \( k \)-flat in affine real space \( \mathbb{R}^n \) is a \( k \)-dimensional affine subspace in \( \mathbb{R}^n \). Throughout the paper we assume that \( \mathbb{R}^n \) is naturally embedded in the real projective space \( \mathbb{P}^n_\mathbb{R} \) via \((x_1, \ldots, x_n) \mapsto (1, x_1, \ldots, x_n) \in \mathbb{P}^n_\mathbb{R} \).

2. Proof of the main theorem

We first illustrate the essential geometric idea underlying our constructions for \((k,n) = (1,3)\), which is the first nontrivial case. Here, Theorem 1 states that there
exists a configuration of four quadrics in $\mathbb{R}^3$ with 32 distinct real common tangent lines.

By (2), the set of lines meeting four given lines in $\mathbb{P}^3$ is the intersection of four hyperplanes on the Grassmannian $G_{1,3}$, and hence there are at most two or infinitely many common lines meeting $\ell_1, \ldots, \ell_4$. If $e_1$ and $e_2$ are opposite edges in a tetrahedron $\Delta \subset \mathbb{R}^3$, then the lines underlying $e_1$ and $e_2$ are the two common transversals of the four lines underlying the other four edges (see Figure 1).

![Figure 1](image-url)

**Figure 1.** A tetrahedron configuration of four lines in $\mathbb{R}^3$ with two real transversals and a configuration of four quadrics with 32 real tangents.

Consider the lines $\ell_1, \ldots, \ell_4$ as (degenerate) infinite circular cylinders with radius $r = 0$. When the radius is slightly increased, then the cylinders intersect pairwise in the regions (combinatorially) given by the four vertices of $\Delta$, and the common tangents roughly have the direction of $e_1$ or $e_2$. Since the neighborhood of a vertex is divided into four regions by the two cylinders, and since each region contains common tangents, this gives $4 \cdot 4$ tangents close to the direction of $e_1$ and $4 \cdot 4$ tangents close to the direction of $e_2$ (see Figure 1).

For the general case, let $1 \leq k \leq n - 2$. By Section 1, the number of $k$-planes in $\mathbb{P}^n$ simultaneously meeting $d_{k,n}$ general $(n-k-1)$-planes is $\#_{k,n}$. We begin with a configuration of $d_{k,n}$ real $(n-k-1)$-flats $U_1, \ldots, U_{d_{k,n}}$ in $\mathbb{R}^n$ having $\#_{k,n}$ real $(n-k-1)$-flats simultaneously meeting $U_1, \ldots, U_{d_{k,n}}$. We then argue that we can replace each of these $(n-k-1)$-flats by a real quadric such that for each of the $k$-flats, there are $2^{d_{k,n}}$ nearby real $k$-flats tangent to each quadric.

**Proposition 3.** For $1 \leq k \leq n - 2$, there exists a configuration of $d_{k,n}$ real $(n-k-1)$-flats $U_1, \ldots, U_{d_{k,n}}$ in $\mathbb{R}^n$ such that there exist exactly $\#_{k,n}$ real $k$-flats simultaneously meeting $U_1, \ldots, U_{d_{k,n}}$.

**Proof.** The corresponding statement for real projective space $\mathbb{P}^n$ was proven for $k = 1$ in [9, Theorem C] and for $k \geq 2$ in [10]. We deduce the affine counterpart above simply by removing a real hyperplane that contains none of the $(n-k-1)$-flats or any of the transversal $k$-flats.
For $k = 1$, the purely existential statement in [7] and Proposition 3 was improved by Eremenko and Gabrielov [2] who gave the following explicit construction of such a collection of $(n-2)$-flats. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$, $\gamma(s) := (1, s, s^2, \ldots, s^{n-1})^T$ be the moment curve in $\mathbb{R}^n$. For each $s \in \mathbb{R}$, set $U(s)$ to be

$$U(s) := \text{affine span}(\gamma(s), \gamma'(s), \ldots, \gamma^{(n-3)}(s)).$$

Geometrically, $U(s)$ is the $(n-2)$-flat osculating the moment curve at the point $\gamma(s)$. By [2], for any distinct $s_1, \ldots, s_{2n-2} \in \mathbb{R}$, the $(n-2)$-flats $U(s_1), U(s_2), \ldots, U(s_{2n-2})$ have exactly $\#_{1,n} = C_{n-1}$ common real transversals, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the $n$-th Catalan number. For general $k$, it is only known that there exist distinct $s_1, \ldots, s_{dk,n} \in \mathbb{R}$ such that there are $\#_{k,n}$ distinct real $k$-flats meeting the osculating $(n-k-1)$-flats to the moment curve at $s_1, \ldots, s_{dk,n}$ [10]. The conjecture on total reality in [8] §1 and §4 conjectures that any choice of distinct $s_1, \ldots, s_{dk,n} \in \mathbb{R}$ implies the reality of all transversal subspaces.

**Definition.** Suppose that $1 \leq k \leq n-2$, and let $U \subset \mathbb{R}^n$ be a $k$-flat and $r > 0$. The $k$-cylinder $Cy(U, r)$ is the set of points having Euclidean distance $r$ from $U$.

This quadratic hypersurface is smooth in $\mathbb{R}^n$ but its extension to $\mathbb{P}^n$ is singular. A $k'$-flat $V \subset \mathbb{R}^n$ is tangent to $Cy(U, r)$ if and only if its Euclidean distance to $U$ is $r$.

We will use the following basic property of intersection multiplicities [9] p. 1].

**Proposition 4.** Let $\mathcal{A}$ be an algebraic curve in $\mathbb{P}^n$, and let $x$ be a singular point on $\mathcal{A}$. For any hyperplane $H \subset \mathbb{P}^n$ such that $x$ is an isolated point in $\mathcal{A} \cap H$, the intersection multiplicity of $\mathcal{A}$ and $H$ in $x$ is greater than 1.

**Theorem 5.** Let $1 \leq k \leq n-2$, and let $U_1, U_2, \ldots, U_{dk,n}$ be $(n-k-1)$-flats in $\mathbb{R}^n$ having exactly $\#_{k,n}$ common transversal $k$-flats, all real. For each $i = 0, 1, \ldots, dk,n$, there exist $r_1, \ldots, r_i > 0$ such that there are exactly $2^i \cdot \#_{k,n}$ distinct $k$-flats, each of them real, that are simultaneously tangent to each of the $(n-k-1)$-cylinders $Cy(U_j, r_j), j = 1, \ldots, i$, and also meet each of the $(n-k-1)$-flats $U_{i+1}, \ldots, U_{dk,n}$.

The case of $i = dk,n$ implies Theorem 1.

**Proof.** We induct on $i$, with the case of $i = 0$ being the hypothesis of the theorem.

Suppose that $i \leq dk,n$ and that there exist $r_1, \ldots, r_{i-1} > 0$ such that there are exactly $2^{i-1} \cdot \#_{k,n}$ distinct $k$-flats $V_1, \ldots, V_{2^{i-1} \#_{k,n}}$ which are simultaneously tangent to $Cy(U_j, r_j)$ for each $j = 1, \ldots, i-1$, and meet each of $U_1, \ldots, U_{dk,n}$, and each of these $k$-flats is real.

Now we drop the condition that the $k$-flats meet $U_i$. Let $\mathcal{A} \subset \mathbb{G}_{k,n}$ be the curve of $k$-flats that are tangent to the cylinders $Cy(U_j, r_j)$ for $j = 1, \ldots, i-1$ and that also meet each of the $(n-k-1)$-flats $U_{i+1}, \ldots, U_{dk,n}$. Since $\mathcal{A}$ is the intersection of $i-1$ quadrics (the tangency conditions) with $dk,n - i$ hyperplanes (conditions to meet the remaining $U_j$) on the Grassmannian, it has degree at most $2^{i-1} \#_{k,n}$. Since its intersection with the hyperplane defined by $U_i$ consists of $2^{i-1} \#_{k,n}$ points, we conclude that the degree of $\mathcal{A}$ is $2^{i-1} \#_{k,n}$ and (by Proposition 3) that each of these points is a smooth point of $\mathcal{A}$.

Let $V \in \{V_1, \ldots, V_{2^{i-1} \#_{k,n}}\}$. Since $V$ is a smooth real point of the real curve $\mathcal{A} \subset \mathbb{G}_{k,n}$ (i.e., $V \subset \mathbb{P}^n$), the real points of $\mathcal{A}$ contain a smooth arc $\alpha$ containing $V$ with $\alpha \cap (\{V_1, \ldots, V_{2^{i-1} \#_{k,n}}\} \setminus V) = \emptyset$. Let $\varphi : (-\delta, \delta) \to \alpha$ be a smooth parametrization
of the arc \( \alpha \) with \( \varphi(0) = V \). Such a parametrization exists, for example, by the Implicit Function Theorem.

Thus, for \( t \in (-\delta, \delta) \setminus \{0\} \), the real \( k \)-flat \( \varphi(t) \) does not meet \( U_i \) and so it has a positive Euclidean distance \( d(t) \) from \( U_i \). Since \( d(t) \) is a continuous function of \( t \), for \( \rho \in \mathbb{R} \) with \( 0 < \rho < \min\{d(-\delta/2), d(\delta/2)\} \) there are at least two distinct real \( k \)-flats in \( \alpha \) whose Euclidean distance to \( U_i \) is \( \rho \).

In this way, we obtain \( 2^{i-1} \cdot \#_{k,n} \) such arcs, each containing one of \( V_1, \ldots, V_{2^{i-1} \cdot \#_{k,n}} \). We may assume that these arcs are pairwise disjoint. Let \( 0 < r_i \) be small enough to ensure that each arc contains two \( k \)-flats having Euclidean distance \( r_i \) from \( U_i \). This gives at least \( 2 \cdot 2^{i-1} \cdot \#_{k,n} \) real \( k \)-planes in \( A \) whose Euclidean distance to \( U_i \) is \( r_i \). Since \( 2^i \cdot \#_{k,n} \) is the maximum number of \( k \)-flats with this property, there are exactly \( 2^i \cdot \#_{k,n} \) distinct \( k \)-flats tangent to \( \text{Cy}(U_j, r_j) \) for \( j = 1, \ldots, i \) and that also meet each of the \((n-k-1)\)-flats \( U_{i+1}, \ldots, U_{d_{k,n}} \).

Since the number of real \( k \)-flats will not change under a small perturbation of the \( k \)-cylinders \( \text{Cy}(U_j, r_j) \), we may replace them by quadrics which are smooth in \( \mathbb{P}^n \). Let \( \text{sign}(Q) \) denote the signature of a quadric \( Q \subset \mathbb{P}^n \).

**Corollary 6.** Let \( 1 \leq k \leq n - 2 \). For

\[
(s_1, \ldots, s_{d_{k,n}}) \in \begin{cases} 
(n-1, n-3, \ldots, 2k-n+1)^{d_{k,n}} & \text{if } k \geq n/2, \\
(n-1, n-3, \ldots, 2 \cdot \left( \frac{n-1}{2} - \left\lfloor \frac{n-1}{2} \right\rfloor \right))^{d_{k,n}} & \text{if } k < n/2
\end{cases}
\]

there exist smooth quadrics \( Q_1, \ldots, Q_{d_{k,n}} \subset \mathbb{P}_R^n \) with \( |\text{sign}(Q_i)| = s_i \), \( 1 \leq i \leq d_{k,n} \), such that the \( \#_{k,n} \) (complex) common tangent \( k \)-flats to \( Q_1, \ldots, Q_{d_{k,n}} \) are all real, distinct, and lie in the affine space \( \mathbb{R}^n \).

**Proof.** Since the absolute value of the signature of an \((n-k-1)\)-cylinder is \( k \), the proof immediately follows from the possible perturbations of the quadratic form in \( \mathbb{P}^n \) of the type

\[-r^2 x_0^2 + x_1^2 + \cdots + x_{k+1}^2.\]

\( \square \)

We conjecture that the reality statement holds for signatures not covered by Corollary 6.

**Conjecture 7.** Let \( 1 \leq k \leq n - 2 \). For

\[
(s_1, \ldots, s_{d_{k,n}}) \in \{n-1, n-3, \ldots, 2 \cdot \left( \frac{n-1}{2} - \left\lfloor \frac{n-1}{2} \right\rfloor \right) \}^{d_{k,n}}
\]

there exist smooth quadrics \( Q_1, \ldots, Q_{d_{k,n}} \subset \mathbb{P}_R^n \) with \( |\text{sign}(Q_i)| = s_i \), \( 1 \leq i \leq d_{k,n} \), such that the \( \#_{k,n} \) (complex) common tangent \( k \)-flats to \( Q_1, \ldots, Q_{d_{k,n}} \) are all real, distinct, and lie in the affine space \( \mathbb{R}^n \).

The first case of this conjecture which is not covered by Corollary 6 is when \( k = 3 \) and \( n = 5 \) and the signature is zero, that is, for 3-flats tangent to 8 smooth quadrics in \( \mathbb{P}^5_R \), with at least one having signature zero. We remark that an argument perturbing cylinders to singular quadrics gives an analog to Corollary 6 concerning \( k \)-flats tangent to singular quadrics. We omit its complicated formulation.
3. A constructive proof for lines in dimension 3

Our proof of Theorem 1 was non-constructive. We close this paper by providing a constructive proof in the first nontrivial case, \((k, n) = (1, 3)\), i.e., the real lines tangent to four quadrics in 3-space. In order to realize the tetrahedral configuration of Figure 1 in \(\mathbb{P}^3_{\mathbb{R}}\), let \(\ell_1, \ldots, \ell_4\) be given by the following equations:
\[
\ell_i : x_0 = x_3 = 0; \quad \ell_2 : x_0 = x_1 = 0; \quad \ell_3 : x_1 = x_2 = 0; \quad \ell_4 : x_2 = x_3 = 0.
\]
The two common transversal lines are given by \(x_2 = x_4 = 0\) and by \(x_1 = x_4 = 0\).

For parameters \(\alpha, \beta \in \mathbb{R}\), consider the four quadrics
\[
Q_1 : x_0^2 + x_3^2 - \beta(x_1^2 + x_2^2) = 0,
Q_2 : x_0^2 + x_1^2 - \beta(x_2^2 + x_3^2) = 0,
Q_3 : x_1^2 + x_2^2 - \alpha(x_0^2 + x_3^2) = 0,
Q_4 : x_2^2 + x_3^2 - \alpha(x_0^2 + x_1^2) = 0.
\]
For \(\alpha = \beta = 0\), the four quadrics become the corresponding lines in \(\mathbb{P}^3_{\mathbb{R}}\). For small \(\alpha, \beta > 0\), these quadrics are deformations of the lines with rank 4 and signature 0—smooth ruled surfaces.

**Theorem 8.** Let \((\alpha, \beta) \in \mathbb{R}^2\) satisfy
\[
\alpha \beta (1 - \alpha \beta)(1 - \beta^2)(1 - \alpha^2)(1 - \alpha)^2(1 - \beta)^2 - 16 \alpha \beta \neq 0.
\]
Then there are 32 distinct (possibly complex) common tangent lines to \(Q_1, \ldots, Q_4\). If \(0 < \alpha, \beta < 3 - 2\sqrt{2}\), then each of these 32 tangent lines is real.

**Proof.** Since the quadrics only contain monomials of the form \(x_i^2\), the tangent equations of \(Q_1, \ldots, Q_4\) only contain monomials of the form \(p_{ij}^2\). Hence, the four tangent equations give the following system of linear equations in \(p_{01}^2, \ldots, p_{23}^2\):
\[
\begin{pmatrix}
-\beta & -\beta & 1 & \beta^2 & -\beta & -\beta \\
1 & -\beta & -\beta & -\beta & \beta^2 & -\beta \\
-\alpha & -\alpha & -\beta & -\beta & -\beta & \beta^2 \\
\alpha^2 & -\alpha & -\alpha & -\alpha & -\alpha & 1
\end{pmatrix}
\begin{pmatrix}
p_{01}^2 \\
p_{02}^2 \\
p_{03}^2 \\
p_{12}^2 \\
p_{13}^2 \\
p_{23}^2
\end{pmatrix} = 0.
\]

Permute the variables into the order \((p_{02}, p_{13}, p_{03}, p_{12}, p_{01}, p_{23})\). For \(\alpha, \beta \in \mathbb{R}\) satisfying
\[
\alpha \beta (1 - \alpha \beta)(1 + \beta)(1 + \alpha) \neq 0,
\]
Gaussian elimination yields the following system:
\[
\begin{pmatrix}
-\beta & -\beta & (1 - \alpha)(1 - \beta) & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha & -\beta & 0 \\
0 & 0 & 0 & -\beta & \alpha & 0 \\
0 & 0 & 0 & 0 & \alpha & -\beta
\end{pmatrix}
\begin{pmatrix}
p_{02}^2 \\
p_{13}^2 \\
p_{03}^2 \\
p_{12}^2 \\
p_{13}^2 \\
p_{23}^2
\end{pmatrix} = 0.
\]
Together with the Plücker equation (7), this gives the following system of equations:

\begin{align}
\beta p_{02}^2 - \beta p_{13}^2 + (1 - \alpha)(1 - \beta)p_{03}^2 &= 0, \\
p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} &= 0, \\
\alpha p_{01}^2 &= \alpha p_{03}^2 = \beta p_{12}^2 = \beta p_{23}^2. 
\end{align}

For \( \alpha, \beta \) satisfying (5) as well as \((1 - \alpha)(1 - \beta) \neq 0\), we distinguish the following three disjoint cases.

**Case 1:** \( p_{02} = 0 \).

Since \( p_{13} = 0 \) would imply that all components are zero and hence contradict \((p_0, \ldots, p_3)^T \in \mathbb{P}^3\), we can assume \( p_{13} = 1 \). Then (6) and (8) imply

\[ \alpha p_{01}^2 = \alpha p_{03}^2 = \beta p_{12}^2 = \beta p_{23}^2 = \frac{\alpha \beta}{(1 - \alpha)(1 - \beta)} \neq 0. \]

Since (7) implies \( p_{01}p_{23} = -p_{03}p_{12} \), only 8 of the \( 2^4 = 16 \) sign combinations for \( p_{01}, p_{03}, p_{12}, p_{23} \) are possible. Namely, the 8 (complex) solutions for \( p_{01}, p_{03}, p_{12}, p_{23} \) are

\[ (p_{01}, p_{03}, p_{12}, p_{23})^T = \frac{1}{\sqrt{(1 - \alpha)(1 - \beta)}} (\gamma_{01} \sqrt{\beta}, \gamma_{03} \sqrt{\beta}, \gamma_{12} \sqrt{\alpha}, -\gamma_{01} \gamma_{03} \gamma_{12} \sqrt{\alpha})^T \]

with \( \gamma_{01}, \gamma_{03}, \gamma_{12} \in \{-1, 1\} \). Hence, for \( \alpha, \beta \in \mathbb{R}^2 \) satisfying (5), this case gives 8 distinct common tangents.

**Case 2:** \( p_{13} = 0 \).

This case is symmetric to case 1. Setting \( p_{02} = 1 \), the resulting 8 solutions for the variables \( p_{01}, p_{03}, p_{12}, p_{23} \) are the same ones as in (9).

**Case 3:** \( p_{02}p_{13} \neq 0 \).

Without loss of generality, we can assume \( p_{02} = 1 \). Solving (7) for \( p_{13} \) and substituting this expression into (8) yields

\[ -\beta - \beta p_{01}^2 p_{23} - \beta p_{03}^2 p_{12} - 2\beta p_{01}p_{03}p_{12}p_{23} + (1 - \alpha)(1 - \beta)p_{03}^2 = 0. \]

We use (8) to write this in terms of \( p_{01} \). This is straightforward for the squared terms, but for the other terms, we observe that, by (5), \( p_{01}p_{23} = \pm p_{03}p_{12} \) and since \( p_{02}p_{13} \neq 0 \), the Plücker equation (7) implies these have the same sign. This gives the quartic equation in \( p_{01} \),

\[ -\beta + (1 - \alpha)(1 - \beta)p_{01}^2 - 4\alpha p_{01}^4 = 0, \]

whose discriminant is

\[ (1 - \alpha)^2(1 - \beta)^2 - 16\alpha. \]

Hence, for \( \alpha, \beta \in \mathbb{R} \) satisfying (5), and for which this discriminant does not vanish, there are two different solutions for \( p_{01}^2 \). For each of these two solutions for \( p_{01}^2 \), there are 8 distinct solutions for \( p_{01}, p_{03}, p_{12}, p_{23} \), namely

\[ (p_{01}, p_{03}, p_{12}, p_{23})^T = \sqrt{p_{01}^2} (\gamma_{01}, \gamma_{03}, \gamma_{12}, \gamma_{01} \gamma_{03} \gamma_{12})^T \]

with \( \gamma_{01}, \gamma_{03}, \gamma_{12} \in \{-1, 1\} \). Since \( p_{13} \) is uniquely determined by \( p_{01}, p_{02}, p_{03}, p_{12} \), case 3 gives 16 distinct common tangents.

In order to determine when all solutions are real, suppose first that \( \alpha = \beta \). Then the discriminant (10) becomes \((\alpha^2 - 6\alpha + 1)(\alpha + 1)^2\), and its smallest positive root is
\( \alpha_0 := 3 - 2\sqrt{2} \approx 0.17157 \). In particular, for \( 0 < \alpha < \alpha_0 \), the discriminant in case 3 is positive and both solutions for \( p_{01}^2 \) are positive. Thus, for \( 0 < \beta = \alpha < \alpha_0 \), the solutions of all three cases are distinct and real. Next, fix \( 0 < \alpha < \alpha_0 \) and suppose that \( 0 < \beta < \alpha \). Then the discriminant (10) is positive: for fixed \( 0 < \alpha < \alpha_0 \), the discriminant (10) is decreasing in \( \beta \) for \( 0 < \beta < \alpha \) and positive when \( \beta = \alpha \). This concludes the proof of Theorem 8. \( \square \)

Figure 2 illustrates the construction and the 32 tangents for \( \alpha = 1/10 \) and \( \beta = 1/20 \).

\[ \text{Figure 2. The configuration of quadrics from Theorem 8} \]

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