

## A NEW REPRESENTATION OF THE DEDEKIND COMPLETION OF $C(K)$ -SPACES

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ABSTRACT. A new representation of the Dedekind completion of  $C(K)$  is given. We present a necessary and sufficient condition on a compact Hausdorff space  $K$  for which the Dedekind completion of  $C(K)$  is  $B(S)$ , the space of real valued bounded functions on some set  $S$ .

### 1. INTRODUCTION

As usual for a topological space  $K$ , the space of real valued continuous (bounded) functions on  $K$  is denoted by  $C(K)$  ( $C_b(K)$ ). The characteristic function of  $k \in K$  is denoted by  $\chi_k$ . For a given set  $S$ ,  $B(S)$  denotes the space of real valued bounded functions on  $S$ . We refer to [6] for classical terminology on Riesz space theory.

Recall that a Dedekind complete Riesz space  $F$  is called a *Dedekind completion* of a Riesz space  $E$  if there exists a Riesz isomorphism  $\pi$  from  $E$  into  $F$  such that for each  $0 < x \in F$  there exist  $0 < a, b \in E$  such that

$$\pi(a) \leq x \leq \pi(b).$$

It is well known that each Archimedean Riesz space has a unique (up to Riesz isomorphism) Dedekind completion. For a compact Hausdorff space  $K$ ,  $C(K)$  is Dedekind complete if and only if  $K$  is Stonean. This is due to Nakano [9]. Recall that a compact Hausdorff space is called *Stonean* if the closure of each open set is also open. In the literature there are several descriptions of the Dedekind completion of the  $C(K)$ -spaces. Some of these are:

- Let  $K$  be a compact Hausdorff space. Dilworth [2] has given a description of the Dedekind completion of  $C(K)$  in terms of lower and upper semicontinuous functions.
- Nakano-Shimagaki [10] showed that the Dedekind completion of  $C(K)$  can be constructed as a semi-ordered linear space.
- In [7] Mack and Johnson have given a necessary and sufficient condition that the Dedekind completion of  $C(X)$ , for  $X$  a realcompact space, be isomorphic to  $C(Y)$  for some  $Y$ . This is obtained for the class of countably paracompact and normal spaces in [12] by Weinberg.

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- Maxey [8] has shown that for any maximal ideal  $I$  of  $C(K)^{\sim\sim}$  with  $C(K) \cap I = \{0\}$ , the quotient space  $C(K)^{\sim\sim}/I$  is the Dedekind completion of  $C(K)$ . This is Theorem 1.49 of [1].

- In [13] and [14] Zaharov has given the Dedekind completion of  $C(K)$  in terms of *quasi normal* functions and functions with *Baire* property. These results are generalized in [11] by Veksler.

- In [4] and [5] Kaplan has presented several descriptions of the Dedekind completion of  $C(K)$ .

We also refer to [3] for some more details about these results.

It is surprise that, to the best of our knowledge, there has been no investigation on the following question, which was asked by the first author in the 2003 Positivity Conference in Rhodes:

*For what type of compact Hausdorff space  $K$ , the Dedekind completion of  $C(K)$  is  $B(S)$  for some set  $S$ ?*

The aim of this paper is to give a new representation of the Dedekind completion of  $C(K)$ -space and then answer the above question.

## 2. MAIN RESULTS

We begin with the following definition.

**Definition 1.** A surjective map  $\varphi$  from a completely regular Hausdorff space  $X$  into another one  $Y$  is called *irreducible* if  $\varphi(A) \neq Y$  for each closed proper subset  $A$  of  $X$ .

Let  $X$  and  $Y$  be completely regular Hausdorff spaces and let  $\varphi$  be a continuous closed surjective function from  $X$  into  $Y$ . Then we define

$$\varphi^* : C_b(Y) \longrightarrow C_b(X), \quad \varphi^*(f) = f\varphi.$$

It is clear that  $\varphi^*$  is an isometry and Riesz isomorphism from  $C_b(Y)$  into  $C_b(X)$ .

**Lemma 2.** *Let  $X$  and  $Y$  be completely regular Hausdorff spaces and let  $\varphi$  be a continuous closed surjection from  $X$  to  $Y$ . Then  $\varphi^*(C_b(Y))$  is order dense in  $C_b(X)$  if and only if  $\varphi$  is irreducible.*

*Proof.* Suppose that  $\varphi^*(C_b(Y))$  is order dense in  $C_b(X)$  and  $\varphi$  is not irreducible. Choose a closed subset  $A$  of  $X$  with  $\varphi(A) = Y$ . Let  $p \in X \setminus A$ ,  $0 \leq f \in C_b(X)$  with  $f(p) = 1$  and  $f(A) = \{0\}$ . Then there exists  $g \in C_b(Y)$  such that  $0 < \varphi^*(g) \leq f$ . As  $\varphi(A) = Y$  we have  $g = 0$ , which is a contradiction.

Conversely, suppose that  $\varphi$  is irreducible. Let  $0 < f \in C_b(X)$ . Choose  $0 < r \in \mathbf{R}$  and a nonempty open set  $U$  in  $X$  such that  $f(U) \subset (r, \infty)$ . As  $X \setminus U$  is a closed proper subset of  $X$ ,  $\varphi(X \setminus U)$  is a proper closed subset of  $Y$ . We can have  $q \in Y \setminus \varphi(X \setminus U)$ ,  $g \in C_b(Y)$  such that

$$g(q) = 1 \quad \text{and} \quad g(\varphi(X \setminus U)) = \{0\}.$$

Then we have  $\varphi \subset [0, r]$  and  $0 < \varphi^*(g) \leq f$ . This completes the proof.

Let  $K$  be a topological space and let  $A \subset K$ . Then  $cl(A)$  denotes the closure of  $A$ . Similarly  $Int(A)$  stands for the interior of  $A$ . A lemma similar to the following may be known, and its proof is routine. We prove only its last part.

**Lemma 3.** *Let  $X$  be a compact Hausdorff space. Then*

**i.**

$$RO(X) = \{O \subset X : O = \text{int}(cl(O))\}$$

*is a Dedekind complete Boolean algebra under the operations*

$$U \vee V = \text{Int}(cl(U \cup V)), \quad U \wedge V = U \cap V$$

*with  $e = X$  and  $0 = \emptyset$ .*

**ii.** *Let  $Y = \text{Stone}RO(X)$  be the set of all maximal filters in  $RO(X)$ . Then  $Y$  is a Stonean space with the basis*

$$\{U^* = \{p : U \in p\} : U \in RO(X) \setminus \{\emptyset\}\}.$$

**iii.** *For each  $p \in Y$ , there exists  $x_p \in X$  with  $\bigcap_{U \in p} cl(U) = \{x_p\}$  and*

$$\varphi : Y \longrightarrow X, \quad \varphi(p) = x_p$$

*is continuous, surjective and irreducible.*

*Proof.* We prove only that  $\varphi$  is irreducible (the proof of the other parts of the lemma is clear). Let  $F$  be a proper closed subspace  $Y$  and let  $q \in Y \setminus F$ . Choose  $U \in RO(X)$  such that

$$U^* \cap F = \emptyset, \quad q \in U.$$

Let  $p \in F$ . Then  $p \notin U^*$ , so  $U \notin p$ . As  $p$  is maximal,  $p \cup \{U\}$  is not a filter. Then there exists  $V \in p$  with  $V \cap U = \emptyset$ . Then

$$\bigcap_{O \in p} cl(O) \subset cl(V) \quad \text{and} \quad U \cap cl(V) = \emptyset.$$

Thus,  $\varphi(p) \notin U$  for each  $p \in F$ . Hence  $\varphi(F) \neq X$ . This completes the proof.

Now we are ready to prove one of the main theorems.

**Theorem 4.** *Let  $X$  be a compact Hausdorff space and let  $Y = \text{Stone}RO(X)$ . Then the Dedekind completion of  $C(X)$  is  $C(Y)$ .*

*Proof.* Consider the isometry and order isomorphism

$$\varphi^* : C(X) \longrightarrow C(Y), \quad \varphi^*(f) = f\varphi,$$

where  $\varphi$  is defined as in the above lemma (iii). Since  $\varphi$  is continuous surjective and irreducible from Lemma 2,  $\varphi^*(C(X))$  is an order dense Riesz subspace of  $C(Y)$  and dominates the elements of  $C(Y)$ . Since  $Y$  is Stonean,  $C(Y)$  is also Dedekind complete. This completes the proof.

It is well known that each AM-space  $E$  with a strong order unit can be represented by  $C(K_E)$  for some compact Hausdorff space  $K_E$  as Riesz isomorphic spaces.  $K_E$  is known as the *Kakutani-Krein space* of  $E$ . The corollary below follows immediately from the above theorem.

**Corollary 5.** *Let  $E$  be an AM-space with a strong unit. Then  $C(Y)$  is the Dedekind completion of  $E$ , where  $Y = \text{Stone}RO(K_E)$  and  $K_E$  is the Kakutani-Krein space of  $E$ .*

Recall that a nonzero positive element  $x$  in an Archimedean Riesz space is called an *atom* if the (order) ideal generated by  $x$  is one dimensional. The following lemma is well known and the proof is trivial, so we omit its proof.

**Lemma 6.** *Let  $K$  be a compact Hausdorff space. Then  $0 < f \in C(K)$  is an atom if and only if  $f = \alpha\chi_a$  for some isolated point  $a \in X$  and  $0 \leq \alpha \in \mathbf{R}$ .*

Now we can give an answer to the question which was asked in the Introduction.

**Theorem 7.** *Let  $K$  be a compact Hausdorff space. Then the Dedekind completion of  $C(K)$  is  $B(S)$  for some  $S$ , if and only if the set of isolated points of  $K$  is dense in  $K$ .*

*Proof.* Suppose that the Dedekind completion of  $C(K)$  is  $B(S)$ , and let  $\pi$  be the required isomorphism from  $C(K)$  into  $B(S)$ . Let  $A$  be the set of isolated points of  $S$ . Let  $s \in S$  be given. Then there exists  $0 < f \in C(K)$  such that  $\pi(f) \leq \chi_s$ , so  $\pi(f) = \alpha\chi_s$  for some positive real number  $\alpha$ . As  $\pi^{-1}(\chi_s)$  is an atom,  $f$  is an atom, so there exists a unique isolated point  $\sigma(s) \in K$  such that  $f = \beta\chi_{\sigma(s)}$  for some real number  $\beta$ . That is, we have a one-to-one map  $\sigma$  from  $S$  into  $A$ . Let  $a \in A$  be given. Then  $\pi(\chi_a) = \alpha\chi_s$  for some  $s \in S$  and real number  $\alpha$ . This observation shows that  $a = \sigma(s)$ , so  $\sigma$  is also onto. So,  $S$  can be replaced by  $A$ . Suppose that  $A$  is not dense in  $K$ . Then there exists  $k \in K$  and  $f \in C(K)$  such that  $f(k) = 1$  and  $f = 0$  on  $A$ . As  $0 < \pi(f)$  there exists  $a \in A$  with  $0 < \alpha\chi_a \leq \pi(f)$  for some  $\alpha$ . Then  $\pi^{-1}(\chi_a)$  is zero on  $A$ . Since it is an atom, there exists an isolated point  $k \in A$  with  $\pi^{-1}(\chi_a) = \alpha\chi_k$  for some nonzero positive real number  $\alpha$ . This is a contradiction, so  $A$  must be dense in  $K$ .

Now suppose that  $A$  is dense in  $K$ . Define  $\pi : C(K) \rightarrow B(A)$  by  $\pi(f) = f|_A$ . Let  $0 < f \in B(A)$ . Choose  $a \in A$  with  $0 < f(a)\chi_a \leq f$ . Then

$$0 < \pi(f(a)\chi_a) \leq f \leq \pi(\|f\|\mathbf{1}).$$

This completes the proof.

We can give an alternative proof of the above theorem as follows: Let  $\pi : C(X) \rightarrow B(S)$  be an order dense Riesz isomorphism with  $\pi(\mathbf{1}) = \mathbf{1}$ . Then there exists  $\varphi : \beta S \rightarrow X$  which is a continuous surjection such that  $\varphi^* = \pi$ . Then from Lemma 2,  $\varphi$  is irreducible and  $cl(\varphi(S)) = X$  (from the fact that  $X = \varphi(\beta S) \subset cl(\varphi(S))$ ). For each  $q \in S$ ,  $\{q\}$  is open in  $\beta S$  and  $\beta S \setminus \{q\}$  is a proper closed subset of  $X$ . Then  $\varphi(\beta S \setminus \{q\})$  is a proper closed subset of  $X$ . As  $X \setminus \varphi(\beta S \setminus \{q\})$  is open and

$$\varphi(q) = X \setminus \varphi(\beta S \setminus \{q\}),$$

$\varphi(q)$  is an isolated point in  $X$  and  $\{\varphi(q) : q \in S\}$  is dense in  $X$ , so the set of isolated points in  $X$  is dense in  $X$ .

The following corollary immediately follows from the above theorem.

**Corollary 8.** *Let  $K$  be a Stonean space. The set of isolated points is dense in  $K$  if and only if  $K$  is a Stone-Cech compactification of a discrete space  $S$ .*

Let  $K$  be a discrete topological space. Then from the above theorem the Dedekind completion of  $C(K_\infty)$  is  $B(K)$ , where  $K_\infty$  is a one-point compactification of  $K$ . Each compact Hausdorff space  $K$  generates a compact Hausdorff space  $A(K)$  such that  $K$  is homeomorphic into  $A(K)$  and the set of isolated points is dense in  $A(K)$ . This construction is called the *Alexandroff Duplicate* of  $K$ , that is,  $A(K)$  is a compact Hausdorff space on  $K \times \{0, 1\}$  generated by sets of the form  $F \times \{1\}$  and  $U \times \{0, 1\} \setminus G \times \{1\}$ , where  $F, G \subset K$  with  $G$  is finite and  $U$  is open

in  $K$ . Then for each  $k \in K$ ,  $(k, 1)$  is an isolated point and  $K \times \{1\}$  is dense in  $A(K)$ . Hence the Dedekind completion of  $C(A(K))$  is  $B(K \times \{1\}) (= B(K))$ . This provides many examples of compact Hausdorff spaces  $K$  such that the Dedekind completion of  $C(K)$  is  $B(S)$ .

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