

CHARACTERIZATION OF BILINEAR SPACES WITH UNIMODULAR ISOMETRY GROUP

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ABSTRACT. We study finite-dimensional bilinear spaces and their isometry groups. To each bilinear space V we associate two canonical filtrations, which yield structural results on V . Prominent among these is an explicit formula for the number of indecomposable and degenerate blocks of V of a given dimension. Equipped with this material, we proceed to characterize those bilinear spaces whose isometry group is contained in the special linear group. This characterization can easily be implemented in practice by means of an algorithm. As an application, we determine the real n -by- n matrices whose congruence class is disconnected.

1. INTRODUCTION

A bilinear space over a field F is a pair (V, φ) , where V is a finite-dimensional F -vector space and $\varphi : V \times V \rightarrow F$ a bilinear form. The main problem concerning bilinear spaces is to determine when two of them are equivalent, i.e. they afford a linear isomorphism preserving the underlying forms. There is a structure theory developed around this problem, with contributions by J. Williamson [19], G.E. Wall [17], C. Riehm [14], P. Gabriel [8] and R. Scharlau [15]. The principal result of this theory is the reduction of the equivalence problem of bilinear spaces to classical problems in linear algebra, such as the similarity of matrices and the equivalence of quadratic and hermitian forms. More recently, a degree of attention has been focused on finding representatives or their numbers for the Gram matrices of bilinear spaces, under certain limitations either on the field or the size of the matrix. See for instance [9, 18, 12, 2, 3, 4].

Of all bilinear spaces, those whose underlying form is symmetric are by far the ones most widely considered, with an extensive theory surrounding them from the algebraic, geometric or arithmetic point of view (see e.g. [13, 16] and their bibliographies). The comparative neglect suffered by general bilinear spaces probably explains why certain basic results took so long to be established. For example, the fact that a square matrix over an arbitrary field is congruent to its transpose was not proved until 1980 by R. Gow [10] for invertible matrices and shortly afterwards by Ballantine and Yip [1] in general.

A more deeply marked difference occurs with the isometry group associated to a bilinear space. On one hand one has the classical linear groups arising as isometry

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groups of zero, alternating or symmetric forms—and subjected to detailed scrutiny (see e.g. [5, 6, 11])—and on the other the general isometry groups of which virtually nothing has been written.

The present paper aims at contributing to the structure theory of bilinear spaces, with an eye on application to the study of the corresponding isometry groups. One such application is dealt with in this paper. Indeed, one of our main results is a characterization of bilinear spaces whose isometry group is unimodular, i.e., contained in the special linear group.

We refer to a bilinear space as totally degenerate if every non-zero orthogonal direct summand of it is degenerate. Any bilinear space, V for short, decomposes as the orthogonal direct sum $V = V_{\text{ndeg}} \perp V_{\text{tdeg}}$, where V_{ndeg} is non-degenerate and V_{tdeg} is totally degenerate. Both V_{ndeg} and V_{tdeg} are unique only up to equivalence. This result is due to P. Gabriel [8]. One of our most interesting results is a formula for the number $m(r)$ of indecomposable degenerate spaces of dimension r (they are all equivalent to each other) present in a given orthogonal decomposition of V into indecomposables. Indeed,

$$m(r) = (-1)^r(l_{r-2} + l_{r-1} - l_r - l_{r+1}), \quad r \geq 1,$$

where $l_{-1} = 0$, $l_i = \dim L^i(V)$, $i \geq 0$, and L^i is the i -iterate of the operator L , which associates to each subspace of V its left orthogonal complement. By means of L and the analogous operator R from the right we obtain structural invariants of V , which are somewhat technical to describe at this point.

In general, neither V_{ndeg} nor V_{tdeg} is stable under the isometry group of V . However, the asymmetry, say σ , of V_{ndeg} , as defined in [14], is uniquely determined by V up to similarity. It is a linear operator that somehow measures how far φ is from being symmetric on V_{ndeg} .

With this notation, we prove that the isometry group G of V is unimodular if and only if $m(r) = 0$ for all odd r and σ has no unipotent Jordan blocks of odd size. This holds for all fields F of characteristic not 2. If $\text{char } F = 2$ but $|F| > 2$, then G is unimodular precisely when every orthogonal direct summand of V_{tdeg} is even dimensional. When $|F| = 2$ the group G is always unimodular.

As an application, we show that a real square matrix A has disconnected congruence class if and only if A has no direct summands of odd size (under congruence transformations).

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2. NOTATION AND PRELIMINARIES

Let (V, φ) be a bilinear space. Explicit reference to φ will be omitted when no confusion is possible. We shall often write $\langle v, w \rangle$ instead of $\varphi(v, w)$.

If U is a subspace of V , then U becomes a bilinear space by restricting φ to $U \times U$. We write $V = U \perp W$ if $V = U \oplus W$ and $\langle U, W \rangle = \langle W, U \rangle = 0$. In this case we refer to U and W as *orthogonal summands* of V . A bilinear space is *indecomposable* if it lacks proper non-zero orthogonal summands. An *isometry* between bilinear spaces is a linear isomorphism which preserves the underlying forms, in which case the spaces are said to be *equivalent*. The group of all isometries of (V, φ) will be denoted by $\text{Iso}(V, \varphi)$.

For a subspace U of V , let

$$L(U) = \{v \in V \mid \langle v, U \rangle = 0\}, \quad R(U) = \{v \in V \mid \langle U, v \rangle = 0\}.$$

Here $L(V)$ and $R(V)$ are the *left* and *right radicals* of V , and $L(V) \cap R(V)$ is the *radical* of V . We have $\dim L(V) = \dim R(V)$, and we say that V is *non-degenerate* whenever this number is 0. Otherwise V is *degenerate*. We shall say that V is *totally degenerate* if all non-zero orthogonal summands of V are degenerate.

We may view L and R as operators which assign to each subspace of V its left and right orthogonal complements, respectively. We may compound these operators, denoting by L^i and R^i their respective i -th iterates. By convention, L^0 and R^0 are the identity operators. When more than one bilinear space is present, we shall write L_V and R_V for L and R . For instance, $R_V = L_{V'}$, where V' is the transpose of the bilinear space V .

For $r \geq 1$, denote by J_r the nilpotent lower Jordan block of size r . Thus

$$J_1 = (0), \quad J_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \dots$$

Write N_r for a bilinear space whose underlying form has matrix J_r relative to some basis. We shall refer to the bilinear space N_r as a *Gabriel block* and to r as its size. We point out that, from the point of view of the theory of matrix pencils, there is another natural choice for a basis of N_r . The corresponding matrices are given explicitly in [18].

We refer the reader to [7, 18] for the following formulation of a theorem due to P. Gabriel [8].

Theorem 2.1. *Let V be a bilinear space over F . Then*

- (a) $V = V_{\text{tdeg}} \perp V_{\text{ndeg}}$, where V_{tdeg} is the orthogonal direct sum of Gabriel blocks and V_{ndeg} is non-degenerate (either of them possibly 0).
- (b) The sizes and multiplicities of the Gabriel blocks appearing in V_{tdeg} are uniquely determined by V .
- (c) The equivalence class of V_{ndeg} is uniquely determined by V .
- (d) Up to equivalence, the only indecomposable and degenerate bilinear space of dimension $r \geq 1$ is N_r .

We refer to V_{tdeg} and V_{ndeg} as the *totally degenerate* and *non-degenerate parts* of V , respectively (this defines them up to equivalence only). We may write $V_{\text{tdeg}} = V_{\text{even}} \perp V_{\text{odd}}$, where V_{even} (resp. V_{odd}) is the orthogonal direct sum of Gabriel blocks of even (resp. odd) size. We refer to them as the *even* and *odd parts* of V (again, they are defined up to equivalence only).

We shall say that a sequence $(U_i)_{i \geq 0}$ of subspaces of V is *nested* if

$$U_1 \subseteq U_3 \subseteq U_5 \subseteq \dots \subseteq U_4 \subseteq U_2 \subseteq U_0 = V.$$

The subsequences $(U_{2i+1})_{i \geq 0}$ and $(U_{2i})_{i \geq 0}$ stabilize. We shall refer to the subspaces U_∞ and U^∞ in which $(U_{2i+1})_{i \geq 0}$ and $(U_{2i})_{i \geq 0}$ stabilize as the *lower* and *upper limits* of $(U_i)_{i \geq 0}$, respectively. Clearly the former is contained in the latter. If these limits coincide, then $(U_i)_{i \geq 0}$ is said to be *convergent*.

As we shall see, the sequences $(L^i(V))_{i \geq 0}$ and $(R^i(V))_{i \geq 0}$ are nested, yielding canonical filtrations for V . We write $L_\infty(V)$ and $L^\infty(V)$ (resp. $R_\infty(V)$ and $R^\infty(V)$) for the lower and upper limits of $(L^i(V))_{i \geq 0}$ (resp. $(R^i(V))_{i \geq 0}$).

Given vectors v_1, \dots, v_s of a vector space, their span will be denoted by $[v_1, \dots, v_s]$.

3. STRUCTURAL RESULTS

In this section we study the structure of arbitrary bilinear spaces and pay special attention to some important orthogonal direct decompositions of these spaces.

Theorem 3.1. *Let N_r be a Gabriel block of size r . Then*

- (a) *The sequences $(L^i(N_r))_{i \geq 0}$ and $(R^i(N_r))_{i \geq 0}$ are nested and convergent.*
- (b) *If r is even, then $N_r = L^\infty(N_r) \oplus R^\infty(N_r)$.*
- (c) *If r is odd, then $L^\infty(N_r) = R^\infty(N_r)$.*
- (d) *If r is even, then $N_r = L^\infty(N_r) \oplus R^\infty(N_r)$ is the only decomposition of N_r as the direct sum of two totally isotropic subspaces.*
- (e) *If r is odd, then $L^\infty(N_r) = R^\infty(N_r)$ is the only totally isotropic subspace of N_r of maximum dimension.*

Proof. Let $\{e_1, \dots, e_r\}$ be a basis for N_r , relative to which the underlying form has matrix J_r . Then

$$\begin{aligned} L^1(N_r) &= [e_1], & L^2(N_r) &= [e_1, e_3, e_4, \dots, e_r], \\ L^3(N_r) &= [e_1, e_3], & L^4(N_r) &= [e_1, e_3, e_5, e_6, \dots, e_r], \dots \end{aligned}$$

and

$$\begin{aligned} R^1(N_r) &= [e_r], & R^2(N_r) &= [e_r, e_{r-2}, e_{r-3}, \dots, e_1], \\ R^3(N_r) &= [e_r, e_{r-2}], & R^4(N_r) &= [e_r, e_{r-2}, e_{r-4}, e_{r-5}, \dots, e_1], \dots \end{aligned}$$

From these identities we easily obtain (a), (b) and (c). It is also clear that $L^\infty(N_r)$ and $R^\infty(N_r)$ are totally isotropic regardless of the parity of r , and that the maximum dimension for a totally isotropic subspace of N_r for r odd is $(r+1)/2$. This is attained by $L^\infty(N_r) = R^\infty(N_r)$.

We next show (d) by induction on r . If $r = 2$, then $[e_1]$ and $[e_2]$ are the only totally isotropic subspaces of N_r , whence the result holds. Suppose $r \geq 4$ and let $N_r = U \oplus W$ be a decomposition of N_r into totally isotropic subspaces. We easily see that

$$L(N_r) = U \cap L(N_r) \oplus W \cap L(N_r).$$

As $L(N_r)$ is one dimensional, either U or W contains $L(N_r)$. Since we also have

$$L(U) = U + L(N_r) \text{ and } L(W) = W + L(N_r),$$

we infer that either $U = L(U)$ or $W = L(W)$ (both possibilities cannot occur because $L(N_r) = L(U) \cap L(W)$ is not zero). Without loss of generality, we may assume that $L(N_r) \subseteq U$ and $U = L(U)$.

Since $L(N_r)$ is contained in the radical of $L^2(N_r)$, we may consider the quotient bilinear space $L^2(N_r)/L(N_r)$, which is easily seen to be equivalent to N_{r-2} (cf. Lemma 3.1 of [7]). From the above discussion we infer that $U \subseteq L^2(N_r)$, whence $U/L(N_r)$ is a totally isotropic subspace of $L^2(N_r)/L(N_r)$. In fact, we have the decomposition

$$L^2(N_r)/L(N_r) = U/L(N_r) \oplus ((W' \oplus L(N_r))/L(N_r))$$

into totally isotropic subspaces, where $W' = L^2(N_r) \cap W$. However, we also have the decomposition

$$L^2(N_r)/L(N_r) = L^\infty(N_r)/L(N_r) \oplus (([e_4, e_6, \dots, e_r] \oplus L(N_r))/L(N_r)).$$

It follows by induction that $U = L^\infty(N_r)$ or $U = [e_1, e_4, e_6, \dots, e_r]$. The latter is impossible since $e_3 \in L([e_1, e_4, e_6, \dots, e_r])$, whereas $U = L(U)$. This establishes $U = L^\infty(N_r)$.

Similarly, either U or W must contain $R(N_r)$, and whichever contains it must equal $R^\infty(N_r)$. Since $e_r \notin U$, it follows that $W = R^\infty(N_r)$.

We finally prove (e) by induction on r . The case $r = 1$ is clear. Suppose $r \geq 3$ and let U be a totally isotropic subspace of N_r of maximum dimension, i.e., $(r + 1)/2$. As U is totally isotropic, we have $U \subseteq L(U)$. On the other hand since the canonical linear map $L(U) \rightarrow (N_r/U)^*$ has the 1-dimensional space $L(N_r)$ as its kernel, we have

$$\dim L(U) - 1 \leq \frac{r - 1}{2}.$$

Therefore $U = L(U)$ and $L(N_r) \subseteq U$. Thus $U/L(N_r)$ is a totally isotropic subspace of $L^2(N_r)/L(N_r)$. Since the latter is equivalent to N_{r-2} , it follows by induction that $U = L^\infty(N_r)$. □

Theorem 3.2. *Let V be a bilinear space over F . Then*

(a) *The sequences $(L^i(V))_{i \geq 0}$ and $(R^i(V))_{i \geq 0}$ are nested. They converge if and only if V is totally degenerate. To simplify the notation, set $V_\infty = L_\infty(V) \cap R_\infty(V)$, $V^\infty = L^\infty(V) \cap R^\infty(V)$, ${}_\infty V = L_\infty(V) + R_\infty(V)$ and ${}^\infty V = L^\infty(V) + R^\infty(V)$.*

(b) *V_∞ is the radical of the following bilinear spaces: $V_\infty, {}_\infty V, V^\infty$ and ${}^\infty V$.*

(c) *In general none of the subspaces $V_{\text{even}}, V_{\text{odd}}, V_{\text{tdeg}}$ and V_{ndeg} is uniquely determined by V . However, V_{odd} has a unique totally isotropic subspace of maximum dimension, and this is uniquely determined by V . In fact, it is equal to V_∞ . Moreover, $V_{\text{even}} \perp V_\infty, V_{\text{ndeg}} \perp V_\infty$ and $V_{\text{ndeg}} \perp V_{\text{even}} \perp V_\infty$ are uniquely determined by V , being respectively equal to ${}_\infty V, V^\infty$, and ${}^\infty V$.*

(d) *If $V_{\text{odd}} = 0$, then V_{tdeg} and V_{ndeg} are uniquely determined by V . In fact, we have $V_{\text{tdeg}} = {}_\infty V$ and $V_{\text{ndeg}} = V^\infty$.*

(e) *We have the following equivalences between bilinear spaces:*

$$\begin{aligned} {}_\infty V/V_\infty &\cong V_{\text{even}}, \\ V^\infty/V_\infty &\cong V_{\text{ndeg}} \quad \text{and} \\ {}^\infty V/V_\infty &\cong V_{\text{even}} \perp V_{\text{ndeg}}. \end{aligned}$$

Proof. We begin by observing that if $V = U \perp W$ and $V' = U' \perp W'$, with $U' \subseteq U$ and $W' \subseteq W$, then

$$L_{V'}(V') = L_U(U') \perp L_W(W').$$

Note further that if W is non-degenerate, then $L_W(W) = 0$ and $L_W^2(W) = W$. Applying these remarks to the decomposition $V = V_{\text{tdeg}} \perp V_{\text{ndeg}}$ of Theorem 2.1, and making use of Theorem 3.1 we immediately obtain (a) and (b).

The first statement from (c) is easily seen by examples. For instance, let V be 4-dimensional with basis $\{e_1, e_2, e_3, e_4\}$ and let

$$\varphi(e_2, e_1) = \varphi(e_3, e_2) = \varphi(e_4, e_4) = 1,$$

while all other $\varphi(e_i, e_j) = 0$. Let $\alpha, \beta \in F$ be arbitrary scalars. Then we can take

$$V_{\text{tdeg}} = V_{\text{odd}} = [e_1, e_3, \alpha e_1 + e_2 - (\alpha + \beta^2)e_3 - \beta e_4], \quad V_{\text{ndeg}} = [\beta e_1 + \beta e_3 + e_4].$$

This shows the non-uniqueness of $V_{\text{ndeg}}, V_{\text{odd}}$, and V_{tdeg} .

In regards to the second, we may assume that $V = V_{\text{odd}}$. The proof is similar to that of part (e) of the previous theorem. We proceed by induction on the

dimension of V . Let $V = V_1 \perp V_2 \perp \dots \perp V_k$, where V_i is a Gabriel block of size $r_i = 2s_i + 1$. Then V_∞ has dimension $d := \sum(s_i + 1)$. Let U be a totally isotropic subspace of V of maximum dimension, and so $\dim U \geq d$. Since the canonical linear map $L(U) \rightarrow (V/U)^*$ has the k -dimensional space $L(V)$ as its kernel, we have $\dim L(U) - k \leq \sum s_i$, i.e., $\dim L(U) \leq \dim U$. As $U \subseteq L(U)$, it follows that $U = L(U)$ and $L(V) \subseteq U$. Thus $U/L(V)$ is a totally isotropic subspace of $L^2(V)/L(V)$. Since the latter is equivalent to

$$\bigoplus_{r_i > 1} N_{r_i-2},$$

we can use induction to finish the proof.

The remaining part of (c) is clear, and it is easy to verify the assertions (d) and (e). □

For $i \geq 0$, let us write l_i for the dimension of $L^i(V)$. For convenience, we set $l_{-1} = 0$. For $r \geq 1$ we shall write $m(r)$ (resp. $n(r)$) for the number of Gabriel blocks of size r (resp. $\geq r$) appearing in a given orthogonal decomposition of V .

Theorem 3.3. *Let V be a bilinear space over F . Then*

$$m(r) = (-1)^r(l_{r-2} + l_{r-1} - l_r - l_{r+1}), \quad r \geq 1.$$

Proof. The formulae at the very beginning of the proofs of Theorems 3.1 and 3.2 show

$$\begin{aligned} l_0 &= \dim V, \\ l_1 - l_{-1} &= n(1), \\ l_0 - l_2 &= n(2), \\ l_3 - l_1 &= n(3), \\ l_2 - l_4 &= n(4), \text{ etc.} \end{aligned}$$

Since $m(r) = n(r) - n(r + 1)$ for all $r \geq 1$, the result follows. □

Combining Theorems 3.2(e) and 3.3 yields another proof of parts (b) and (c) of Theorem 2.1 (note that part (a) follows from the other parts).

4. WHEN IS THE ISOMETRY GROUP UNIMODULAR?

In this section we consider a bilinear space (V, φ) over a field F and its isometry group $G = \text{Iso}(V, \varphi)$. We say that G is *unimodular* if it is contained in $\text{SL}(V)$. We shall find necessary and sufficient conditions for G to be unimodular. If $|F| = 2$, then $\text{SL}(V) = \text{GL}(V)$ and $G \subseteq \text{SL}(V)$ holds trivially. Thus we shall assume that $|F| > 2$.

Lemma 4.1. *If G is unimodular and $|F| > 2$, then $V_{\text{odd}} = 0$.*

Proof. Assume that $V = N_r \perp W$ with r odd. Then $\text{Iso}(N_r)$ is a subgroup of G . Choose a basis of N_r in which its bilinear form has matrix J_r . If $a \in F$ and $a \neq 0, 1$, then

$$\text{diag}(a, a^{-1}, a, a^{-1}, \dots, a^{-1}, a) \in \text{Iso}(N_r)$$

has determinant a . Hence G is not contained in $\text{SL}(V)$. □

Let us make two trivial but useful observations. First, if V is non-degenerate, then every $g \in G$ has determinant ± 1 . In particular, when the characteristic is 2, G is unimodular. Second, we have

$$(4.1) \quad \text{Iso}(V, \varphi) = \text{Iso}(V, \varphi') \subseteq \text{Iso}(V, \varphi - \varphi').$$

This is useful because, when $\varphi - \varphi'$ is non-degenerate, we have $\text{Iso}(V, \varphi - \varphi') \subseteq \text{SL}(V)$. We refer to $\varphi - \varphi'$ as the *alternating part* of (V, φ) .

The characteristic 2 case is completely covered by the following theorem.

Theorem 4.2. *Assume that $|F| > 2$. Then G is unimodular if and only if $V_{\text{odd}} = 0$ and $\text{Iso}(V_{\text{ndeg}})$ is unimodular. If F has characteristic 2, then the second condition is automatically satisfied.*

Proof. The first condition is necessary by Lemma 4.1.

Assume that the first condition is satisfied. Then by Theorem 3.2(d) we have $V = V_{\text{even}} \perp V_{\text{ndeg}}$, where the summands are uniquely determined by V . Hence G is the direct product of $\text{Iso}(V_{\text{ndeg}})$ and $\text{Iso}(V_{\text{even}})$. The alternating part of V_{even} is non-degenerate, since that is the case for N_r with r even. Therefore by applying (4.1) to V_{even} , we see that the latter group is unimodular. Hence, G is unimodular if and only if so is $\text{Iso}(V_{\text{ndeg}})$.

The second assertion follows from the first observation above. □

We have reduced the problem of unimodularity of G to the non-degenerate case. For the remainder of this section we assume that the characteristic of F is not 2 and that V is non-degenerate. Following Riehm [14], we define the *asymmetry* of (V, φ) as the linear operator $\sigma \in \text{GL}(V)$ such that $\varphi'(v, w) = \varphi(v, \sigma(w))$ for all $v, w \in V$. We view V as a module over $F[t]$ via the action of σ . For an irreducible polynomial $p \in F[t]$, let V_p denote the primary component of V associated to p . Write $V_{\text{unip}} = V_{t-1}$.

Lemma 4.3. *$\text{Iso}(V)$ is unimodular if and only if $\text{Iso}(V_{\text{unip}})$ is unimodular.*

Proof. By [14, Proposition 3], we have $V = V_{\text{unip}} \perp W$, where both summands are uniquely determined by V . Hence G is the direct product of $\text{Iso}(V_{\text{unip}})$ and $\text{Iso}(W)$. Now $\varphi(v, w) - \varphi'(v, w) = \varphi(v, w - \sigma(w))$, with $\varphi|_{W \times W}$ non-degenerate and $(1 - \sigma)|_W$ invertible. Therefore $(\varphi - \varphi')|_{W \times W}$ is non-degenerate. Hence by applying (4.1) to W , we conclude that $\text{Iso}(W)$ is unimodular. Thus the result follows. □

Suppose V has an odd-dimensional orthogonal summand W . Then $-I_W$ can be extended to an isometry of V with determinant -1 , so V is not unimodular. We next derive an equivalent condition to the existence of such W in terms of the elementary divisors of V_{unip} .

Lemma 4.4. *Let $\varepsilon = \pm 1$. Suppose $V = V_{t-\varepsilon}$ and that V is a free module over $F[t]/(t-\varepsilon)^s$ for some $s \geq 1$. Let $\pi = \sigma - \sigma^{-1}$ and $\bar{V} = V/\pi V$. Define $g_s : \bar{V} \times \bar{V} \rightarrow F$ by the formula*

$$g_s(\bar{u}, \bar{v}) = \varphi(\pi^{s-1}u, v), \quad u, v \in V,$$

where $\bar{u} = u + \pi V$ and $\bar{v} = v + \pi V$. Then

- (a) g_s is a well-defined non-degenerate bilinear form.
- (b) g_s has scalar asymmetry $(-1)^{s-1}\varepsilon$.
- (c) $\dim V = s \cdot \dim \bar{V}$.

Proof. The first two conditions are explained by [14, pp. 50, 53], while the third is clear. \square

Lemma 4.5. *V has an odd-dimensional orthogonal summand if and only if V has an elementary divisor of the form $(t - 1)^s$ with s odd.*

Proof. Suppose V has an elementary divisor of the form $(t - 1)^s$ with s odd. Due to [14, Corollary to Proposition 3 and Lemma 6] we may assume that $V = V_{t-1}$ is a free module over $F[t]/(t - 1)^s$. Construct the non-degenerate symmetric bilinear form g_s of Lemma 4.4. Since F has characteristic not 2, we infer that g_s has a non-isotropic vector, i.e., there is a $v \in V$ such that $\varphi(\pi^{s-1}v, v) \neq 0$. It follows that the $F[t]$ -submodule, say W , of V generated by v is non-degenerate and has dimension s . We conclude from [14, page 47] that W is an odd-dimensional orthogonal summand of V .

Suppose conversely that W is an odd-dimensional orthogonal summand of V . We may assume that $W = V$ is indecomposable. By [14, section 1] the minimal polynomial of σ is a power of an irreducible monic polynomial $p(t)$ of odd degree. By [14, Proposition 1], we have $p(t) = t \pm 1$. By [14, Lemma 6], V is a free module over $F[t]/(t \pm 1)^s$ for some odd s . Now $p(t)$ cannot equal $t + 1$ due to Lemma 4.4, since skew-symmetric forms on odd-dimensional vector spaces over a field of characteristic not 2 do not exist. Thus $(t - 1)^s$ is an elementary divisor of V . \square

We shall now prove the main result of this section.

Theorem 4.6. *Let (V, φ) be a bilinear space over a field F of characteristic not 2. Then $G = \text{Iso}(V, \varphi)$ is unimodular if and only if $V_{\text{odd}} = 0$ and the asymmetry σ of V_{ndeg} has no elementary divisor $(t - 1)^d$ with d odd.*

Proof. Necessity follows from Lemmas 4.1 and 4.5. To prove sufficiency we may assume by Theorem 4.2 and Lemma 4.3 that $V = V_{\text{ndeg}} = V_{\text{unip}}$. Consider the nilpotent operator $\pi = \sigma - \sigma^{-1}$ and the associated increasing filtration:

$$(4.2) \quad 0 = V(0) \subset V(1) \subset \cdots \subset V(t-1) \subset V(t) = V,$$

where $V(s) = \ker(\pi^s)$ for each s . These are G -modules since σ is in the center of G . We have $t = 2r$ by our hypothesis on σ .

On the G -module $M(s) = V(s)/(V(s-1) + \pi V(s+1))$, $s \geq 1$, consider the non-degenerate bilinear form h_s of [14, p. 54]. Thus

$$h_s(\bar{v}, \bar{w}) = \varphi(\pi^{s-1}v, w), \quad v, w \in V(s),$$

where, e.g., \bar{v} is the coset of v in $M(s)$. Recall that symplectic transformations have determinant 1. Since h_s is G -invariant and has scalar asymmetry $(-1)^{s-1}$, the action of G on each $M(2s)$ must be unimodular (*).

Let $N(s) = V(s)/V(s-1)$ and $P(s) = (V(s-1) + \pi V(s+1))/V(s-1)$, which are G -modules for $s \geq 1$. Note that $P(s) \subseteq N(s)$ with $N(s)/P(s) \cong M(s)$ as G -modules. From [14, formula (4)] and our hypothesis on σ we see that $P(s) = N(s)$ for odd s . Let $f : N(s+1) \rightarrow P(s)$ be defined by $f(v + V(s)) = \pi v + V(s-1)$, $v \in V(s+1)$. This is a well-defined epimorphism of G -modules, and we see from [14, formula (4)] that $\dim N(s+1) = \dim P(s)$, whence $N(s+1) \cong P(s)$. If s is odd, this and $P(s) = N(s)$ yield $N(s) \cong N(s+1)$.

Note that the $N(s)$, $1 \leq s \leq 2r$, are the factors of the normal series (4.2) of V , so it suffices to show that G is unimodular on each $N(s)$. We do this by induction on s . If $s = 2r$, it is easy to check that $N(2r) = M(2r)$, so G is unimodular by (*).

So assume that $s < 2r$ and that G is unimodular on $N(k)$ for all k , $s < k \leq 2r$. If s is odd, then $N(s) \cong N(s + 1)$ as G -modules, so G is unimodular on $N(s)$ by the induction hypothesis. Suppose s is even. Because of the exact sequence

$$0 \rightarrow P(s) \rightarrow N(s) \rightarrow M(s) \rightarrow 0$$

of G -modules and (*), it suffices to show that G is unimodular on $P(s)$, and this follows from $P(s) \cong N(s + 1)$. □

Corollary 4.7. *Let (V, φ) be a bilinear space over a field F of characteristic not 2. Then G is unimodular if and only if (V, φ) has no orthogonal summands of odd dimension.*

The main theorem can be easily applied in practice. Indeed a bilinear space (V, φ) is given by an n -by- n matrix over F , where $n = \dim V$. Then it is easy to write a program (say in Maple) to compute the sequences $(L^i(V))_{i \geq 0}$ and $(R^i(V))_{i \geq 0}$. Hence by using Theorem 3.3 we can find the multiplicities $m(r)$ and verify whether or not $V_{\text{odd}} = 0$. Next we can use Theorem 3.2(e) to compute the matrix of V_{ndeg} and its asymmetry σ . Finally one can check whether or not σ has an elementary divisor $(t - 1)^d$ with d odd.

5. AN APPLICATION

Suppose here that the field F is arbitrary and that V is a finite-dimensional F -vector space. Write $B(V)$ for the space of all bilinear forms on V . There is a canonical left action of $\text{GL}(V)$ on $B(V)$ such that the stabilizer of $\varphi \in B(V)$ equals $\text{Iso}(V, \varphi)$. As usual, we have a bijection between the space of left cosets $\text{GL}(V)/\text{Iso}(V, \varphi)$ and the orbit O_φ of φ . In the real case this bijection is a homeomorphism (see [20, Theorems 3.1.3 and 2.1.14]). Hence O_φ is either connected or it has two connected components, depending on whether $\text{Iso}(V, \varphi)$ contains operators with negative determinant or not, respectively. It is natural to inquire under what conditions O_φ will actually be disconnected. As a corollary of the results in the previous section, we obtain the following characterization, written in matrix terms.

Theorem 5.1. *Let $A \in M_n(\mathbf{R})$ and $O_A = \{X'AX : X \in \text{GL}_n(\mathbf{R})\}$. Define the bilinear form φ on the column space $V = \mathbf{R}^n$ by $\varphi(x, y) = x'Ay$. Let $m(r)$ be the multiplicity in (V, φ) of the Gabriel block N_r , as defined prior to Theorem 3.3. Let B be the matrix of V_{ndeg} relative to some basis, and set $S = B^{-1}B'$. Then O_A is disconnected if and only if $m(r) = 0$ for odd r 's and S has no elementary divisors of the form $(t - 1)^d$ with d odd.*

Let us make three remarks that are useful for the applications of this theorem.

- 1) If $\det(A - A') \neq 0$, then O_A is disconnected (see the remark following (4.1)).
- 2) If A is nonsingular and $S = A^{-1}A'$, then the number of unipotent Jordan blocks of S is equal to the nullity of $A - A'$. This follows from the equality $I_n - S = A^{-1}(A - A')$.
- 3) If O_A is disconnected, then $\dim V$, $\dim V_{\text{ndeg}}$ and $\dim V_{\text{tdeg}}$ must be even.

As an example, we shall apply the above theorem to determine for which real square matrices A , of size $n \leq 4$, the congruence class O_A is disconnected. Of course, this can only occur for $n = 2$ or $n = 4$. In the former case exactly the non-symmetric A will have disconnected orbit.

Now let $n = 4$ and let $V = \mathbf{R}^4$ and φ be as in the theorem.

If $V = V_{\text{tdeg}}$, then O_A is disconnected if and only if A is congruent to J_4 or $J_2 \oplus J_2$.

Next let V_{ndeg} and V_{tdeg} each have dimension 2. Then we may assume that $A = J_2 \oplus B$, where $\det(B) \neq 0$. The orbit O_A will be disconnected if and only if B is not symmetric.

Finally let A be non-singular and $S = A^{-1}A'$. Clearly, O_A is connected if $A - A'$ has rank 0 and disconnected if this rank is 4. Thus assume that its rank is 2. Then we may assume that

$$A = \begin{pmatrix} B + K & C \\ C' & D \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where B and D are symmetric matrices of size 2. If $D \neq 0$ it is easy to see that V has a 1-dimensional orthogonal summand and so O_A is connected. We may now assume that $D = 0$ and so C is invertible. It is now easy to see that A is congruent to

$$(5.1) \quad \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

In this case the two unipotent Jordan blocks of S each have size 2 and so O_A is disconnected.

This last case can be easily generalized to obtain the following interesting

Corollary 5.2. *If n is divisible by 4, say $n = 4k$, then there exists a unique disconnected congruence class $O_A \subseteq M_n(\mathbf{R})$ such that $A - A'$ has rank $2k$. Such A is congruent to the direct sum of k copies of the matrix (5.1), which is itself indecomposable.*

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