

A REMARK ON A CONJECTURE OF BORWEIN AND CHOI

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ABSTRACT. We prove the remaining case of a conjecture of Borwein and Choi concerning an estimate on the square of the number of solutions to $n = x^2 + Ny^2$ for a squarefree integer N .

1. INTRODUCTION

We consider the positive definite quadratic form $Q(x, y) = x^2 + Ny^2$ for a square-free integer N . Let $r_{2,N}(n)$ denote the number of solutions to $n = Q(x, y)$ (counting signs and order). In this note, we estimate

$$\sum_{n \leq x} r_{2,N}(n)^2.$$

A positive squarefree integer N is called solvable (or more classically “numerus idoneus”) if $x^2 + Ny^2$ has one form per genus. Note that this means the class number of the form class group of discriminant $-4N$ equals the number of genera, 2^t , where t is the number of distinct prime factors of N . Concerning $r_{2,N}(n)$, Borwein and Choi [1] proved the following:

Theorem 1.1. *Let N be a solvable squarefree integer. Let $x > 1$ and $\epsilon > 0$. We have*

$$\sum_{n \leq x} r_{2,N}(n)^2 = \frac{3}{N} \left(\prod_{p|2N} \frac{2p}{p+1} \right) (x \log x + \alpha(N)x) + O(N^{\frac{1}{4}+\epsilon} x^{\frac{3}{4}+\epsilon}),$$

where the product is over all primes dividing $2N$ and

$$\alpha(N) = -1 + 2\gamma + \sum_{p|2N} \frac{\log p}{p+1} + \frac{2L'(1, \chi_{-4N})}{L(1, \chi_{-4N})} - \frac{12}{\pi^2} \zeta'(2),$$

where γ is the Euler-Mascheroni constant and $L(1, \chi_{-4N})$ is the L -function corresponding to the quadratic character mod $-4N$.

Based on this result, Borwein and Choi posed the following:

Conjecture 1.2. *For any squarefree N ,*

$$\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left(\prod_{p|2N} \frac{2p}{p+1} \right) x \log x.$$

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The main result in [10] is the following.

Theorem 1.3. *Let $Q(x, y) = x^2 + Ny^2$ for a squarefree integer N with $-N \not\equiv 1 \pmod{4}$. Let $r_{2,N}(n)$ denote the number of solutions to $n = Q(x, y)$ (counting signs and order). Then*

$$\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left(\prod_{p|2N} \frac{2p}{p+1} \right) x \log x.$$

In this note, we settle the conjecture in the remaining case, namely

Theorem 1.4. *For $-N \equiv 1 \pmod{4}$, we have*

$$\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left(\prod_{p|2N} \frac{2p}{p+1} \right) x \log x.$$

2. PRELIMINARIES

Let $Q(x, y) = ax^2 + bxy + cy^2$ denote a positive definite integral quadratic form with discriminant $D = b^2 - 4ac$ and $\gcd(a, b, c) = 1$. Given Q , let κ be the largest positive integer with D/κ^2 an integer congruent to 0 or 1 modulo 4. We call κ the conductor of Q and set $d = D/\kappa^2$. Let $r(Q, n)$ be the number of representations of the integer n by the form Q . We now relate $r(Q, n)$ to counting the number of integral ideals of norm n in a given class in a generalized ideal class group.

Given $D = \kappa^2 d$ we consider ideals in \mathcal{O}_K , where $K = \mathbb{Q}(\sqrt{d})$. Let I_κ be the group of fractional ideals of \mathcal{O}_K which are quotients of ideals coprime to κ and let P_κ be the subgroup of fractional ideals which are quotients of principal ideals $\langle \alpha \rangle \in I_\kappa$, where $\alpha \in \mathbb{Z} + \kappa\mathcal{O}$. Then set $CL_\kappa(K) = I_\kappa/P_\kappa$. The elements of $CL_\kappa(K)$ correspond bijectively to proper equivalence classes of positive definite quadratic forms of discriminant $D = \kappa^2 d$. If the proper equivalence class of Q corresponds to the ideal class \mathfrak{c} , then by [3], page 219, we have

$$r(Q, n) = \sum_{r|\kappa} w((\kappa/r)^2 d) J(\mathfrak{c}_r, n/r^2),$$

where

$$w(D) = \begin{cases} 6 & \text{if } D = -3, \\ 4 & \text{if } D = -4, \\ 2 & \text{otherwise.} \end{cases}$$

Also $J(\mathfrak{c}_r, n)$ is the number of integral ideals of norm n in the class \mathfrak{c}_r , where \mathfrak{c}_r is the image of \mathfrak{c} under the natural homomorphism $CL_\kappa(K) \rightarrow CL_{\kappa/r}(K)$. For the form $Q(x, y) = x^2 + Ny^2$ where $-N \equiv 1 \pmod{4}$, the conductor $\kappa = 2$ and so we have

$$\begin{aligned} r_{2,N}(n) &= w(-4N)J(\mathfrak{c}, n) + w(-N)J(\mathfrak{c}_2, n/4) \\ &= 2J(\mathfrak{c}, n) + w(-N)J(\mathfrak{c}_2, n/4), \end{aligned}$$

where \mathfrak{c}_2 is the image under $CL_2(K) \rightarrow CL_1(K)$, that is, \mathfrak{c}_2 is a class in the ideal class group of $K = \mathbb{Q}(\sqrt{-N})$.

We now discuss a classical result of Rankin [11] and Selberg [12] which estimates the size of Fourier coefficients of a modular form. Specifically, if $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$ is a nonzero cusp form of weight k on $\Gamma_0(N)$, then

$$\sum_{n \leq x} |a(n)|^2 = \alpha \langle f, f \rangle x^k + O(x^{k-\frac{2}{5}}),$$

where $\alpha > 0$ is an absolute constant and $\langle f, f \rangle$ is the Petersson scalar product. In particular, if f is a cusp form of weight 1, then $\sum_{n \leq x} |a(n)|^2 = O(x)$. One can adapt their result to say the following. Given two cusp forms of weight k on a suitable congruence subgroup of $\Gamma = SL_2(\mathbb{Z})$, say $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ and $g(z) = \sum_{n=1}^{\infty} b(n)e^{2\pi inz}$, then

$$\sum_{n \leq x} a(n)\overline{b(n)}n^{1-k} = Ax + O(x^{\frac{3}{5}}),$$

where A is a constant. In particular, if f and g are cusp forms of weight 1, then $\sum_{n \leq x} a(n)\overline{b(n)} = O(x)$.

We conclude this section with a relationship between genus characters of generalized ideal class groups and the poles of the Rankin-Selberg convolution of L-functions. Recall that a group homomorphism $\chi : I_2 \rightarrow S^1$ is an ideal class character if it is trivial on P_2 , i.e.

$$\chi(\langle a \rangle) = 1$$

for $a \equiv 1 \pmod{\langle 2 \rangle}$. Thus an ideal class character is a character on the generalized class group I_2/P_2 . Recall also that a genus character (see Chapter 12, section 5 in [5]) is an ideal class character of order at most two.

Let us also recall the notion of the Rankin-Selberg convolution of two L-functions. For squarefree N , consider two ideal class characters χ_1, χ_2 for $CL_2(K)$, the generalized ideal class group of $K = \mathbb{Q}(\sqrt{-N})$, and their associated Hecke L-series

$$L_2(s, \chi_1) = \sum_{(\mathfrak{a}, 2)=1} \frac{\chi_1(\mathfrak{a})}{N(\mathfrak{a})^s},$$

$$L_2(s, \chi_2) = \sum_{(\mathfrak{a}, 2)=1} \frac{\chi_2(\mathfrak{a})}{N(\mathfrak{a})^s}$$

which converge absolutely in some right half-plane. We form the convolution L-series by multiplying the coefficients

$$L_2(s, \chi_1 \otimes \chi_2) = \sum_{(\mathfrak{a}, 2)=1} \frac{\chi_1(\mathfrak{a})\chi_2(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

The following result describes a relationship between genus characters χ and the orders of poles of $L_2(s, \chi \otimes \chi)$. The proof is similar to that of Proposition 2.4 in [10].

Proposition 2.1. *Let χ be an ideal class character for $CL_2(K)$, $-N \equiv 1 \pmod{4}$, and let $L_2(s, \chi)$ be the associated Hecke L-series. Then χ is a genus character if and only if $L_2(s, \chi \otimes \chi)$ has a double pole at $s = 1$.*

Remark 2.2. By Proposition 2.1, if χ is a non-genus character, then $L_2(s, \chi \otimes \chi)$ has at most a simple pole at $s = 1$.

3. PROOF OF THEOREM 1.4

Proof. As the proof is similar to that of Theorem 1.3 in [10], we sketch the relevant details. If $-N \equiv 1 \pmod{4}$, then the discriminant of $K = \mathbb{Q}(\sqrt{-N})$ is $-N$. We also

assume that t is the number of distinct prime factors of N and so the discriminant $-N$ also has t distinct prime factors. For $K = \mathbb{Q}(\sqrt{-N})$, consider the zeta function

$$\zeta_K(s, 2) = \sum_{(\mathfrak{a}, 2)=1} \frac{1}{N(\mathfrak{a})^s},$$

where the sum is over those ideals \mathfrak{a} of \mathcal{O}_K prime to 2. We now split up $\zeta_K(s, 2)$, according to the classes \mathfrak{c}_i of the generalized ideal class group $CL_2(K)$, into the partial zeta functions (see page 161 of [7])

$$\zeta_{\mathfrak{c}_i}(s) = \sum_{\mathfrak{a} \in \mathfrak{c}_i} \frac{1}{N(\mathfrak{a})^s}$$

so that $\zeta_K(s, 2) = \sum_{i=0}^{h_2-1} \zeta_{\mathfrak{c}_i}(s)$, where h_2 is the order of $CL_2(K)$.

Let \mathfrak{c} be the ideal class in $CL_2(K)$ which corresponds to the proper equivalence class of $Q(x, y) = x^2 + Ny^2$. Now let χ be an ideal class character of $CL_2(K)$ and consider the Hecke L-series for χ , namely

$$L_2(s, \chi) = \sum_{(\mathfrak{a}, 2)=1} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

We may now rewrite the Hecke L-series as

$$L_2(s, \chi) = \sum_{i=0}^{h_2-1} \chi(\mathfrak{c}_i) \zeta_{\mathfrak{c}_i}(s).$$

Therefore, summing over all ideal class characters of $CL_2(K)$, we have

$$\sum_{\chi} \overline{\chi}(\mathfrak{c}) L_2(s, \chi) = \sum_{i=0}^{h_2-1} \zeta_{\mathfrak{c}_i}(s) \left(\sum_{\chi} \overline{\chi}(\mathfrak{c}) \chi(\mathfrak{c}_i) \right).$$

The inner sum is non-zero precisely when $\mathfrak{c} = \mathfrak{c}_i$. Thus we have

$$\zeta_{\mathfrak{c}}(s) = \frac{1}{h_2} \sum_{\chi} \overline{\chi}(\mathfrak{c}) L_2(s, \chi)$$

and so

$$\zeta_{\mathfrak{c}}(s) = \frac{1}{h_2} (L_2(s, \chi_0) + \overline{\chi_1}(\mathfrak{c}) L_2(s, \chi_1) + \cdots + \overline{\chi_{h_2-1}}(\mathfrak{c}) L_2(s, \chi_{h_2-1})).$$

As χ_0 is the trivial character, $L_2(s, \chi_0) = \zeta_K(s, 2)$. Comparing n^{th} coefficients, we have

$$J(\mathfrak{c}, n) = \frac{1}{h_2} (a_n + b_1(n) + \cdots + b_{h_2-1}(n)),$$

where a_n is the number of integral ideals of \mathcal{O}_K prime to 2 and of norm n , and the b_i 's are coefficients of weight 1 cusp forms (see [2]). Recall that we also have

$$r_{2,N}(n) = 2J(\mathfrak{c}, n) + w(-N)J(\mathfrak{c}_2, n/4)$$

and so

$$r_{2,N}(n) = \frac{2}{h_2} (a_n + b_1(n) + \cdots + b_{h_2-1}(n)) + w(-N)J(\mathfrak{c}_2, n/4).$$

Thus

$$\begin{aligned} \sum_{n \leq x} r_{2,N}(n)^2 &= \frac{4}{h_2^2} \left(\sum_{n \leq x} a_n^2 + \sum_{\substack{i \\ n \leq x}} b_i(n)^2 + 2 \sum_{\substack{i \\ n \leq x}} a_n b_i(n) + \sum_{\substack{i \neq j \\ n \leq x}} b_i(n) b_j(n) \right) \\ &\quad + \frac{4}{h_2} \sum_{n \leq x} \left(a_n + b_1(n) + \cdots + b_{h_2-1}(n) \right) w(-N) J(\mathfrak{c}_2, n/4) \\ &\quad + \sum_{n \leq x} w(-N)^2 J(\mathfrak{c}_2, n/4)^2. \end{aligned}$$

Assume $-N \equiv 1 \pmod 8$. Applying the main theorem in [6] to the Dirichlet series $\sum_{n=1}^\infty \frac{a_n^2}{n^s}$, we obtain

$$\sum_{n \leq x} a_n^2 \sim Ax \log x,$$

where $A = \frac{1}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p|N} \frac{p}{p+1}$. As $-N$ has t distinct prime factors, we have 2^t genus characters for $CL(K)$ where $K = \mathbb{Q}(\sqrt{-N})$. By [7] (see Theorem 1, page 127), we have 2^t genus characters for $CL_2(K)$. We now must estimate $\sum_{n \leq x} b_i(n)^2$.

Let us now assume that the first $2^t - 1$ terms arise from L-functions associated to genus characters. By Proposition 2.1 and an application of Perron’s formula, we obtain

$$\sum_{n \leq x} b_i(n)^2 \sim Ax \log x.$$

As this estimate holds for each i such that $1 \leq i \leq 2^t - 1$, the term $Ax \log x$ appears 2^t times in the estimate of $\sum_{n \leq x} r_{2,N}(n)^2$. By Remark 2.2 and the Rankin-Selberg estimate, the remaining terms are all $O(x)$. Thus

$$\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{4}{h_2^2} \left(2^t \frac{1}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p|N} \frac{p}{p+1} \right) x \log x.$$

By [4], we have $L(1, \chi_{-N}) = \frac{h\pi}{\sqrt{N}}$, where h is the class number of K and $h_2 = h$. Thus

$$\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left(\prod_{p|2N} \frac{2p}{p+1} \right) x \log x.$$

For $-N \equiv 5 \pmod 8$, we have $h_2 = 3h$ and again by [6],

$$\sum_{n \leq x} a_n^2 \sim \left(\frac{9}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p|N} \frac{p}{p+1} \right) x \log x.$$

Thus

$$\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left(\prod_{p|2N} \frac{2p}{p+1} \right) x \log x.$$

□

Remark 3.1. We would like to mention another approach which confirms Theorems 1.3 and 1.4. Let $Q \in \mathbb{Z}^{2 \times 2}$ be a non-singular symmetric matrix with even diagonal entries and $q(\mathbf{x}) = \frac{1}{2} Q[\mathbf{x}] = \frac{1}{2} \mathbf{x}^T Q \mathbf{x}$, $\mathbf{x} \in \mathbb{Z}^2$, the associated quadratic form in two

variables. Let $r(Q, n)$ denote the number of representations of n by the quadratic form Q . Now consider the theta function

$$\theta_Q(z) = \sum_{\mathbf{x} \in \mathbb{Z}^2} e^{\pi i z Q[\mathbf{x}]}$$

The Dirichlet series associated with the automorphic form θ_Q is

$$(4\pi)^{-1/2} \zeta_Q\left(\frac{1}{2} + s\right),$$

where

$$\zeta_Q(s) = \sum_{n=1}^{\infty} \frac{r(Q, n)}{n^s} = \sum_{\mathbf{x} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}} q(\mathbf{x})^{-s}$$

for $\Re(s) > 1$. A careful and involved application of the Rankin-Selberg method to the above Dirichlet series (see Theorems 2.1 and 5.1 in [8] and Theorem 5.2 in [9]) combined with a Tauberian argument yields the following (see Theorem 6.1 in [8]):

$$\sum_{n \leq x} r(Q, n)^2 \sim A_Q x \log x,$$

where

$$A_Q = 12 \frac{A(q)}{q} \prod_{p|q} \left(1 + \frac{1}{p}\right)^{-1}.$$

Here $q = \det Q$ and $A(q)$ denotes the multiplicative function defined by

$$A(p^e) = 2 + \left(1 - \frac{1}{p}\right)(e - 1),$$

where p is an odd prime, $e \geq 1$, and

$$A(2^e) = \begin{cases} 1 & \text{if } e \leq 1, \\ 2 & \text{if } e = 2, \\ e - 1 & \text{if } e \geq 3. \end{cases}$$

Let us now turn to our situation. Consider $q(\mathbf{x}) = x^2 + Ny^2 = \frac{1}{2} \mathbf{x}^T Q \mathbf{x}$, where $Q = \begin{pmatrix} 2 & 0 \\ 0 & 2N \end{pmatrix}$, N squarefree. Thus $q = 4N$. Suppose N has t distinct prime factors. Then $A(4N) = 2^{t+1}$ and so

$$A_Q = \frac{3}{N} 2^{t+1} \prod_{p|2N} \left(1 + \frac{1}{p}\right)^{-1} = \frac{3}{N} \prod_{p|2N} \frac{2p}{p+1}.$$

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