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A NOTE ON THE ZERO MACH NUMBER LIMIT OF COMPRESSIBLE EULER EQUATIONS

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ABSTRACT. This note presents a short and elementary justification of the classical zero Mach number limit for isentropic compressible Euler equations with prepared initial data. We also show the existence of smooth compressible flows, with the Mach number sufficiently small, on the (finite) time interval where the incompressible Euler equations have smooth solutions.

1. INTRODUCTION

In a suitable nondimensional form (see, e.g., [5]), the compressible Euler equations for an isentropic fluid read as

$$\begin{aligned} &\rho_t + \operatorname{div}(\rho v) = 0, \\ &(\rho v)_t + \operatorname{div}(\rho v \otimes v) + \epsilon^{-2}\nabla p = 0. \end{aligned} \tag{1.1}$$

Here $\rho = \rho(x,t)$ is the density function of $(x,t) \in \Omega \times [0,\infty)$ with $\Omega \subset \mathbb{R}^d$, $v = v(x,t) = (v_1, v_2, \cdots, v_d)(x,t)$ is the fluid velocity, ϵ is the Mach number, and the pressure $p = p(\rho)$ is a given strictly increasing smooth function of $\rho > 0$. Throughout this paper, Ω is assumed to be $\mathbb{R}^d$ or the d-dimensional torus and (1.1) is supplemented with initial data

$$(\rho, v)(x,0) = (\bar{\rho}(x,\epsilon), \bar{v}(x,\epsilon)). \tag{1.2}$$

The interest is to investigate the limit when ϵ goes to zero. This limit problem was first studied in [3, 4, 5] and has attracted much attention since then. The interested reader is referred to [6] for a comprehensive survey of the literature.

In this note, we present a short and elementary approach to the above limit problem. This approach is based on the convergence-stability lemma (Lemma 9.1) in [9] — a continuation principle first formulated in [9] for general (hyperbolic) singular limt problems.

Our result can be roughly stated as follows. Suppose the initial data in (1.2) are smooth and have the form

$$\bar{\rho}(x,\epsilon) = \rho_0 + O(\epsilon^2), \qquad \bar{v}(x,\epsilon) = \bar{v}(x,0) + O(\epsilon)$$

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with $\rho_0 > 0$ a constant and $\bar{v}(x,0)$ solenoidal. Let $[0,T_*]$ be a (finite) time interval where the incompressible Euler equations

$$\begin{aligned} \operatorname{div} w = 0, \\ \rho_0(w_t + w\cdot\nabla w) + \nabla\pi = 0 \end{aligned} \tag{1.3}$$

with $w(x,0) = \bar{v}(x,0)$ have a smooth solution (π, w). Then, for ϵ sufficiently small, the compressible Euler equations (1.1) with (1.2) have a unique smooth solution $(\rho^\epsilon, v^\epsilon)$ defined for $(x,t) \in \Omega \times [0,T_*]$ and satisfying

$$\rho^\epsilon = \rho_0 + O(\epsilon^2), \qquad v^\epsilon = w + O(\epsilon).$$

For details and comparison to existing results, see the next section. For the existence of the smooth solution (π, w), see [2, 7].

Note that our analysis can be extended to the same limit problem of the Navier-Stokes equations and requires T_* to be finite. However, for the Navier-Stokes equations of heat-conducting fluid flows, it is possible to study the Mach number limit for $T_* = \infty$ [1].

Notation. $|U|$ denotes some norm of a vector or matrix U. For a nonnegative integer k, $H^k = H^k(\Omega)$ denotes the usual L^2-type Sobolev space of order k. We write $||\cdot||_k$ for the standard norm of H^k and $||\cdot||$ for $||\cdot||_0$. When U is a function of another variable t as well as x, we write $||U(\cdot,t)||_k$ to recall that the norm is taken with respect to x while t is viewed as a parameter. In addition, we denote by $C\big([0,T],\mathbf{X}\big)$ (resp. $C^1\big([0,T],\mathbf{X}\big)$) the space of continuous (resp. continuously differentiable) functions on $[0,T]$ with values in a Banach space $\mathbf{X}$.

2. Results

First of all, we rewrite the compressible Euler equations (1.1) as a symmetrizable hyperbolic system. Since $p = p(\rho)$ is strictly increasing, it has an inverse $\rho = \rho(p)$. Set $q(p) = [\rho(p)p'(\rho(p))]^{-1}$. Then the Euler equations (1.1) for smooth solutions can be rewritten as

$$\begin{aligned} q(p)\big(p_t + v\cdot\nabla p\big) + \operatorname{div} v = 0, \\ \rho(p)\big(v_t + v\cdot\nabla v\big) + \epsilon^{-2}\nabla p = 0. \end{aligned} \tag{2.1}$$

Following [5], we introduce

$$\tilde{p} = (p - p_0)/\epsilon \qquad \text{and} \qquad \tilde{v} = v$$

with $p_0 = p(\rho_0) > 0$. Then (2.1) can be rewritten as

$$\begin{aligned} q(p_0 + \epsilon\tilde{p})\big(\tilde{p}_t + \tilde{v}\cdot\nabla\tilde{p}\big) + \epsilon^{-1}\operatorname{div}\tilde{v} = 0, \\ \rho(p_0 + \epsilon\tilde{p})\big(\tilde{v}_t + \tilde{v}\cdot\nabla\tilde{v}\big) + \epsilon^{-1}\nabla\tilde{p} = 0. \end{aligned} \tag{2.2}$$

In the vector form, we arrive at, for $U = (\tilde{p}, \tilde{v})$,

$$A_0(\epsilon\tilde{p})U_t + \sum_{j=1}^{d}\big(\tilde{v}_j A_0(\epsilon\tilde{p}) + \epsilon^{-1}C_j\big)U_{x_j} = 0. \tag{2.3}$$

Here $A_0 = A_0(\epsilon\tilde{p})$ is diagonal and positive definite, and C_j is constant and symmetric.

For $\epsilon \in (0,1]$, the state space for (2.3) is obviously $G = (-p_0, +\infty) \times \mathbb{R}^d$. Assume the initial data $\bar{U} = (\bar{p}(x,\epsilon), \bar{v}(x,\epsilon))$ for (2.3) satisfies

$$\left\{\bar{U}(x,\epsilon) : (x,\epsilon) \in \Omega \times (0,1]\right\} \in [-p_0/2, C] \times [-C, C]^d \equiv G_0 \subset\subset G$$

with $C > 0$ a constant and $\bar{U}(\cdot,\epsilon) \in H^s$ with $s > d/2+1$ an integer. Fix ϵ. According to the local existence theory for IVPs of symmetrizable hyperbolic systems (see Theorem 2.1 in [5]), for each convex open subset G_1 satisfying $G_0 \subset\subset G_1 \subset\subset G$, there exists $T > 0$ so that (2.3) has a unique classical solution

$$U^\epsilon \in C([0,T], H^s) \qquad \text{and} \qquad U^\epsilon(x,t) \in G_1 \quad \forall (x,t) \in \Omega \times [0,T].$$

Define

$$T_\epsilon = \sup\left\{T > 0 : U^\epsilon \in C([0,T], H^s), \quad U^\epsilon(x,t) \in G_1 \quad \forall (x,t) \in \Omega \times [0,T]\right\}. \tag{2.4}$$

Namely, $[0, T_\epsilon)$ is the maximal time interval of H^s existence. Note that T_ϵ depends on G_1 and may tend to zero as ϵ goes to 0.

To show that $\underline{\lim}_{\epsilon\to 0} T_\epsilon > 0$, we will prove the following theorem in the next section.

Theorem 2.1. *Suppose the initial data $\bar{U}(x,\epsilon)$ satisfy $\|\bar{p}(\cdot,\epsilon),\ \bar{v}(\cdot,\epsilon) - \bar{v}(\cdot,0)\|_s = O(\epsilon)$ and $\operatorname{div} \bar{v}(\cdot,0) = 0$. Let (π, w) be a smooth solution to the incompressible Euler equations* (1.3) *with $w(x,0) = \bar{v}(x,0)$. If*

$$(\pi, w) \in C([0,T_*], H^{s+1}) \cap C^1([0,T_*], H^s)$$

with $T_* > 0$ finite, then there are positive constants ϵ_0 and K such that, for $\epsilon \le \epsilon_0$,

$$\|U^\epsilon(\cdot,t) - (\epsilon\pi, w)(\cdot,t)\|_s \le K\epsilon$$

for $t \in [0, \min\{T_*, T_\epsilon\})$.

Having this theorem, we slightly modify the argument in [8] to prove

Theorem 2.2. *Under the conditions of Theorem* 2.1, *for any G_1 satisfying*

$$G_0 \bigcup_{x,t,\epsilon} \{(\epsilon\pi, w)(x,t)\} \subset\subset G_1 \subset\subset G, \tag{2.5}$$

$T_\epsilon(G_1) > T_*$ holds for $\epsilon > 0$ sufficiently small.

Proof. Otherwise, there is a G_1 satisfying (2.5) and a sequence $\{\epsilon_k\}_{k\ge 1}$ such that $\lim_{k\to\infty} \epsilon_k = 0$ and $T_{\epsilon_k} = T_{\epsilon_k}(G_1) \le T_*$. Then there exists $\tilde{G}$ satisfying

$$\bigcup_{x,t,\epsilon} \{(\epsilon\pi, w)(x,t)\} \subset\subset \tilde{G} \subset\subset G_1.$$

Moreover, we deduce from Sobolev's embedding theorem and Theorem 2.1 that

$$|U^\epsilon(x,t) - (\epsilon\pi, w)(x,t)| \le \text{const.} \|U^\epsilon(\cdot,t) - (\epsilon\pi, w)(\cdot,t)\|_s \le \text{const.} K\epsilon.$$

Thus, there is a k such that $U^{\epsilon_k}(x,t) \in \tilde{G}$ for all $(x,t) \in \Omega \times [0, T_{\epsilon_k})$. On the other hand, it follows from

$$\|U^\epsilon(\cdot,t)\|_s \le \|U^\epsilon(\cdot,t) - (\epsilon\pi, w)(\cdot,t)\|_s + \|(\epsilon\pi, w)(\cdot,t)\|_s \le K\epsilon_0 + \|(\epsilon\pi, w)(\cdot,t)\|_s$$

that $\|U^{\epsilon_k}(\cdot,t)\|_s$ is bounded uniformly with respect to $t \in [0, T_{\epsilon_k})$. Now we could apply Theorem 2.1 in [5], beginning at a time t less than T_{ϵ_k} (k is fixed here!), to continue the solution beyond $T_{\epsilon_k}(G_1)$. This contradicts the definition of $T_{\epsilon_k}(G_1)$ in (2.4) and, hence, the proof is complete. □

We make two remarks about the above theorems.

Remark 2.1. In case (π, w) is defined globally in time and the conditions of Theorem 2.1 hold for $T_* = \infty$, we actually prove the following existence result for (2.3): For any $T < \infty$, there is a neigborhood of $\epsilon = 0$ such that for all ϵ in the neighborhood, (2.3) with initial data $\bar{U}(x, \epsilon)$ has a unique classical solution

$$U^\epsilon \in C([0, T], H^s).$$

Moreover, the error estimate in Theorem 2.1 holds, for $t \le T$, with K depending on T.

Remark 2.2. In comparison with previous works [3, 4, 5, 6], especially [4, 5], our approximate solution $(\epsilon\pi, w)$ is the simplest. Thus, the regularity requirement on the initial data and approximate solution is the least. Moreover, the establishment of Theorem 2.2 simplifies the analysis considerably.

Finally, we mention that Theorem 2.2 is a special case of the convergence-stability lemma (Lemma 9.1) in [9] — a continuation principle formulated in [9] for general (hyperbolic) singular limit problems.

3. A proof of Theorem 2.1

This section is devoted to proving Theorem 2.1. Note that

$$p_\epsilon = \epsilon\pi \qquad \text{and} \qquad v_\epsilon = w$$

satisfy

$$q(p_0 + \epsilon p_\epsilon)(p_{\epsilon t} + v_\epsilon \cdot \nabla p_\epsilon) + \epsilon^{-1} \operatorname{div} v_\epsilon = \epsilon q(p_0 + \epsilon^2\pi)(\pi_t + w \cdot \nabla\pi),$$
$$\rho(p_0 + \epsilon p_\epsilon)(v_{\epsilon t} + v_\epsilon \cdot \nabla v_\epsilon) + \epsilon^{-1}\nabla p_\epsilon = [\rho(p_0 + \epsilon^2\pi) - \rho(p_0)](w_t + w \cdot \nabla w).$$

Namely, $U_\epsilon = (\epsilon\pi, w)$ satisfies

$$U_{\epsilon t} + \sum_{j=1}^{d} A_j(U_\epsilon, \epsilon)U_{\epsilon x_j} = \begin{pmatrix} \epsilon(\pi_t + w \cdot \nabla\pi) \\ \left(1 - \dfrac{\rho(p_0)}{\rho(p_0 + \epsilon^2\pi)}\right)(w_t + w \cdot \nabla w) \end{pmatrix} \equiv R, \tag{3.1}$$

where $A_j(U, \epsilon) = v_j I_{d+1} + \epsilon^{-1} A_0(\epsilon p)^{-1} C_j$ with I_{d+1} the unit matrix of order $(d+1)$. Since $(\pi, w) \in C([0, T_*], H^{s+1}) \cap C^1([0, T_*], H^s)$ is assumed in Theorem 2.1, we have

$$\max_{t\in[0,T_*]} \|R(\cdot, t)\|_s \le C\epsilon. \tag{3.2}$$

Here and below, C denotes a generic constant that can change from line to line.

From (2.3) and (3.1) we compute that $E = U_\epsilon - U^\epsilon$ satisfies

$$E_t + \sum_j A_j(U^\epsilon, \epsilon)E_{x_j} = R + \sum_j [A_j(U^\epsilon, \epsilon) - A_j(U_\epsilon, \epsilon)]U_{\epsilon x_j}.$$

Differentiating this equation with ∂^α for any multi-index α satisfying $|\alpha| \le s$ and setting $E_\alpha = \partial^\alpha E$, we get

$$E_{\alpha t} + \sum_j A_j(U^\epsilon, \epsilon)E_{\alpha x_j} = R_\alpha + F^\alpha \tag{3.3}$$

with

$$F^\alpha = \Big\{ \sum_j [A_j(U^\epsilon, \epsilon) - A_j(U_\epsilon, \epsilon)]U_{\epsilon x_j} \Big\}_\alpha + \sum_j \big\{ A_j(U^\epsilon, \epsilon)E_{\alpha x_j} - [A_j(U^\epsilon, \epsilon)E_{x_j}]_\alpha \big\}.$$

Recall that $A_0^\epsilon \equiv A_0(h_0+\epsilon h^\epsilon)$ and $A_0^\epsilon A_j(U^\epsilon,\epsilon)(j=1,2,\cdots,d)$ are all symmetric. Let E_α^T be the transpose of E_α. Multiplying (3.3) with $E_\alpha^T A_0^\epsilon$ from the left and setting $e(E_\alpha)=E_\alpha^T A_0^\epsilon E_\alpha$, we get

$$\begin{aligned} e(E_\alpha)_t+\sum_j \left\{E_\alpha^T A_0^\epsilon A_j(U^\epsilon,\epsilon)E_\alpha\right\}_{x_j} &= 2\mathrm{Re}E_\alpha^T A_0^\epsilon(R_\alpha+F^\alpha) \\ &+E_\alpha^T\Big\{\frac{\partial A_0^\epsilon}{\partial t}+\sum_j \frac{\partial[A_0^\epsilon A_j(U^\epsilon,\epsilon)]}{\partial x_j}\Big\}E_\alpha. \end{aligned} \tag{3.4}$$

Now we estimate various terms in (3.4). Note that our estimates only need to be done for $t\in[0,\min\{T_*,T_\epsilon\})$, in which both U^ϵ and U_ϵ are regular enough and take values in a convex compact subset of the state space. In particular, we have

$$C^{-1}|E_\alpha|^2 \le e(E_\alpha)\le C|E_\alpha|^2, \tag{3.5}$$

$$2\mathrm{Re}E_\alpha^T A_0^\epsilon(R_\alpha+F^\alpha)\le C|E_\alpha|^2+C|R_\alpha|^2+C|F^\alpha|^2. \tag{3.6}$$

Moreover, since U^ϵ is an exact solution to (2.3) or (2.2), it is not difficult to get

$$\frac{\partial A_0^\epsilon}{\partial t}+\sum_j \frac{\partial[A_0^\epsilon A_j(U^\epsilon,\epsilon)]}{\partial x_j}=A_0^\epsilon \operatorname{div} v^\epsilon + A_{0t}^\epsilon + v^\epsilon\cdot\nabla A_0^\epsilon=(A_0^\epsilon-\frac{A_0'(\epsilon h^\epsilon)}{q(h_0+\epsilon h^\epsilon)})\operatorname{div} v^\epsilon.$$

Because $s>d/2+1$, we use Sobolev's embedding theorem to obtain

$$\frac{\partial A_0^\epsilon}{\partial t}+\sum_j \frac{\partial[A_0^\epsilon A_j(U^\epsilon,\epsilon)]}{\partial x_j}\le C|\operatorname{div} v^\epsilon|\le C|\operatorname{div} v_\epsilon|+C|\operatorname{div}(v_\epsilon-v^\epsilon)|\le C+C\|E\|_s. \tag{3.7}$$

Next we estimate $\|F^\alpha\|$ with the help of the Moser-type calculus inequalities in Sobolev spaces [5]. For the first term, we use the relation

$$\begin{aligned} A_j(U^\epsilon,\epsilon)-A_j(U_\epsilon,\epsilon)&=(v_j^\epsilon-v_{j\epsilon})I_{d+1}+\epsilon^{-1}[(A_0^\epsilon)^{-1}-A_0(\epsilon p_\epsilon)^{-1}]C_j \\ =&(v_j^\epsilon-v_{j\epsilon})I_{d+1}-(p^\epsilon-p_\epsilon)(A_0^{-2}A_0')(\epsilon p_\epsilon+\epsilon\theta(p^\epsilon-p_\epsilon))C_j, \end{aligned}$$

with $\theta\in[0,1]$, and the boundedness of $\|(\epsilon\pi,w)(\cdot,t)\|_{s+1}$ to conclude that

$$\begin{aligned} \|\big\{[A_j(U^\epsilon,\epsilon)-A_j(U_\epsilon,\epsilon)]U_{\epsilon x_j}\big\}_\alpha\| &\le C\|U_{\epsilon x_j}\|_s\|A_j(U^\epsilon,\epsilon)-A_j(U_\epsilon,\epsilon)\|_{|\alpha|} \\ &\le C(1+\|p_\epsilon+\theta(p^\epsilon-p_\epsilon)\|_s^s)\|E\|_{|\alpha|} \\ &\le C(1+\|E\|_s^s)\|E\|_{|\alpha|}. \end{aligned}$$

For the second term, since

$$\begin{aligned} A_j(U^\epsilon,\epsilon)E_{\alpha x_j}-[A_j(U^\epsilon,\epsilon)E_{x_j}]_\alpha &= -\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta A_j(U^\epsilon,\epsilon)\partial^{\alpha-\beta}E_{x_j} \\ =-\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta[v_j^\epsilon I_{d+1}+\epsilon^{-1}(A_0^\epsilon)^{-1}C_j]\partial^{\alpha-\beta}E_{x_j}, \end{aligned}$$

we have

$$\begin{aligned} \|A_j(U^\epsilon,\epsilon)E_{\alpha x_j}-[A_j(U^\epsilon,\epsilon)E_{x_j}]_\alpha\| &\le C(\|E\|_s+\|p^\epsilon\|_s^s)\|E_{x_j}\|_{|\alpha|-1} \\ &\le C(1+\|E\|_s^s)\|E\|_{|\alpha|}. \end{aligned}$$

Putting the above two estimates together, we get

$$\|F^\alpha\|\le C(1+\|E\|_s^s)\|E\|_{|\alpha|}. \tag{3.8}$$

Substituting (3.6)–(3.8) into (3.4) and integrating it over $x \in \Omega$ yields

$$\frac{d}{dt}\int_\Omega e(E_\alpha)dx \le C\|R_\alpha\|^2 + C(1+\|E\|_s^{2s})\|E\|_{|\alpha|}^2. \tag{3.9}$$

With (3.5), we integrate (3.9) from 0 to T with $T < \min\{T_\epsilon, T_*\}$ to obtain

$$\|E_\alpha(T)\|^2 \le C\|E_\alpha(0)\|^2 + C\int_0^T \|R_\alpha(t)\|^2 dt + C\int_0^T (1+\|E(t)\|_s^{2s})\|E(t)\|_{|\alpha|}^2 dt.$$

Summing up this inequality for all α with $|\alpha| \le s$, we get

$$\|E(T)\|_s^2 \le C\|E(0)\|_s^2 + C\int_0^{T_*} \|R(t)\|_s^2 dt + C\int_0^T (1+\|E(t)\|_s^{2s})\|E(t)\|_s^2 dt. \tag{3.10}$$

Since $\|E(0)\|_s^2 + \int_0^{T_*} \|R(t)\|_s^2 dt = O(\epsilon^2)$, we apply Gronwall's lemma to (3.10) to get

$$\|E(T)\|_s^2 \le C\epsilon^2 \exp\left[C\int_0^T (1+\|E(t)\|_s^{2s})dt\right] \equiv \Phi(T). \tag{3.11}$$

Thus, we have

$$\Phi'(t) = C(1+\|E(t)\|_s^{2s})\Phi(t) \le C\Phi(t) + C\Phi^{s+1}(t).$$

Applying the nonlinear Gronwall-type inequality in [8] to this inequality yields

$$\|E(t)\|_s^2 \le \Phi(t) \le \exp(CT_*)$$

for all $t \in [0, \min\{T_\epsilon, T_*\})$ if $\Phi(0) = C\epsilon^2 < \exp(-CT_*)$. Because of (3.11), there exists a constant K, independent of ϵ, such that

$$\|E(t)\|_s \le K\epsilon$$

for all $t \in [0, \min\{T_\epsilon, T_*\})$. This completes the proof.

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