**Abstract.** Let $A$ and $B$ be $C^*$-algebras and let $X$ be an $A$-$B$-imprimitivity bimodule. Then it is shown that if the spectrum $\hat{A}$ of $A$ (resp. $\hat{B}$ of $B$) is discrete, then every closed $A$-$B$-submodule of $X$ is orthogonally closed in $X$, and conversely that if $\hat{A}$ (resp. $\hat{B}$) is a $T_1$-space and if every closed $A$-$B$-submodule of $X$ is orthogonally closed in $X$, then $\hat{A}$ (resp. $\hat{B}$) is discrete.

1. Introduction

Let $A$ be a $C^*$-algebra and let $X$ be a Hilbert $A$-module with an $A$-valued inner product $\langle \cdot , \cdot \rangle$. For any closed Hilbert $A$-submodule $Y$ of $X$, we denote by $Y^\perp$ the orthogonally complemented subspace of $Y$ in $X$, i.e.,

$$Y^\perp = \{ x \in X \mid \langle x , y \rangle = 0 \text{ for all } y \in Y \}.$$ 

We say that a closed $A$-submodule $Y$ of a Hilbert $A$-module $X$ is **orthogonally complemented** in $X$ if $X$ coincides with $Y \oplus Y^\perp$, and that a closed $A$-submodule $Y$ of a Hilbert $A$-module $X$ is **orthogonally closed** in $X$ if $Y^{\perp\perp} = Y$. If $Y$ is orthogonally complemented in $X$, then it is orthogonally closed in $X$. But the converse is not necessarily true in general. Suppose that $X$ is a full (right) Hilbert $A$-module. Then Schweitzer [7, Theorem 1] has shown that every closed right $A$-submodule of $X$ is orthogonally closed if and only if every closed right $A$-submodule of $X$ is orthogonally complemented in $X$. Here we remark that every full right Hilbert $A$-module is a $\mathcal{K}(X)$-$A$-imprimitivity bimodule, where $\mathcal{K}(X)$ is the $C^*$-algebra generated by those operators $\theta_{x,y}$ with all $x, y \in X$ defined by $\theta_{x,y}(z) = x \langle y , z \rangle$ for all $z \in X$.

Let $A$ and $B$ be $C^*$-algebras and let $X$ be an $A$-$B$-imprimitivity bimodule. Note that $A$ is isomorphic to $\mathcal{K}(X)$. In this paper, we show that every closed $A$-$B$-submodule of $X$ is orthogonally closed if and only if every closed $A$-$B$-submodule of $X$ is orthogonally complemented in $X$. As a corollary, we show that if the spectrum $\hat{A}$ of $A$ (resp. $\hat{B}$ of $B$) is discrete, then every closed $A$-$B$-submodule of $X$ is orthogonally closed in $X$, and conversely that if $\hat{A}$ (resp. $\hat{B}$) is a $T_1$-space and if every closed $A$-$B$-submodule of $X$ is orthogonally closed in $X$, then $\hat{A}$ (resp. $\hat{B}$) is discrete.
2. Results

Let $A$ and $B$ be $C^*$-algebras and let $X$ be an $A$-$B$-imprimitivity bimodule (see [4] or [6] for the definition of an imprimitivity bimodule). We denote by $\langle a, b \rangle$ the $A$-valued inner product on $X$ as a left Hilbert $A$-module, and by $\langle a, b \rangle_B$ the $B$-valued inner product on $X$ as a right Hilbert $B$-module, respectively.

Throughout this paper, by an $A$-$B$-submodule we mean a left $A$-submodule and right $B$-submodule.

Two $C^*$-algebras $A$ and $B$ are said to be Morita equivalent if there exists an $A$-$B$-imprimitivity bimodule. We remark that in this paper, Morita equivalence means strong Morita equivalence in the sense of Rieffel (cf. [6, Remark 3.15]).

Let $A$ be a $C^*$-algebra and let $I$ be a closed ideal of $A$. Throughout this paper, unless otherwise stated, by an ideal of $A$ we always mean a two-sided ideal. Then $I^\perp$ is defined by

$$I^\perp = \{a \in A \mid ax = 0 \text{ for all } x \in I\}.$$  

It is easily seen that $I^\perp$ is a closed ideal of $A$. Note here that $A$ can be regarded as an $A$-$A$-imprimitivity bimodule, for the bimodule structure given by the multiplication in $A$, with $\langle a, b \rangle = ab^*$ and $\langle a, b \rangle_A = a^*b$ for $a, b \in A$ (see [6] Example 3.5). Then $I$ is a closed $A$-$A$-submodule of $A$.

Let $A$ and $B$ be $C^*$-algebras and let $X$ be an $A$-$B$-imprimitivity bimodule. For a closed $A$-$B$-submodule $Y$ of $X$, its orthogonally complemented subspace $Y^\perp$ in $X$ is defined by

$$Y^\perp = \{x \in X \mid \langle a, y \rangle = \langle x, y \rangle_B = 0 \text{ for all } y \in Y\}.$$  

**Lemma 2.1.** Let $A$ be a $C^*$-algebra and let $I$ be any closed ideal $I$ of $A$. If every closed ideal of $A$ is orthogonally closed in $A$, then we have $A = I \oplus I^\perp$.

**Proof.** Using the fact that for an ideal $J$ of $A$ we have $J \cap J^\perp = \{0\}$, it follows that

$$(I \oplus I^\perp)^\perp = I^\perp \cap (I^\perp)^\perp = \{0\}.$$  

Hence, using the hypothesis, we have $I \oplus I^\perp = (I \oplus I^\perp)^\perp = A$. \hfill \Box

Now we recall the Rieffel correspondence (see [6] Theorem 3.22)). Let two $C^*$-algebras $A$ and $B$ be Morita equivalent and let $X$ be an $A$-$B$-imprimitivity bimodule. We denote by $\mathcal{I}(A)$ (resp. $\mathcal{I}(B)$) the set of all closed (two-sided) ideals of $A$ (resp. $B$), and by $\mathcal{S}(X)$ the set of closed $A$-$B$-submodules of $X$. Note that $\mathcal{I}(A)$, $\mathcal{I}(B)$ and $\mathcal{S}(X)$ can be partially ordered by inclusion, and are then lattices. Then there are natural lattice isomorphisms among $\mathcal{I}(A)$, $\mathcal{I}(B)$ and $\mathcal{S}(X)$ given by

$\mathcal{I}(A) \ni I \longmapsto _I X \in \mathcal{S}(X)$, where $_I X = \{y \in X \mid \langle a, y \rangle = 0 \text{ for all } x \in X\}$;

$\mathcal{S}(X) \ni Y \longmapsto _Y I \in \mathcal{I}(A)$, $\gamma I \in \mathcal{I}(B)$,

where $_Y I$ is the closed linear span of $\{\langle a, y \rangle, x \} \mid y \in Y$ and $x \in X$;

and $\gamma I$ is the closed linear span of $\{\langle x, y \rangle_B \mid y \in Y$ and $x \in X\}$;

$\mathcal{I}(B) \ni J \longmapsto _J X \in \mathcal{S}(X)$, where $_J X = \{y \in X \mid \langle x, y \rangle_B \in J \text{ for all } x \in X\}$.

We refer to such lattice isomorphisms among $\mathcal{I}(A)$, $\mathcal{I}(B)$ and $\mathcal{S}(X)$ as the Rieffel correspondences (see [6, 3.3]). Note that $_I X$ is the closed linear span of $I \cdot X$ (cf. [6, Lemma 3.23]), and we will employ this fact in the proof of Lemma 2.2.

**Lemma 2.2.** Let $A$ and $B$ be $C^*$-algebras and let $X$ be an $A$-$B$-imprimitivity bimodule. For any closed ideal $I$ of $A$, we have $(_I X)^\perp = _I X$. 

Proof. Since \( iX \) is the closed linear span of \( IX \), we see from \( \lambda(I\perp X, IX) = I\perp, \lambda(X, X) = I \) \( \Rightarrow \) \( \lambda(I\perp X, IX) = 0 \) that \( iX \subset (I\perp X)\perp \). For the reverse inclusion, let \( Y = (IX)\perp \) and note that \( \{0\} = \lambda(Y, IX) = \lambda(Y, X) \cdot I \) implies that \( \lambda(Y, Y) \) is contained in \( I\perp \), hence \( Y \subset I\perp X \) by a well-known argument. \( \square \)

Now we are in a position to establish the main result.

**Theorem 2.3.** Let \( A \) and \( B \) be \( C^* \)-algebras and let \( X \) be an \( A-B \)-imprimitivity bimodule. Then every closed \( A-B \)-submodule of \( X \) is orthogonally closed in \( X \) if and only if every closed \( A-B \)-submodule of \( X \) is orthogonally complemented in \( X \).

Proof. Suppose that every closed \( A-B \)-submodule of \( X \) is orthogonally closed in \( X \). Let \( I \) be any closed ideal of \( A \). We claim that \( A = I \oplus I\perp \). Since it follows from Lemma 2.2 that \( (iX)\perp = i\perp X \), we obtain \((iX)\perp \perp = (i\perp X)\perp \), and with \( I \) replaced by \( I\perp \) we have \((i\perp X)\perp = i\perp X \). Since \( iX \) is a closed \( A-B \)-submodule, by assumption we see that \( i\perp X = (i\perp X)\perp = (iX)\perp \perp = iX \). By the Rieffel correspondence, we then see that \( I\perp \perp = I \). Thus we see that every closed ideal of \( A \) is orthogonally closed in \( A \). Hence Lemma 2.1 yields that \( A = I \oplus I\perp \) for any closed ideal \( I \).

Let \( Y \) be any closed \( A-B \)-submodule of \( X \). Since \( I_Y \) is a closed ideal of \( A \), it follows from the above claim that \( A = I_Y \oplus (I_Y)\perp \). We remark that \( I_Y \perp = (I_Y)\perp \) (see the proof of [3] Theorem 2.3 for the details). Since the Rieffel correspondences are natural lattice isomorphisms among \( \mathcal{I}(A), \mathcal{I}(B) \) and \( \mathcal{S}(X) \), a least upper bound \( Y \vee Y\perp (= Y \oplus Y\perp) \) of \( Y \) and \( Y\perp \) corresponds to a least upper bound \( I_Y \vee I_Y\perp (= I_Y \oplus (I_Y)\perp = A) \) of \( I_Y \) and \( I_Y\perp \). Since \( A \) and \( X \) correspond by the Rieffel correspondence, we conclude that \( X = Y \oplus Y\perp \). \( \square \)

We denote by \( \widehat{A} \) the spectrum of \( A \), that is, the set of (unitary) equivalence classes of nonzero irreducible representations of \( A \) equipped with the Jacobson topology. We note that \( \widehat{A} \) is a locally compact space, not necessarily a \( T_0 \)-space. The reader is referred to [4] for the spectrum of a \( C^* \)-algebra.

**Corollary 2.4.** Let two \( C^* \)-algebras \( A \) and \( B \) be Morita equivalent and let \( X \) be an \( A-B \)-imprimitivity bimodule. Consider the following conditions:

1. The spectrum \( \widehat{A} \) of \( A \) is discrete in the Jacobson topology.
2. The spectrum \( \widehat{B} \) of \( B \) is discrete in the Jacobson topology.
3. Every closed \( A-B \)-submodule of \( X \) is complemented in \( X \).
4. Every closed \( A-B \)-submodule of \( X \) is orthogonally closed in \( X \).

Then we have \( (1) \iff (2) \iff (3) \iff (4) \). If either \( \widehat{A} \) or \( \widehat{B} \) is a \( T_1 \)-space, then conditions \( (1) \sim (4) \) are equivalent.

**Proof.** This easily follows from [3] Theorem 2.3] and Theorem 2.3 above. \( \square \)

Note that in the implication \( (4) \Rightarrow (1) \) above, the assumption that either \( \widehat{A} \) or \( \widehat{B} \) be a \( T_1 \)-space is necessary (see [3] Remark 2.4 (2))). We end this paper by stating a remark.

**Remark 2.5.** Let \( A \) and \( B \) be \( C^* \)-algebras and let \( X \) be an \( A-B \)-imprimitivity bimodule. Consider the following conditions:

1. Every closed right \( B \)-submodule of \( X \) is orthogonally closed in \( X \).
2. Every closed right \( B \)-submodule of \( X \) is orthogonally complemented in \( X \).
3. Every closed \( A-B \)-submodule of \( X \) is orthogonally closed in \( X \).
(4) Every closed $A$-$B$-submodule of $X$ is orthogonally complemented in $X$.

Then the equivalence of (1) and (2) follows from [7], the implication $(2) \implies (4)$ is obvious, and the equivalence of (3) and (4) is nothing but Theorem 2.3. When $B$ is simple, it follows from an easy application of the Rieffel correspondence that condition (4) is true. In this case, if $B$ is not an elementary $C^*$-algebra, both conditions (1) and (2) are false (see [5] and [7]), that is, condition (4) does not necessarily imply condition (2). Thus we see that $(1) \iff (2) \iff (3) \iff (4)$.

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**References**


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