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GENERALIZED SUBDIFFERENTIAL OF THE DISTANCE FUNCTION

S. DUTTA

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ABSTRACT. We derive the *proximal normal formula* for almost proximinal sets in a smooth and locally uniformly convex Banach space. Our technique leads us to show the generic Fréchet smoothness of the distance function in the case the norm is Fréchet smooth, and we derive a necessary and sufficient condition for the convexity of a Chebyshev set in a Banach space X with norms on X and X^* locally uniformly convex.

1. Introduction

Let X be a real Banach space. The closed unit ball of X will be denoted by X_1. For a closed set K in X and $x \in X$, we denote the distance function of K at x by $d_K(x) = \inf\{\|x-k\| : k \in K\}$. d_K is a 1-Lipschitz function on X. The metric projection of x onto K is $P_K(x) = \{k \in K : \|x-k\| = d_K(x)\}$. The set K is called proximinal (Chebyshev) if for every $x \in X \setminus K$, $P_K(x)$ is nonempty (singleton). K will be called almost proximinal if $P_K(x)$ is nonempty for a dense set of $x \in X \setminus K$.

Let $h : X \to \mathbb{R}$ be a Lipschitz function. For $x, y \in X$ we define

$$h^0(x,y) = \limsup_{z\to x,\, t\to 0^+} \frac{h(z+ty)-h(z)}{t}$$

and the generalized subdifferential of h at $x \in X$ is defined as

$$\partial h(x) = \{f \in X^* : f(y) \le h^0(x,y)\ \forall y \in X\}.$$

Given a nonempty closed set K in X and the distance function d_K at x there is a geometrical object called the *normal cone* at x which is defined as

$$N_K(x) = \overline{\bigcup_{\lambda\ge 0} \lambda \partial d_K(x)}^{w^*},$$

the w^*-closed convex cone generated by $\partial d_K(x)$. In $\mathbb{R}^n$, the generalized subdifferential of d_K at $x \in bdyK$ has a geometrical formulation as the convex hull of the origin and the cluster points of $\frac{v_n}{\|v_n\|}$ where $v_n \perp K$ at points $x_n \in bdyK$ as $x_n \to x$ and $\|v_n\| \to 0$. The corresponding expression for the normal cone $N_K(x)$ in terms of approximating normals to K in $\mathbb{R}^n$ is called the *proximal normal formula*.

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The best known infinite-dimensional formulation of the *proximal normal formula* in Banach space was given by Borwein and Giles in [1]. In particular they derived the formula for two different cases: For an almost proximinal set using uniform Gâteux smoothness of norm on X and for nonempty closed sets in a smooth reflexive Banach space with Kadec norm.

In the present note, we show that one can obtain the *proximal normal formula* for an almost proximinal set if the norm on X is smooth and locally uniformly convex. Our technique is based on observing density of the set $E_1(K)$, where $E_1(K)$ denotes the set of points in $X \setminus K$ for which every minimizing sequence in K converges to a unique nearest point. In the first section, we show that in a smooth Banach space if $E_1(K)$ is dense, then we can give an explicit description of the generalized subdifferential of the distance function and thereby obtain the *proximal normal formula*. In the second section, we show that for any almost proximinal set K in X, a sufficient condition for $E_1(K)$ to be dense is the local uniform convexity (LUR) of the norm on X.

Differentiability properties of the distance function had been of great interest in studies in optimization theory as it relates to the famous problem of convexity of Chebyshev sets. The landmark theorem in this direction is that of Vlasov [5]: *If X^* is strictly convex, then every Chebyshev set in X with continuous metric projection is convex.* Fitzpatrick [4] observed that in a Banach space with both the norms on X and X^* Fréchet smooth, a differentiability condition on the distance function implies the convexity of Chebyshev sets. Note that Fréchet smoothness of the norm on X^* already implies reflexivity of X. Here we show that our technique leads to generic (Fréchet) smoothness of the distance function in the case the norm is (Fréchet) smooth. This improves a lot on the result of [1] where the authors derived it for Hilbert spaces, and this helps us to give a necessary and sufficient condition for the convexity of a Chebyshev set in Banach spaces where the norms on X and X^* are LUR.

2. The proximal normal formula

We take $E(K)$ to be the set of points in $X \setminus K$ which has nearest points in K and $E_1(K)$ to be the set $x \in E(K)$ such that every minimizing sequence for x in K converges to a unique nearest point of x. By $\mathcal{D}$ we will mean the duality map on X, that is, $\mathcal{D}(x) = \{f \in X_1^* : f(x) = \|x\|\}$. For explanations of the properties of the generalized subdifferential we use in this note, see [2].

Lemma 1. *Let $x \in E_1(K)$. Then*

$$\partial d_K(x) \subseteq \mathcal{D}(x - P_K(x)).$$

Equality holds if the norm on X is smooth at $x - P_K(x)$. Moreover, if the norm on X is Fréchet smooth at $x - P_K(x)$, d_K is Fréchet smooth at x.

Proof. Let $f \in \partial d_K(x)$. Since $\mathcal{D}(x - P_K(x))$ is a w^*-compact convex set, it is enough to show that given any $y \in X$, $\|y\| = 1$, there is a $g \in \mathcal{D}(x - P_K(x))$ such that $f(y) \leq g(y)$.

By definition of $\partial d_K(x)$, given any $\varepsilon > 0$ there exist $z_n \to x$ and $t_n \to 0^+$ such that

$$f(y) - \varepsilon \leq \frac{d_K(z_n + t_n y) - d_K(z_n)}{t_n}.$$

Get $k_n \in K$ such that $\|z_n - k_n\| < d_K(z_n) + t_n^2$. Then $\{k_n\}$ is minimizing for x as well and hence $k_n \to P_K(x)$. Now

$$f(y) - \varepsilon < \frac{\|z_n + t_n y - k_n\| - \|z_n - k_n\| + t_n^2}{t_n}.$$

By the mean value property of subdifferentials (see [2, Theorem 2.3.7]), there exist $g_n \in \mathcal{D}(w_n)$ such that $g_n(t_n y) = \|z_n + t_n y - k_n\| - \|z_n - k_n\|$, where w_n lies on the line $[z_n - k_n, z_n + t_n y - k_n]$. Thus $f(y) - \varepsilon < g_n(y) + t_n$. Let g be a w^*-cluster point of $\{g_n\}$. Since $w_n \to x - P_K(x)$ in norm, by the upper semicontinuity of $\mathcal{D}$, we have $g \in \mathcal{D}(x - P_K(x))$ and $f(y) - \varepsilon \le g(y)$. Thus we have the desired result.

If the norm on X is smooth at $x - P_K(x)$, $\mathcal{D}(x - P_K(x))$ is a singleton and so is $\partial d_K(x)$. Note that this implies d_K is smooth at x.

If the norm is Fréchet smooth at $x - P_K(x)$, note that for $f \in \partial d_K(x)$ and any $z \in X$, $\|z\| = 1$, we have

$$f(z) = \lim_{n\to\infty} \frac{d_K(x + \frac{1}{n}z) - d_K(x)}{1/n} \le \lim_{n\to\infty} \frac{\|x - P_K(x) + \frac{1}{n}z\| - \|x - P_K(x)\|}{1/n}$$

where the right-hand side converges to $f(z)$ uniformly over all $\|z\| = 1$. Thus d_K is Fréchet smooth at x. □

Let $E_1(K)$ be dense in $X \setminus K$. For $x \in X \setminus K$, denote by $D_x(K)$ the w^*-cluster points of $\mathcal{D}(y_n - P_K(y_n))$, where $y_n \in E_1(K)$ and $y_n \to x$. Our main result in this section is the following:

Theorem 2. *Let X be smooth. Let K be a closed set in X such that $E_1(K)$ is dense in $X \setminus K$. Then for any $x \in X \setminus K$ we have*

$$\partial d_K(x) = \overline{co}^{w^*}\{D_x(K)\}.$$

Thus we have a proximal normal formula in X for X smooth. This reads as follows:

Suppose K is a closed set in X such that $E_1(K)$ is dense in $X \setminus K$. Then for $x \in bdyK$, $N_K(x)$ is the w^*-closed convex cone generated by the origin and $D_x(K)$.

Proof. Let $f \in D_x(K)$. Then there is a sequence $\{y_n\} \in E_1(K)$, $y_n \to x$, $f_n \in \mathcal{D}(y_n - P_K(y_n))$ such that $f_n \to f$ in the w^*-topology. By Lemma 1, $f_n \in \partial d_K(y_n)$. By upper semicontinuity of ∂d_K, we have $f \in \partial d_K(x)$. Since $\partial d_K(x)$ is a w^*-closed convex set, $\overline{co}^{w^*}\{D_x(K)\} \subseteq \partial d_K(x)$.

Conversely, let $f \in \partial d_K(x)$. As before, it is enough to show that for any $y \in X$, $\|y\| = 1$, there exists $g \in \overline{co}^{w^*}\{D_x(K)\}$ such that $f(y) \le g(y)$.

Given $\varepsilon > 0$ there are $z_n \in X \setminus K$, $z_n \to x$ and $t_n \to 0^+$ such that for each n,

$$f(y) - \varepsilon/2 \le \frac{d_K(z_n + t_n y) - d_K(z_n)}{t_n}.$$

Choose $y_n \in E_1(K)$ such that $\|z_n + t_n y - y_n\| < t_n^2$. Then $d_K(z_n + t_n y) \le d_K(y_n) + t_n^2$ and $d_K(z_n) > d_K(y_n - t_n y) - t_n^2$. Thus for all sufficiently large n,

$$f(y) - \varepsilon/2 \le \frac{d_K(y_n) - d_K(y_n - t_n y)}{t_n} + 2t_n \le d_K^0(y_n, y) + \varepsilon/2 + 2t_n.$$

Since by Lemma 1, $\partial d_K(y_n) = \mathcal{D}(y_n - P_K(y_n))$ is a singleton we have $d_K^0(y_n, y) = g_n(y)$ where $g_n(y_n - P_K(y_n)) = \|y_n - P_K(y_n)\|$. Let g be a w^*-cluster point of g_n. We then have $f(y) - \varepsilon \le g(y)$ and the result follows. □

3. Density of $E_1(K)$

In this section we investigate sufficient conditions on X such that for every almost proximinal set K, $E_1(K)$ is dense in $X \setminus K$. As mentioned in the introduction, this sufficient condition turns out to be local uniform convexity (LUR) of the norm.

In the following proposition we collect some properties of the set $E_1(K)$. Given $x \in X \setminus K$ and $\delta > 0$ we define $P_K(x, \delta) = \{k \in K : \|x - k\| < d_K(x) + \delta\}$.

Proposition 3. *Suppose K is a closed set in X.*

- (a) *$x \in E_1(K)$ if and only if given any $\varepsilon > 0$ there is a $\delta > 0$ such that diameter $P_K(x, \delta) < \varepsilon$.*
- (b) *If $x \in E_1(K)$, then the metric projection P_K is single-valued and continuous at x.*
- (c) *$E_1(K)$ is a G_δ in X.*

Proof. (a). This follows from the definition of $E_1(K)$.

(b). Again, P_K is a singleton follows from the definition. To show continuity, let $x_n \to x$. Then any $k_n \in P_K(x_n)$ is a minimizing sequence for x as well and hence converges to $P_K(x)$.

(c). For each $n \geq 1$, consider the set $G_n = \{x \in X \setminus K :$ there exists $\delta > 0$, such that diameter $P_K(x, \delta) < 1/n\}$. From (a) it follows that $E_1(K) = \bigcap_{n \geq 1} G_n$. We need to show each G_n is open.

Let $x \in G_n$. Thus there is a $\delta > 0$ such that dia$P_K(x, \delta) < 1/n$. Choose $0 < \alpha < \delta/2$ and $\beta = \delta - 2\alpha$. Then for $y \in X \setminus K$, $\|y - x\| < \alpha$ and $k \in P_K(y, \beta)$ we have $\|x - k\| < \|y - k\| + \alpha < d_K(y) + \beta + \alpha < d_K(x) + \beta + 2\alpha = d_K(x) + \delta$. Thus $P_K(y, \beta) \subseteq P_K(x, \delta)$ and thus $y \in G_n$ as well. This shows G_n is open. □

We now present our main theorem of this section:

Theorem 4. *Let X be a Banach space with LUR norm and K be a closed almost proximinal set in X. Then $E_1(K)$ is a dense G_δ in $X \setminus K$.*

Remark 5. In conjunction with Theorem 2 we observe that if the norm on X is both LUR and smooth, for every almost proximinal set K we have the *proximal normal formula*.

The proof of the following corollary follows from Lemma 1 and Proposition 3. In Theorem 10 of [1] the authors showed this for Hilbert spaces.

Corollary 6. *Let the norm on X be LUR and (Fréchet) smooth. Then the distance function generated by an almost proximinal set K is generically (Fréchet) smooth on $X \setminus K$.*

The proof of Theorem 4 is based on the observation that if we consider our set K to be X minus the open unit ball, then any point in the open unit ball has a nearest point on the unit sphere and if the space is LUR, for any such point except the origin, every minimizing sequence converges.

Proof of Theorem 4. By Proposition 3, it suffices to show that $E_1(K)$ is dense in $E(K)$. So, let $x \in E(K)$ and $k_0 \in P_K(x)$. We show that given any $0 < \varepsilon < 1/3$, the point $x_0 = x - \varepsilon(x - k_0) \in E_1(K)$.

We note that $d_K(x_0) = (1 - \varepsilon)d_K(x)$. Let $\{k_n\}$ be a minimizing sequence for x_0. It is easy to observe that $\|x - k_n\| \to d_K(x)$ as well. Now, $\|x_0 - k_n\| \to d_K(x_0)$, that is,

$$\|k_n - x + \varepsilon(x - k_0)\| \to (1-\varepsilon)d_K(x).$$

Since

$$\begin{aligned} 1-\varepsilon &\leq \left\| \frac{k_n - x}{\|x - k_n\|} - \varepsilon \frac{k_0 - x}{d_K(x)} \right\| \\ &\leq \frac{1}{d_K(x)} \|k_n - x - \varepsilon(k_0 - x)\| + \|k_n - x\| \left[\frac{1}{d_K(x)} - \frac{1}{\|x - k_n\|} \right], \end{aligned}$$

we have

$$\left\| \frac{k_n - x}{\|x - k_n\|} - \varepsilon \frac{k_0 - x}{d_K(x)} \right\| \to 1 - \varepsilon.$$

Let $u_n = (k_n - x)/\|x - k_n\|$, $u_0 = (k_0 - x)/d_K(x)$ and $\lambda = (1-2\varepsilon)/(1-\varepsilon)$. Note that, since $e < 1/3$, $1/2 < \lambda < 1$. Then $\|u_n\| = \|u_0\| = 1$ and

$$\|2u_n - [\lambda u_n + (1-\lambda)u_0]\| \to 1.$$

Since

$$\|2u_n - (\lambda u_n + (1-\lambda)u_0)\| \geq 2 - \|\lambda u_n + (1-\lambda)u_0\| \geq 1$$

we have $\|\lambda u_n + (1-\lambda)u_0)\| \to 1$ as well. Let

$$f_n(\lambda) = 1 - \|\lambda u_n + (1-\lambda)u_0\|.$$

Using convexity of the norm, we get that

$$\|\lambda u_n + (1-\lambda)u_0\| \leq (2-2\lambda) \left\| \frac{u_n + u_0}{2} \right\| + (2\lambda - 1).$$

It follows that

$$f_n(\lambda) \geq (2-2\lambda) f_n(1/2) \geq 0.$$

Since $f_n(\lambda) \to 0$, we have that $f_n(1/2) \to 0$, that is, $\|u_n + u_0\| \to 2$. Since X is LUR, $u_n \to u_0$ and hence, $k_n \to k$. □

We conclude this note with a result on the continuity of metric projection on Chebyshev sets. Our result, in conjunction with the result of Vlasov quoted in the introduction, gives a necessary and sufficient condition for convexity of Chebyshev sets in a Banach space X such that both X and X^* are LUR. We believe this improves upon the known results in this direction.

Proposition 7. *Suppose the norm on X is both LUR and Fréchet smooth. Then for a Chebyshev set $K \subseteq X$ and $x \in X \setminus K$, the metric projection P_K is continuous at x if and only if $\partial d_K(x)$ is a singleton.*

Proof. Let $K \subseteq X$ be a Chebyshev set and $x \in X \setminus K$ be such that P_K is continuous at x. From Theorem 2 and Theorem 4, we have $\partial d_K(x) = \overline{co}^{w^*}\{D_x(K)\}$.

Now, let $f \in D_x(K)$. By definition of $D_x(K)$, there exists $\{x_n\} \subseteq E_1(K)$, $x_n \to x$ and $f_n \in \mathcal{D}(x_n - P_K(x_n))$ such that $f_n \to f$ in the w^*-topology. But by continuity of P_K, $x_n - P_K(x_n) \to x - P_K(x)$. Therefore, $f \in \mathcal{D}(x - P_K(x))$. That is, $D_x(K) \subseteq \mathcal{D}(x - P_K(x))$. Hence, $\partial d_K(x) = \overline{co}^{w^*}\{D_x(K)\} \subseteq \mathcal{D}(x - P_K(x))$ as well. Now, since X is smooth, $\partial d_K(x)$ must be a singleton.

Conversely, let $K \subseteq X$ be a Chebyshev set and $x \in X \setminus K$ be such that $\partial d_K(x)$ is a singleton. By [1, Lemma 1], this implies $\partial d_K(x) \subseteq \mathcal{D}(x - P_K(x))$. Since X is smooth, we actually have $\partial d_K(x) = \mathcal{D}(x - P_K(x))$.

Now, let $y_n \in X \setminus K$, $y_n \to x$. We want to show $P_K(y_n)$ converges to $P_K(x)$. Define $x_n = y_n - \frac{1}{n}(y_n - P_K(y_n))$. By the proof of Theorem 4, $x_n \in E_1(K)$ and

$P_K(x_n) = P_K(y_n)$ for all $n > 3$. Note that $x_n \to x$ as well and by Theorem 1, $\partial d_K(x_n) \subseteq \mathcal{D}(x_n - P_K(x_n))$. Let $f_n \in \partial d_K(x_n)$ and let f be a w^*-cluster point of f_n. Then by upper semicontinuity of ∂d_K, $f \in \partial d_K(x) = \mathcal{D}(x - P_K(x))$. Since X is Fréchet smooth, this would imply that $f_n \to f$ in norm as well. Hence,

$$f\left(\frac{x_n - P_K(x_n)}{\|x_n - P_K(x_n)\|}\right) \to 1.$$

Now since the norm on X is LUR, f strongly exposes $\frac{x - P_K(x)}{\|x - P_K(x)\|}$. Thus

$$\frac{x_n - P_K(x_n)}{\|x_n - P_K(x_n)\|} \to \frac{x - P_K(x)}{\|x - P_K(x)\|}.$$

Since $\|x_n - P_K(x_n)\| = d_K(x_n) \to d_K(x) = \|x - P_K(x)\|$, we have $P_K(x_n) = P_K(y_n) \to P_K(x)$ as desired. □

Theorem 8. *Suppose the norms on X and X^* are LUR. Then a Chebyshev set K is convex in X if and only if $\partial d_K(x)$ is a singleton for all $x \in X \setminus K$.*

Proof. If K is convex, then ∂d_K coincides with the usual subdifferential of d_K, and if the norm on X^* is LUR, then d_K is Fréchet smooth at each $x \in X \setminus K$ (see [3, page 365]). Thus $\partial d_K(x)$ is a singleton for each such x.

Conversely, let $\partial d_K(x)$ be a singleton for each $x \in X \setminus K$. By Proposition 7, we have that the metric projection on K is continuous. Thus by Vlasov's Theorem, K is convex. □

Remark 9. (a) d_K being a Lipschitz function, the condition $\partial d_K(x)$ is a singleton for all $x \in X \setminus K$ reduces to strict differentiability of d_K (see [2], page 30 for definition and Proposition 2.2.4 for the equivalence of these two). In particular, this is satisfied when d_K is continuously differentiable on $X \setminus K$.

(b) In [4, Theorem 3.6], the author showed that for a closed set K in a Banach space X with the norms of X and X^* Fréchet differentiable, if for each $x \in X \setminus K$ there exists a unit vector $u \in X$ such that the directional derivative $D_u d_K(x) = 1$, then K is convex. A close look at the proof given in that paper actually shows that this condition implies $E_1(K) = X \setminus K$ and thus the set is Chebyshev and by Lemma 1 we also have that $\partial d_K(x)$ is a singleton for each $x \in X \setminus K$. Thus the result follows as a simple corollary of Theorem 8.

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Stat–Math Division, Indian Statistical Institute, 203, B. T. Road, Kolkata 700 108, India

E-mail address: sudipta_r@isical.ac.in

Current address: Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Beer-Sheva 84105, Israel

E-mail address: sudipta@math.bgu.ac.il