

ON WANDERING VECTOR MULTIPLIERS FOR UNITARY GROUPS

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ABSTRACT. A wandering vector multiplier is a unitary operator which maps the set of wandering vectors for a unitary system acting on a separable Hilbert space \mathcal{H} into itself. It is proved that the wandering vector multipliers for a unitary group form a group, which gives a positive answer for a problem of Han and Larson. Furthermore, non-abelian unitary groups of order 6 are considered. We prove that the wandering vector multipliers of such a unitary group can not generate $\mathcal{B}(\mathcal{H})$. This negatively answers another of their problems.

1. INTRODUCTION

Let \mathcal{H} be a separable Hilbert space. $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on \mathcal{H} . A unitary system \mathcal{U} is a set of unitary operators acting on \mathcal{H} which contains the identity I in $\mathcal{B}(\mathcal{H})$.

A unit vector ξ in \mathcal{H} is called a wandering vector for \mathcal{U} if $\mathcal{U}\xi = \{U\xi : U \in \mathcal{U}\}$ is an orthonormal set, that is, $(U\xi, V\xi) = 0$ if $U, V \in \mathcal{U}$ and $U \neq V$. If $\mathcal{U}\xi$ is an orthonormal basis for \mathcal{H} , then ξ is called a complete wandering vector for \mathcal{U} . The set of all complete wandering vectors for \mathcal{U} is denoted by $\mathcal{W}(\mathcal{U})$. Let A be a unitary operator on \mathcal{H} . We call A a wandering vector multiplier for \mathcal{U} if $A\mathcal{W}(\mathcal{U}) \subseteq \mathcal{W}(\mathcal{U})$ ([1]). We use $\mathfrak{M}_{\mathcal{U}}$ to denote the set of all such operators. The wandering vectors and their multipliers for unitary systems have been systematically studied (see [1, 2, 3, 4]). One important class of unitary systems is that of unitary groups. Han and Larson in [3] considered the wandering vector multipliers for unitary groups and proved that for many unitary groups, for example, finite, abelian and ICC unitary groups, the wandering vector multipliers are groups. They asked in [3]:

(1) Do the wandering vector multipliers $\mathfrak{M}_{\mathcal{U}}$ for any unitary group \mathcal{U} constitute a group?

If the unitary group is either abelian or ICC, then the set of wandering vector multipliers $\mathfrak{M}_{\mathcal{U}}$ generates $\mathcal{B}(\mathcal{H})$ ([3]). Thus they also asked:

(2) Does $\mathfrak{M}_{\mathcal{U}}$ generate $\mathcal{B}(\mathcal{H})$ in general?

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In this note we consider these two problems and give a positive answer for the first problem, a negative answer for the second problem.

Let \mathcal{S} be a subset of $\mathcal{B}(\mathcal{H})$. We use $W^*(\mathcal{S})$ to denote the von Neumann algebra generated by \mathcal{S} . The commutant of \mathcal{S} is $\mathcal{S}' = \{T \in \mathcal{B}(\mathcal{H}) : TA = AT, \forall A \in \mathcal{S}\}$. Let $x \in \mathcal{H}$ be a nonzero vector. The local commutant $\mathcal{C}_x(\mathcal{S})$ of \mathcal{S} at x is defined by $\{T \in \mathcal{B}(\mathcal{H}) : TA x = AT x, \forall A \in \mathcal{S}\}$. A very useful result is the one-to-one correspondence between the complete wandering vectors and the unitary operators in $\mathcal{C}_\xi(\mathcal{U})$. In particular, if $\xi \in \mathcal{W}(\mathcal{U})$, then $\mathcal{W}(\mathcal{U}) = \{T\xi : T \in \mathcal{C}_\xi(\mathcal{U}) \text{ is unitary}\}$ (see [1, Proposition 1.3]).

Let \mathcal{M} be a finite von Neumann algebra on \mathcal{H} and τ a faithful normal finite trace on \mathcal{M} . For $1 \leq p \leq \infty$, let $\mathbb{L}^p(\mathcal{M})$ denote the non-commutative L^p -space associated with \mathcal{M} and τ (see [8]). We use $\|\cdot\|_p$ to denote the norm of $\mathbb{L}^p(\mathcal{M})$. Then $\mathbb{L}^2(\mathcal{M})$ is a Hilbert space with the inner product $(x, y) = \tau(y^*x)$, for all $x, y \in \mathbb{L}^2(\mathcal{M})$. Moreover, for any $A \in \mathcal{M}$, we define the operators L_A and R_A on $\mathbb{L}^2(\mathcal{M})$ by the equations

$$L_A x = Ax, \quad R_A x = xA, \quad \forall x \in \mathbb{L}^2(\mathcal{M}).$$

Then we know that \mathcal{M} is $*$ -isomorphic (resp. $*$ -anti-isomorphic) to $L(\mathcal{M}) = \{L_A : A \in \mathcal{M}\}$ (resp. $R(\mathcal{M}) = \{R_A : A \in \mathcal{M}\}$) and $(L(\mathcal{M}))' = R(\mathcal{M})$. In particular, if \mathcal{M} has a cyclic and separate unit vector ξ and we define $\tau(A) = (A\xi, \xi)$, for all $A \in \mathcal{M}$, then \mathcal{H} is isomorphic to $\mathbb{L}^2(\mathcal{M})$ and \mathcal{M} is unitarily equivalent to $L(\mathcal{M})$ (see [8, Theorem 7.2.9]). In this case, we may assume that $\mathcal{H} = \mathbb{L}^2(\mathcal{M})$ and \mathcal{M} identifies with $L(\mathcal{M})$.

2. AN ANSWER FOR THE FIRST PROBLEM

Let \mathcal{U}_1 and \mathcal{U}_2 acting on Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , respectively, be two unitary systems. We say that \mathcal{U}_1 and \mathcal{U}_2 are unitarily equivalent if there is a unitary operator $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that $T\mathcal{U}_1 T^* = \mathcal{U}_2$. Then we easily have

Proposition 1 (See [3]). *Let \mathcal{U}_1 and \mathcal{U}_2 be two unitarily equivalent unitary systems on Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 respectively. If one has a completely wandering vector, so does the other and their wandering vector multipliers $M_{\mathcal{U}_1}$ and $M_{\mathcal{U}_2}$ are unitarily equivalent.*

Let \mathcal{U} be a unitary group on \mathcal{H} with a completely wandering vector ξ and put $\mathcal{M} = W^*(\mathcal{U})$. Then by Proposition 1.1 in [3], we have that \mathcal{M} is finite with the cyclic and separate vector ξ . Letting $\xi, \eta \in \mathcal{W}(\mathcal{U})$, we have $\tau(T) = (T\xi, \xi) = (T\eta, \eta)$ for all $T \in \mathcal{M}$ and τ is a faithful normal finite trace of \mathcal{M} . Thus we may assume that $\mathcal{H} = \mathbb{L}^2(\mathcal{M})$ and \mathcal{M} identifies with $L(\mathcal{M})$ by Proposition 1. Now we give a positive answer for the first problem, which is posed by Han and Larson in [3].

Theorem 1. *Let \mathcal{U} be a unitary group with a completely wandering vector ξ . Then $\mathfrak{M}_{\mathcal{U}}$ is a group.*

Proof. By Proposition 1, without loss of generality, we may assume $\mathcal{H} = \mathbb{L}^2(\mathcal{M})$ and $I \in \mathbb{L}^2(\mathcal{M})$ is a completely wandering vector for \mathcal{U} , that is, $I \in \mathcal{W}(\mathcal{U})$.

Let $\Phi \in \mathfrak{M}_{\mathcal{U}}$. It suffices to show that Φ^{-1} is also a wandering vector multiplier. $\forall x \in \mathcal{W}(\mathcal{U})$, by Lemma 1.1 in [1], there is a unitary operator $T \in \mathcal{U}'$ such that $x = TI$. We note that $\mathcal{U}' = \mathcal{M}' = R(\mathcal{M})$. Then there is a unitary operator $U \in \mathcal{M}$ such that $x = R_U I = IU = U$, that is, $\mathcal{W}(\mathcal{U}) = \{x : x \text{ is a unitary operator in } \mathcal{M}\}$ (see Lemma 2.2 in [3]). Thus Φ is a unitary-preserving linear map on \mathcal{M} . It

follows that $\Phi(I)$ is unitary in \mathcal{M} . Replacing Φ by $(\Phi(I))^{-1}\Phi$, we can assume that $\Phi(I) = I$. By Proposition 2.3, Proposition 2.5 and Corollary 3.6 in [3], we know that Φ is an isometry as well as a Jordan homomorphism from \mathcal{M} into \mathcal{M} . Let \mathcal{B} be the von Neumann algebra generated by $\Phi(\mathcal{M})$. Then we have that $\mathcal{B} = \mathcal{M}$ by the fact that $\Phi(\mathcal{M})$ is dense in $\mathbb{L}^2(\mathcal{M})$ (see the first part of the proof of Theorem 2.9 in [3]). By Lemma 1.4 in [3], we know that there exists a projection P in the center $\mathcal{Z}(\mathcal{M})$ of \mathcal{M} , such that

$$a \rightarrow \Phi(a)P$$

is a $*$ -homomorphism, and

$$a \rightarrow \Phi(a)(I - P)$$

is a $*$ -antihomomorphism.

Let $\Phi_1(a) = \Phi(a)P$ and $\Phi_2(a) = \Phi(a)(I - P)$ for all $a \in \mathcal{M}$ and denote by $\ker(\Phi_i)$ the kernel of Φ_i for $i = 1, 2$. We claim that $\ker(\Phi_i)$ ($i = 1, 2$) is weakly closed.

Let x_λ be a net in $\ker(\Phi_1)$ such that $\lim_\lambda x_\lambda = x$ in the weak operator topology for some $x \in \mathcal{M}$. We next show that $x \in \ker(\Phi_1)$. For any $y \in \mathbb{L}^2(\mathcal{M})$, we have

$$\begin{aligned} \lim_\lambda (\Phi(x_\lambda), y) &= \lim_\lambda (x_\lambda, \Phi^*(y)) \\ &= \lim_\lambda (x_\lambda I, \Phi^*(y)) = (x, \Phi^*(y)) = (\Phi(x), y), \end{aligned}$$

which implies that $\Phi(x_\lambda) \rightarrow \Phi(x)$ weakly in $\mathbb{L}^2(\mathcal{M})$. We then have $P\Phi(x_\lambda) \rightarrow P\Phi(x)$ weakly. Therefore $P\Phi(x) = 0$ since $P\Phi(x_\lambda) = 0$. Thus $x \in \ker(\Phi_1)$, that is, $\ker(\Phi_1)$ is weakly closed. Similarly we also have that $\ker(\Phi_2)$ is weakly closed. Since Φ_1 (resp. Φ_2) is a $*$ -homomorphism (resp. $*$ -antihomomorphism), $\ker \Phi_1$ (resp. $\ker \Phi_2$) is a weakly closed two-sided ideal of \mathcal{M} . By Theorem 6.8.8 in [7], there are two central projections $E, F \in \mathcal{Z}(\mathcal{M})$ such that $\ker(\Phi_1) = E\mathcal{M}$ and $\ker(\Phi_2) = F\mathcal{M}$. Since $\Phi = \Phi_1 + \Phi_2$ and $\Phi(I) = I$, We have $\Phi_1(I - E) = P$ and $\Phi_2(I - F) = I - P$. We easily have $EF = 0$ since $\Phi(EF) = \Phi_1(EF) + \Phi_2(EF) = 0$. Thus E and F are orthogonal to each another.

Next we prove that $E + F = I$. We first claim that Φ_1 (resp. Φ_2) may be extended to a contraction on $\mathbb{L}^2(\mathcal{M})$. In fact, we have $\Phi_1(x) = \Phi(x)P$ for all $x \in \mathcal{M}$. Then for any $x \in \mathbb{L}^2(\mathcal{M})$, there is a net x_λ in \mathcal{M} such that $\lim_\lambda x_\lambda = x$ in $\mathbb{L}^2(\mathcal{M})$. It follows that $\lim_\lambda \Phi(x_\lambda) = \Phi(x)$ in $\mathbb{L}^2(\mathcal{M})$. Thus $\lim_\lambda \Phi_1(x_\lambda) = \lim_\lambda \Phi(x_\lambda)P = \Phi(x)P$ in $\mathbb{L}^2(\mathcal{M})$. We have that Φ_1 may be extended to a bounded linear operator on $\mathbb{L}^2(\mathcal{M})$ and we also denote it by Φ_1 . It is clear that $\|\Phi_1\| \leq \|\Phi\| \leq 1$. We similarly have that Φ_2 may be extended to a contraction on $\mathbb{L}^2(\mathcal{M})$ and we also denote it by Φ_2 . Then we have that $\Phi = \Phi_1 + \Phi_2$ on $\mathbb{L}^2(\mathcal{M})$.

Note that Φ is a unitary operator on $\mathbb{L}^2(\mathcal{M})$ and $\mathcal{M} \subseteq \mathbb{L}^2(\mathcal{M})$. Then there is a vector $x \in \mathbb{L}^2(\mathcal{M})$ such that $\Phi(x) = P$. Now, $\Phi(x) = \Phi(x)P + \Phi(x)(I - P) = \Phi_1(x) + \Phi_2(x) = P$, so we have $\Phi_2(x) = 0$. Then $x \in \ker(\Phi_2) = F\mathbb{L}^2(\mathcal{M})$ in $\mathbb{L}^2(\mathcal{M})$. Then $x = Fx = xF$. On the other hand, $\Phi(x) = P = \Phi_1(I - E) = \Phi_1(x)$; then $I - E - x \in \ker(\Phi_1) = \mathbb{L}^2(\mathcal{M})E$. It now follows that $(I - E - x)(I - E) = 0$, that is, $x(I - E) = I - E$. Thus we have $x = xE + (I - E) = xF$. Hence $x = F$ since $EF = 0$. Now we have $\Phi(F) = \Phi_1(F) = \Phi_1((I - E))$ and $F \leq I - E$. It follows that $F = I - E$ since $\ker(\Phi_1) = E\mathcal{M}$. We thus have $E + F = I$. It also follows that $\Phi(\mathcal{M}) = \Phi_1(\mathcal{M}) + \Phi_2(\mathcal{M})$.

It now follows that Φ_1 is an injective $*$ -homomorphism from $F\mathcal{M}$ into $P\mathcal{M}$. Thus $\Phi_1(F\mathcal{M})$ is a $*$ -subalgebra of $P\mathcal{M}$. We claim that $\Phi_1(F\mathcal{M}) = P\mathcal{M}$. By the proof above we know that the von Neumann algebra generated by $\Phi(\mathcal{M})$ is \mathcal{M} . Then $\Phi_1(F\mathcal{M})$ must be strongly dense in $P\mathcal{M}$ since $\Phi(\mathcal{M}) = \Phi_1(\mathcal{M}) + \Phi_2(\mathcal{M}) \subset P\mathcal{M} + (I - P)\mathcal{M} = \mathcal{M}$. Let $B \in P\mathcal{M}$ with norm one. Then there is a net $\{A_\lambda\}$ in $F\mathcal{M}$ such that $\Phi_1(A_\lambda)$ is in the unit ball of $\Phi_1(F\mathcal{M})$ and $\Phi_1(A_\lambda)$ converges to B in the strong operator topology by Kaplansky's density theorem. Thus $\{A_\lambda\}$ is contained in the unit ball of $F\mathcal{M}$ because Φ is an isometry for the norm topology of \mathcal{M} . Note that $\|\Phi_1(A_\lambda) - B\|_2 = \|\Phi(A_\lambda)P - BP\|_2 \rightarrow 0$ in $\mathbb{L}^2(\mathcal{M})$. We have that $\Phi_1(A_\lambda)$ is a Cauchy net in $\mathbb{L}^2(\mathcal{M})$. However we have $\Phi_1(X) = \Phi(X)$ for all $X \in F\mathcal{M}$ and Φ is a unitary operator on $\mathbb{L}^2(\mathcal{M})$. Then A_λ is also a Cauchy net in $\mathbb{L}^2(\mathcal{M})$. Letting $A_\lambda \rightarrow A$ in $\mathbb{L}^2(\mathcal{M})$, we have $\Phi_1(A_\lambda) = \Phi(A_\lambda) \rightarrow \Phi(A)$ in $\mathbb{L}^2(\mathcal{M})$. It follows that $B = \Phi(A) = \Phi_1(A)$. On the other hand, since the net $\{A_\lambda\}$ is in the unit ball of $F\mathcal{M}$, we may assume that A_λ converges to an operator $A' \in F\mathcal{M}$ in the weak operator topology. Then we easily have $A = A' \in F\mathcal{M}$ and thus $\Phi_1(F\mathcal{M}) = P\mathcal{M}$.

Similarly we have $\Phi_2((I - F)\mathcal{M}) = (I - P)\mathcal{M}$. Then Φ is a surjective isometry on \mathcal{M} . So the inverse of Φ is also an isometry. By Lemma 12 in [6], Φ^{-1} preserves unitary elements. Then Φ^{-1} is a wandering vector multiplier, that is, $\Phi^{-1} \in \mathfrak{M}_{\mathcal{U}}$. The proof is complete. \square

Corollary 1. *Let $\Phi \in \mathfrak{M}_{\mathcal{U}}$. Then Φ is a weakly continuous and surjective isometry on \mathcal{M} . Moreover, if $\Phi(I) = I$, then $\tau(\Phi(T)) = \tau(T)$ for all $T \in \mathcal{M}$.*

Proof. By the proof of Theorem 1, we know that Φ_1 (resp. Φ_2) is a $*$ -isomorphism (resp. $*$ -anti-isomorphism) from the von Neumann algebra $F\mathcal{M}$ (resp. $(I - F)\mathcal{M}$) onto $P\mathcal{M}$ (resp. $(I - P)\mathcal{M}$). Then both Φ_1 and Φ_2 are ultraweakly continuous. Hence $\Phi = \Phi_1 + \Phi_2$ is also ultraweakly continuous. Ultraweak topology coincides with weak topology on \mathcal{M} since \mathcal{M} has a cyclic and separate vector (see [7]); then Φ is weakly continuous. We now show that $\tau(\Phi(T)) = \tau(T)$ for all $T \in \mathcal{M}$ if Φ is a Jordan homomorphism such that $\Phi(I) = I$. In fact, since $\tau(\cdot)$ is a trace and P is a central projection, we have

$$\begin{aligned} \tau(\Phi(T^*T)) &= \tau(\Phi(T^*T)P) + \tau(\Phi(T^*T)(I - P)) \\ &= \tau(\Phi(T^*)\Phi(T)P) + \tau(\Phi(T)\Phi(T^*)(I - P)) \\ &= \tau(\Phi(T)^*\Phi(T)P) + \tau(\Phi(T)\Phi(T)^*(I - P)) \\ &= \tau(\Phi(T^*)\Phi(T)P) + \tau(\Phi(T)^*\Phi(T)(I - P)) \\ &= \tau(\Phi(T^*)\Phi(T)) \\ &= \|\Phi(T)\|_2^2 = \|T\|_2^2 \\ &= \tau(T^*T). \end{aligned}$$

It follows that $\tau(\Phi(T)) = \tau(T)$ for all positive elements $T \in \mathcal{M}$ and therefore for all elements $T \in \mathcal{M}$. The proof is complete. \square

Let $\mathcal{J}_{\mathcal{U}}$ be the set of all weakly continuous and surjective Jordan homomorphisms Φ on \mathcal{M} such that $\tau \circ \Phi = \tau$.

Theorem 2. *$\mathfrak{M}_{\mathcal{U}}$ is unitarily equivalent to the group generated by $\mathcal{J}_{\mathcal{U}}$ and all the unitary elements in \mathcal{M} .*

Proof. By Proposition 1, Theorem 1 and Corollary 1, it is sufficient to show that $\mathcal{J}_{\mathcal{U}} \subset \mathfrak{M}_{\mathcal{U}}$. Let $\Phi \in \mathcal{J}_{\mathcal{U}}$. Then it is clear that Φ preserves the unitary elements of \mathcal{M} . So it is enough to prove that Φ can be extended to a unitary operator on $\mathbb{L}^2(\mathcal{M})$. Letting $X \in \mathcal{M}$, as in the proof of Corollary 1, we have

$$\|\Phi(X)\|_2^2 = \tau(\Phi(X)^*\Phi(X)) = \tau(\Phi(X^*X)) = \tau(X^*X) = \|X\|_2^2.$$

Then Φ can be extended to a unitary operator on $\mathbb{L}^2(\mathcal{M})$ and $\Phi \in \mathfrak{M}_{\mathcal{U}}$. The proof is complete. \square

Let $U, V \in \mathcal{B}(\mathcal{H})$ be two unitary operators. Let $\theta \in (0, 1)$ and let $\lambda = \exp(2\pi i\theta)$. If $UV = \lambda VU$, then we call the unitary system $\mathcal{U}_{U,V} = \{U^m V^n : m, n \in \mathbb{Z}\}$ a rotation system. It is not a unitary group. If θ is an irrational number, it is called an irrational rotation system, which has been studied systematically in [2, 3]. It is proved that the wandering vector multipliers of $\mathcal{U}_{U,V}$ form a group since $W^*(\mathcal{U}_{U,V})$ is a factor (Theorem 4.1 in [3]). Contrasting this with the irrational rotation system, if θ is a rational number in $(0, 1)$, we note that the von Neumann algebra generated by U and V is not a factor in general. In [4, Theorem 4.3], Han and Larson proved that $\mathcal{W}(\mathcal{U}_{U,V})$ is a closed and connected subset of \mathcal{H} . We similarly have the following result whose proof is similar to that of Theorem 1 above. We give the theorem here without proof.

Theorem 3. *Let θ be a rational number in $(0, 1)$. If $\mathcal{U}_{U,V}$ has a completely wandering vector, then $\mathfrak{M}_{\mathcal{U}_{U,V}}$ is a group.*

3. AN ANSWER FOR THE SECOND PROBLEM

We give a counterexample in this section to show that the second problem is false in general. Let G be a finite group. If G is non-abelian, then it is at least of order 6 ([5, p. 98]). Furthermore, any two non-abelian groups of order 6 are isomorphic. Let \mathcal{U} be a non-abelian unitary group of order 6 acting on the Hilbert space \mathcal{H} such that $\mathcal{W}(\mathcal{U}) \neq \emptyset$. Then $\dim \mathcal{H} = 6$. We first note that, up to unitary equivalence, such a unitary group is unique.

Proposition 2. *Let \mathcal{U}_i be two non-abelian unitary groups of order 6 acting on the Hilbert space \mathcal{H}_i such that $\mathcal{W}(\mathcal{U}_i) \neq \emptyset$ for $i = 1, 2$. Then $W^*(\mathcal{U}_1)$ and $W^*(\mathcal{U}_2)$ are unitarily equivalent.*

Proof. Let ξ_i ($i = 1, 2$) be the completely wandering vector for \mathcal{U}_i . Then ξ_i is cyclic and separate for $W^*(\mathcal{U}_i)$ by Proposition 1.1 in [3]. Noting that \mathcal{U}_1 and \mathcal{U}_2 are group isomorphic, then we easily have that $W^*(\mathcal{U}_1)$ and $W^*(\mathcal{U}_2)$ are $*$ -isomorphic. Thus we have the result by Theorem 7.2.9 in [7]. The proof is complete. \square

By Proposition 2 above, it is sufficient to consider a concrete non-abelian unitary group of order 6 with a completely wandering vector. Let \mathbb{C}^n be the complex n -dimensional Hilbert space, and we denote by M_n the set of all complex $n \times n$ matrices. $I_n \in M_n$ is the identity matrix.

Let $\mathcal{H} = \mathbb{C}^6$ and let $\{e_i : i = 1, 2, \dots, 6\}$ be an orthonormal basis of \mathbb{C}^6 , and for $x, y \in \mathbb{C}^6$, define $x \otimes y(z) = (z, y)x$, for all $z \in \mathbb{C}^6$. Then $x \otimes x$ is a rank-1 projection for all unit vectors x .

Let $\lambda = \exp(\frac{2}{3}\pi i) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and let $A_1 = \lambda e_1 \otimes e_1 + \bar{\lambda} e_2 \otimes e_2$ and $A_2 = e_1 \otimes e_2 + e_2 \otimes e_1$ be two unitary matrices in M_2 . It is easy to check that $A_1^3 = I_2$,

$A_1^2 \neq I_2$, $A_2^2 = I_2$ and $A_2A_1 = A_1^{-1}A_2$. Then by Theorem 6.13 in [5], we know that the group G generated by A_1 and A_2 is non-abelian with order 6. In fact, $G = \{I_2, A_1, A_2, A_3, A_4, A_5\}$, where $A_3 = \bar{\lambda}e_1 \otimes e_1 + \lambda e_2 \otimes e_2$, $A_4 = \bar{\lambda}e_1 \otimes e_2 + \lambda e_2 \otimes e_1$ and $A_5 = \lambda e_1 \otimes e_2 + \bar{\lambda}e_2 \otimes e_1$.

Now we identify $\mathcal{H} = \mathbb{C}^6 = \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C}$. For a matrix A , we denote $A^{(2)} = A \oplus A$. Put $U_1 = A_1^{(2)} \oplus 1 \oplus 1$, $U_2 = A_2^{(2)} \oplus 1 \oplus (-1)$, $U_3 = A_3^{(2)} \oplus 1 \oplus 1$, $U_4 = A_4^{(2)} \oplus 1 \oplus (-1)$ and $U_5 = A_5^{(2)} \oplus 1 \oplus (-1)$ respectively. Then $\mathcal{U} = \{I_6, U_1, U_2, U_3, U_4, U_5\}$ is a non-abelian unitary group of order 6.

Let $\xi = \frac{1}{\sqrt{6}}(-1, 1, 1, 1, 1, 1) \in \mathbb{C}^6$; then it is easy to show that $\xi \in \mathcal{W}(\mathcal{U})$. As in Section 2, we let $\mathcal{M} = W^*(\mathcal{U})$. The following result is evident.

Proposition 3. $\mathcal{M} = M_2^{(2)} \oplus \mathbb{C} \oplus \mathbb{C}$.

It now follows from Proposition 3 that $E = I_2^{(2)} \oplus 0 \oplus 0$ is a minimal central projection of \mathcal{M} such that $E\mathcal{M}$ is a non-commutative factor, while $(I_6 - E)\mathcal{M}$ is abelian.

Proposition 4. *Let \mathcal{M} and \mathcal{U} be as above. Then $\mathfrak{M}_{\mathcal{U}}$ cannot generate $\mathcal{B}(\mathcal{H})$.*

Proof. We know \mathbb{C}^6 is unitarily equivalent to $\mathbb{L}^2(\mathcal{M})$. Then without loss of generality, we may assume that $\mathcal{M} = L(\mathcal{M})$ acts on $\mathbb{L}^2(\mathcal{M})$. Let $\Phi \in \mathfrak{M}_{\mathcal{U}}$ and E be as above.

As in the proof of Theorem 1, we may assume that Φ is a Jordan homomorphism on \mathcal{M} such that $\Phi(I) = I$. Since $E \in \mathcal{M}'$, it is sufficient to show that $\Phi L_E = L_E \Phi$. From the proof of Theorem 1 again, there are two central projections F and P of \mathcal{M} such that $\Phi : F\mathcal{M} \rightarrow P\mathcal{M}$ is a *-isomorphism, while $\Phi : (I - F)\mathcal{M} \rightarrow (I - P)\mathcal{M}$ is a *-anti-isomorphism. If $F \geq E$, we must have $P \geq E$. Then $\Phi(E) = E$ since $EF\mathcal{M} = EP\mathcal{M}$ is the common non-commutative factor part of $F\mathcal{M}$ and $P\mathcal{M}$. Thus for any $X \in \mathcal{M}$, $L_E \Phi(X) = E\Phi(X) = \Phi(E)\Phi(X) = \Phi(EX) = \Phi(L_EX) = \Phi L_E(X)$, which implies that $L_E \Phi = \Phi L_E$. If $I - F \geq E$, we similarly have the same result. The proof is complete. \square

Proposition 4 answered the second problem negatively.

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