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DISCRETE SPECTRA OF C^*-ALGEBRAS AND ORTHOGONALLY CLOSED SUBMODULES IN HILBERT C^*-MODULES

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Abstract. Let A and B be C^*-algebras and let X be an A-B-imprimitivity bimodule. Then it is shown that if the spectrum $\widehat{A}$ of A (resp. $\widehat{B}$ of B) is discrete, then every closed A-B-submodule of X is orthogonally closed in X, and conversely that if $\widehat{A}$ (resp. $\widehat{B}$) is a T_1-space and if every closed A-B-submodule of X is orthogonally closed in X, then $\widehat{A}$ (resp. $\widehat{B}$) is discrete.

1. Introduction

Let A be a C^*-algebra and let X be a Hilbert A-module with an A-valued inner product $\langle \cdot\ ,\ \cdot \rangle$. For any closed Hilbert A-submodule Y of X, we denote by $Y^\perp$ the orthogonally complemented subspace of Y in X, i.e.,

$$Y^\perp = \{\ x \in X \mid \langle x\ ,\ y\rangle = 0 \text{ for all } y \in Y\ \}.$$

We say that a closed A-submodule Y of a Hilbert A-module X is *orthogonally complemented* in X if X coincides with $Y \oplus Y^\perp$, and that a closed A-submodule Y of a Hilbert A-module X is *orthogonally closed* in X if $Y^{\perp\perp} = Y$. If Y is orthogonally complemented in X, then it is orthogonally closed in X. But the converse is not necessarily true in general. Suppose that X is a full (right) Hilbert A-module. Then Schweitzer [7, Theorem 1] has shown that every closed right A-submodule of X is orthogonally closed if and only if every closed right A-submodule of X is orthogonally complemented in X. Here we remark that every full right Hilbert A-module is a $\mathcal{K}(X)$-A-imprimitivity bimodule, where $\mathcal{K}(X)$ is the C^*-algebra generated by those operators $\theta_{x,y}$ with all $x, y \in X$ defined by $\theta_{x,y}(z) = x\cdot\langle y\ ,\ z\rangle$ for all $z \in X$.

Let A and B be C^*-algebras and let X be an A-B-imprimitivity bimodule. Note that A is isomorphic to $\mathcal{K}(X)$. In this paper, we show that every closed A-B-submodule of X is orthogonally closed if and only if every closed A-B-submodule of X is orthogonally complemented in X. As a corollary, we show that if the spectrum $\widehat{A}$ of A (resp. $\widehat{B}$ of B) is discrete, then every closed A-B-submodule of X is orthogonally closed in X, and conversely that if $\widehat{A}$ (resp. $\widehat{B}$) is a T_1-space and if every closed A-B-submodule of X is orthogonally closed in X, then $\widehat{A}$ (resp. $\widehat{B}$) is discrete.

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2. Results

Let A and B be C^*-algebras and let X be an A-B-imprimitivity bimodule (see [4] or [6] for the definition of an imprimitivity bimodule). We denote by ${}_A\langle\cdot\ ,\ \cdot\rangle$ the A-valued inner product on X as a left Hilbert A-module and by $\langle\cdot\ ,\ \cdot\rangle_B$ the B-valued inner product on X as a right Hilbert B-module, respectively.

Throughout this paper, by an *A-B-submodule* we mean a left A-submodule and right B-submodule.

Two C^*-algebras A and B are said to be *Morita equivalent* if there exists an A-B-imprimitivity bimodule. We remark that in this paper, Morita equivalence means strong Morita equivalence in the sense of Rieffel (cf. [6, Remark 3.15]).

Let A be a C^*-algebra and let I be a closed ideal of A. Throughout this paper, unless otherwise stated, by an ideal of A we always mean a two-sided ideal. Then $I^\perp$ is defined by

$$I^\perp = \{a \in A \mid ax = 0 \text{ for all } x \in I\ \}.$$

It is easily seen that $I^\perp$ is a closed ideal of A. Note here that A can be regarded as an A-A-imprimitivity bimodule, for the bimodule structure given by the multiplication in A, with ${}_A\langle a\ ,\ b\rangle = ab^*$ and $\langle a\ ,\ b\rangle_A = a^*b$ for $a, b \in A$ (see [6, Example 3.5]). Then I is a closed A-A-submodule of A.

Let A and B be C^*-algebras and let X be an A-B-imprimitivity bimodule. For a closed A-B-submodule Y of X, its orthogonally complemented subspace $Y^\perp$ in X is defined by

$$Y^\perp = \{\ x \in X \mid {}_A\langle x\ ,\ y\rangle = \langle x\ ,\ y\rangle_B = 0 \text{ for all } y \in Y\ \}.$$

Lemma 2.1. *Let A be a C^*-algebra and let I be any closed ideal I of A. If every closed ideal of A is orthogonally closed in A, then we have $A = I \oplus I^\perp$.*

Proof. Using the fact that for an ideal J of A we have $J \cap J^\perp = \{0\}$, it follows that

$$(I \oplus I^\perp)^\perp = I^\perp \cap (I^\perp)^\perp = \{0\}.$$

Hence, using the hypothesis, we have $I \oplus I^\perp = (I \oplus I^\perp)^{\perp\perp} = A$. □

Now we recall the Rieffel correspondence (see [6, Theorem 3.22]). Let two C^*-algebras A and B be Morita equivalent and let X be an A-B-imprimitivity bimodule. We denote by $\mathcal{I}(A)$ (resp. $\mathcal{I}(B)$) the set of all closed (two-sided) ideals of A (resp. B), and by $\mathcal{S}(X)$ the set of closed A-B-submodules of X. Note that $\mathcal{I}(A)$, $\mathcal{I}(B)$ and $\mathcal{S}(X)$ can be partially ordered by inclusion, and are then lattices. Then there are natural lattice isomorphisms among $\mathcal{I}(A)$, $\mathcal{I}(B)$ and $\mathcal{S}(X)$ given by

$$\mathcal{I}(A) \ni I \longmapsto {}_IX \in \mathcal{S}(X), \text{ where } {}_IX = \{y \in X | {}_A\langle y\ ,\ x\rangle \in I \text{ for all } x \in X\ \};$$

$$\mathcal{S}(X) \ni Y \longmapsto I_Y \in \mathcal{I}(A),\ {}_YI \in \mathcal{I}(B),$$

where I_Y is the closed linear span of $\{{}_A\langle y\ ,\ x\rangle \mid y \in Y \text{ and } x \in X\ \}$

and ${}_YI$ is the closed linear span of $\{\langle x\ ,\ y\rangle_B \mid y \in Y \text{ and } x \in X\ \}$;

$$\mathcal{I}(B) \ni J \longmapsto X_J \in \mathcal{S}(X), \quad \text{where } X_J = \{y \in X \mid \langle x\ ,\ y\rangle_B \in J \text{ for all } x \in X\ \}.$$

We refer to such lattice isomorphisms among $\mathcal{I}(A)$, $\mathcal{I}(B)$ and $\mathcal{S}(X)$ as the *Rieffel correspondences* (see [6, 3.3]). Note that ${}_IX$ is the closed linear span of $I \cdot X$ (cf. [6, Lemma 3.23]), and we will employ this fact in the proof of Lemma 2.2.

Lemma 2.2. *Let A and B be C^*-algebras and let X be an A-B-imprimitivity bimodule. For any closed ideal I of A, we have $({}_IX)^\perp = {}_{I^\perp}X$.*

Proof. Since ${}_IX$ is the closed linear span of IX, we see from ${}_A\langle I^\perp X\ ,\ IX\rangle = I^\perp\cdot{}_A\langle X\ ,\ X\rangle\cdot I = \{0\}$ that ${}_{I^\perp}X \subset ({}_IX)^\perp$. For the reverse inclusion, let $Y = (IX)^\perp$ and note that $\{0\} = {}_A\langle Y\ ,\ IX\rangle = {}_A\langle Y\ ,\ X\rangle\cdot I$ implies that ${}_A\langle Y\ ,\ Y\rangle$ is contained in $I^\perp$, hence $Y \subset I^\perp X$ by a well-known argument. □

Now we are in a position to establish the main result.

Theorem 2.3. *Let A and B be C^*-algebras and let X be an A-B-imprimitivity bimodule. Then every closed A-B-submodule of X is orthogonally closed in X if and only if every closed A-B-submodule of X is orthogonally complemented in X.*

Proof. Suppose that every closed A-B-submodule of X is orthogonally closed in X. Let I be any closed ideal of A. We claim that $A = I \oplus I^\perp$. Since it follows from Lemma 2.2 that $({}_IX)^\perp = {}_{I^\perp}X$, we obtain $({}_IX)^{\perp\perp} = ({}_{I^\perp}X)^\perp$, and with I replaced by $I^\perp$ we have $({}_{I^\perp}X)^\perp = {}_{I^{\perp\perp}}X$. Since ${}_IX$ is a closed A-B-submodule, by assumption we see that ${}_{I^{\perp\perp}}X = ({}_{I^\perp}X)^\perp = ({}_IX)^{\perp\perp} = {}_IX$. By the Rieffel correspondence, we then see that $I^{\perp\perp} = I$. Thus we see that every closed ideal of A is orthogonally closed in A. Hence Lemma 2.1 yields that $A = I \oplus I^\perp$ for any closed ideal I.

Let Y be any closed A-B-submodule of X. Since I_Y is a closed ideal of A, it follows from the above claim that $A = I_Y \oplus (I_Y)^\perp$. We remark that $I_{Y^\perp} = (I_Y)^\perp$ (see the proof of [3, Theorem 2.3] for the details). Since the Rieffel correspondences are natural lattice isomorphisms among $\mathcal{I}(A), \mathcal{I}(B)$ and $\mathcal{S}(X)$, a least upper bound $Y \vee Y^\perp (= Y \oplus Y^\perp)$ of Y and $Y^\perp$ corresponds to a least upper bound $I_Y \vee I_{Y^\perp}$ $(= I_Y \vee (I_Y)^\perp = I_Y \oplus (I_Y)^\perp = A)$ of I_Y and $I_{Y^\perp}$. Since A and X correspond by the Rieffel correspondence, we conclude that $X = Y \oplus Y^\perp$. □

We denote by $\widehat{A}$ the spectrum of A, that is, the set of (unitary) equivalence classes of nonzero irreducible representations of A equipped with the Jacobson topology. We note that $\widehat{A}$ is a locally compact space, not necessarily a T_0-space. The reader is referred to [1] for the spectrum of a C^*-algebra.

Corollary 2.4. *Let two C^*-algebras A and B be Morita equivalent and let X be an A-B-imprimitivity bimodule. Consider the following conditions:*

(1) *The spectrum $\widehat{A}$ of A is discrete in the Jacobson topology.*
(2) *The spectrum $\widehat{B}$ of B is discrete in the Jacobson topology.*
(3) *Every closed A-B-submodule of X is complemented in X.*
(4) *Every closed A-B-submodule of X is orthogonally closed in X.*

Then we have (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) $\Longleftrightarrow$ (4). *If either $\widehat{A}$ or $\widehat{B}$ is a T_1-space, then conditions* (1) − (4) *are equivalent.*

Proof. This easily follows from [3, Theorem 2.3] and Theorem 2.3 above. □

Note that in the implication (4) $\Longrightarrow$ (1) above, the assumption that either $\widehat{A}$ or $\widehat{B}$ be a T_1-space is necessary (see [3, Remark 2.4 (2)]). We end this paper by stating a remark.

Remark 2.5. Let A and B be C^*-algebras and let X be an A-B-imprimitivity bimodule. Consider the following conditions:

(1) Every closed right B-submodule of X is orthogonally closed in X.
(2) Every closed right B-submodule of X is orthogonally complemented in X.
(3) Every closed A-B-submodule of X is orthogonally closed in X.

(4) Every closed A-B-submodule of X is orthogonally complemented in X.

Then the equivalence of (1) and (2) follows from [7], the implication (2) $\Longrightarrow$ (4) is obvious, and the equivalence of (3) and (4) is nothing but Theorem 2.3. When B is simple, it follows from an easy application of the Rieffel correspondence that condition (4) is true. In this case, if B is not an elementary C^*-algebra, both conditions (1) and (2) are false (see [5] and [7]), that is, condition (4) does not necessarily imply condition (2). Thus we see that (1) $\Longleftrightarrow$ (2) $\underset{\not\Leftarrow}{\Rightarrow}$ (3) $\Longleftrightarrow$ (4).

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References

1. J. Dixmier, *C^*-algebras*, North Holland, New York, 1982. MR0458185 (56:16388)
2. M. Kusuda, *Three-space problems in discrete spectra of C^*-algebras and dual C^*-algebras*, Proc. Royal Soc. Edinburgh **131A** (2001), 701–707. MR1838507 (2002m:46087)
3. M. Kusuda, *Discrete spectra of C^*-algebras and complemented submodules in Hilbert C^*-modules*, Proc. Amer. Math. Soc. **131** (2003), 3075–3081. MR1993216 (2004d:46067)
4. E. C. Lance, *Hilbert C^*-modules*, London Math. Soc. Lecture Note Series Vol. 210, Cambridge Univ. Press, Cambridge, 1994. MR1325694 (96k:46100)
5. B. Magajna, *Hilbert C^*-modules in which all closed submodules are complemented*, Proc. Amer. Math. Soc. **125** (1997), 849–852. MR1346981 (97e:46079)
6. I. Raeburn and D. P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Math. Surveys Monographs Vol. 60, Amer. Math. Soc., Providence, 1998. MR1634408 (2000c:46108)
7. J. Schweitzer, *A description of Hilbert C^*-modules in which all closed submodules are orthogonally closed*, Proc. Amer. Math. Soc. **127** (1999), 2123–2125. MR1646207 (99j:46068)

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