

DISCRETE SPECTRA OF  $C^*$ -ALGEBRAS  
AND ORTHOGONALLY CLOSED SUBMODULES  
IN HILBERT  $C^*$ -MODULES

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ABSTRACT. Let  $A$  and  $B$  be  $C^*$ -algebras and let  $X$  be an  $A$ - $B$ -imprimitivity bimodule. Then it is shown that if the spectrum  $\widehat{A}$  of  $A$  (resp.  $\widehat{B}$  of  $B$ ) is discrete, then every closed  $A$ - $B$ -submodule of  $X$  is orthogonally closed in  $X$ , and conversely that if  $\widehat{A}$  (resp.  $\widehat{B}$ ) is a  $T_1$ -space and if every closed  $A$ - $B$ -submodule of  $X$  is orthogonally closed in  $X$ , then  $\widehat{A}$  (resp.  $\widehat{B}$ ) is discrete.

1. INTRODUCTION

Let  $A$  be a  $C^*$ -algebra and let  $X$  be a Hilbert  $A$ -module with an  $A$ -valued inner product  $\langle \cdot, \cdot \rangle$ . For any closed Hilbert  $A$ -submodule  $Y$  of  $X$ , we denote by  $Y^\perp$  the orthogonally complemented subspace of  $Y$  in  $X$ , i.e.,

$$Y^\perp = \{ x \in X \mid \langle x, y \rangle = 0 \text{ for all } y \in Y \}.$$

We say that a closed  $A$ -submodule  $Y$  of a Hilbert  $A$ -module  $X$  is *orthogonally complemented* in  $X$  if  $X$  coincides with  $Y \oplus Y^\perp$ , and that a closed  $A$ -submodule  $Y$  of a Hilbert  $A$ -module  $X$  is *orthogonally closed* in  $X$  if  $Y^{\perp\perp} = Y$ . If  $Y$  is orthogonally complemented in  $X$ , then it is orthogonally closed in  $X$ . But the converse is not necessarily true in general. Suppose that  $X$  is a full (right) Hilbert  $A$ -module. Then Schweitzer [7, Theorem 1] has shown that every closed right  $A$ -submodule of  $X$  is orthogonally closed if and only if every closed right  $A$ -submodule of  $X$  is orthogonally complemented in  $X$ . Here we remark that every full right Hilbert  $A$ -module is a  $\mathcal{K}(X)$ - $A$ -imprimitivity bimodule, where  $\mathcal{K}(X)$  is the  $C^*$ -algebra generated by those operators  $\theta_{x,y}$  with all  $x, y \in X$  defined by  $\theta_{x,y}(z) = x \cdot \langle y, z \rangle$  for all  $z \in X$ .

Let  $A$  and  $B$  be  $C^*$ -algebras and let  $X$  be an  $A$ - $B$ -imprimitivity bimodule. Note that  $A$  is isomorphic to  $\mathcal{K}(X)$ . In this paper, we show that every closed  $A$ - $B$ -submodule of  $X$  is orthogonally closed if and only if every closed  $A$ - $B$ -submodule of  $X$  is orthogonally complemented in  $X$ . As a corollary, we show that if the spectrum  $\widehat{A}$  of  $A$  (resp.  $\widehat{B}$  of  $B$ ) is discrete, then every closed  $A$ - $B$ -submodule of  $X$  is orthogonally closed in  $X$ , and conversely that if  $\widehat{A}$  (resp.  $\widehat{B}$ ) is a  $T_1$ -space and if every closed  $A$ - $B$ -submodule of  $X$  is orthogonally closed in  $X$ , then  $\widehat{A}$  (resp.  $\widehat{B}$ ) is discrete.

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## 2. RESULTS

Let  $A$  and  $B$  be  $C^*$ -algebras and let  $X$  be an  $A$ - $B$ -imprimitivity bimodule (see [4] or [6] for the definition of an imprimitivity bimodule). We denote by  ${}_A\langle \cdot, \cdot \rangle$  the  $A$ -valued inner product on  $X$  as a left Hilbert  $A$ -module and by  $\langle \cdot, \cdot \rangle_B$  the  $B$ -valued inner product on  $X$  as a right Hilbert  $B$ -module, respectively.

Throughout this paper, by an  $A$ - $B$ -submodule we mean a left  $A$ -submodule and right  $B$ -submodule.

Two  $C^*$ -algebras  $A$  and  $B$  are said to be *Morita equivalent* if there exists an  $A$ - $B$ -imprimitivity bimodule. We remark that in this paper, Morita equivalence means strong Morita equivalence in the sense of Rieffel (cf. [6, Remark 3.15]).

Let  $A$  be a  $C^*$ -algebra and let  $I$  be a closed ideal of  $A$ . Throughout this paper, unless otherwise stated, by an ideal of  $A$  we always mean a two-sided ideal. Then  $I^\perp$  is defined by

$$I^\perp = \{a \in A \mid ax = 0 \text{ for all } x \in I\}.$$

It is easily seen that  $I^\perp$  is a closed ideal of  $A$ . Note here that  $A$  can be regarded as an  $A$ - $A$ -imprimitivity bimodule, for the bimodule structure given by the multiplication in  $A$ , with  ${}_A\langle a, b \rangle = ab^*$  and  $\langle a, b \rangle_A = a^*b$  for  $a, b \in A$  (see [6, Example 3.5]). Then  $I$  is a closed  $A$ - $A$ -submodule of  $A$ .

Let  $A$  and  $B$  be  $C^*$ -algebras and let  $X$  be an  $A$ - $B$ -imprimitivity bimodule. For a closed  $A$ - $B$ -submodule  $Y$  of  $X$ , its orthogonally complemented subspace  $Y^\perp$  in  $X$  is defined by

$$Y^\perp = \{x \in X \mid {}_A\langle x, y \rangle = \langle x, y \rangle_B = 0 \text{ for all } y \in Y\}.$$

**Lemma 2.1.** *Let  $A$  be a  $C^*$ -algebra and let  $I$  be any closed ideal  $I$  of  $A$ . If every closed ideal of  $A$  is orthogonally closed in  $A$ , then we have  $A = I \oplus I^\perp$ .*

*Proof.* Using the fact that for an ideal  $J$  of  $A$  we have  $J \cap J^\perp = \{0\}$ , it follows that

$$(I \oplus I^\perp)^\perp = I^\perp \cap (I^\perp)^\perp = \{0\}.$$

Hence, using the hypothesis, we have  $I \oplus I^\perp = (I \oplus I^\perp)^{\perp\perp} = A$ .  $\square$

Now we recall the Rieffel correspondence (see [6, Theorem 3.22]). Let two  $C^*$ -algebras  $A$  and  $B$  be Morita equivalent and let  $X$  be an  $A$ - $B$ -imprimitivity bimodule. We denote by  $\mathcal{I}(A)$  (resp.  $\mathcal{I}(B)$ ) the set of all closed (two-sided) ideals of  $A$  (resp.  $B$ ), and by  $\mathcal{S}(X)$  the set of closed  $A$ - $B$ -submodules of  $X$ . Note that  $\mathcal{I}(A)$ ,  $\mathcal{I}(B)$  and  $\mathcal{S}(X)$  can be partially ordered by inclusion, and are then lattices. Then there are natural lattice isomorphisms among  $\mathcal{I}(A)$ ,  $\mathcal{I}(B)$  and  $\mathcal{S}(X)$  given by

$$\mathcal{I}(A) \ni I \longmapsto {}_IX \in \mathcal{S}(X), \text{ where } {}_IX = \{y \in X \mid {}_A\langle y, x \rangle \in I \text{ for all } x \in X\};$$

$$\mathcal{S}(X) \ni Y \longmapsto I_Y \in \mathcal{I}(A), \quad {}_YI \in \mathcal{I}(B),$$

where  $I_Y$  is the closed linear span of  $\{{}_A\langle y, x \rangle \mid y \in Y \text{ and } x \in X\}$

and  ${}_YI$  is the closed linear span of  $\{\langle x, y \rangle_B \mid y \in Y \text{ and } x \in X\}$ ;

$$\mathcal{I}(B) \ni J \longmapsto X_J \in \mathcal{S}(X), \text{ where } X_J = \{y \in X \mid \langle x, y \rangle_B \in J \text{ for all } x \in X\}.$$

We refer to such lattice isomorphisms among  $\mathcal{I}(A)$ ,  $\mathcal{I}(B)$  and  $\mathcal{S}(X)$  as the *Rieffel correspondences* (see [6, 3.3]). Note that  ${}_IX$  is the closed linear span of  $I \cdot X$  (cf. [6, Lemma 3.23]), and we will employ this fact in the proof of Lemma 2.2.

**Lemma 2.2.** *Let  $A$  and  $B$  be  $C^*$ -algebras and let  $X$  be an  $A$ - $B$ -imprimitivity bimodule. For any closed ideal  $I$  of  $A$ , we have  $({}_IX)^\perp = {}_{I^\perp}X$ .*

*Proof.* Since  ${}_I X$  is the closed linear span of  $IX$ , we see from  ${}_A \langle I^\perp X, IX \rangle = I^\perp \cdot {}_A \langle X, X \rangle \cdot I = \{0\}$  that  ${}_I X \subset ({}_I X)^\perp$ . For the reverse inclusion, let  $Y = (IX)^\perp$  and note that  $\{0\} = {}_A \langle Y, IX \rangle = {}_A \langle Y, X \rangle \cdot I$  implies that  ${}_A \langle Y, Y \rangle$  is contained in  $I^\perp$ , hence  $Y \subset I^\perp X$  by a well-known argument.  $\square$

Now we are in a position to establish the main result.

**Theorem 2.3.** *Let  $A$  and  $B$  be  $C^*$ -algebras and let  $X$  be an  $A$ - $B$ -imprimitivity bimodule. Then every closed  $A$ - $B$ -submodule of  $X$  is orthogonally closed in  $X$  if and only if every closed  $A$ - $B$ -submodule of  $X$  is orthogonally complemented in  $X$ .*

*Proof.* Suppose that every closed  $A$ - $B$ -submodule of  $X$  is orthogonally closed in  $X$ . Let  $I$  be any closed ideal of  $A$ . We claim that  $A = I \oplus I^\perp$ . Since it follows from Lemma 2.2 that  $({}_I X)^\perp = {}_{I^\perp} X$ , we obtain  $({}_I X)^{\perp\perp} = ({}_{I^\perp} X)^\perp$ , and with  $I$  replaced by  $I^\perp$  we have  $({}_{I^\perp} X)^\perp = {}_{I^{\perp\perp}} X$ . Since  ${}_I X$  is a closed  $A$ - $B$ -submodule, by assumption we see that  ${}_{I^{\perp\perp}} X = ({}_{I^\perp} X)^\perp = ({}_I X)^{\perp\perp} = {}_I X$ . By the Rieffel correspondence, we then see that  $I^{\perp\perp} = I$ . Thus we see that every closed ideal of  $A$  is orthogonally closed in  $A$ . Hence Lemma 2.1 yields that  $A = I \oplus I^\perp$  for any closed ideal  $I$ .

Let  $Y$  be any closed  $A$ - $B$ -submodule of  $X$ . Since  $I_Y$  is a closed ideal of  $A$ , it follows from the above claim that  $A = I_Y \oplus (I_Y)^\perp$ . We remark that  $I_{Y^\perp} = (I_Y)^\perp$  (see the proof of [3, Theorem 2.3] for the details). Since the Rieffel correspondences are natural lattice isomorphisms among  $\mathcal{I}(A), \mathcal{I}(B)$  and  $\mathcal{S}(X)$ , a least upper bound  $Y \vee Y^\perp (= Y \oplus Y^\perp)$  of  $Y$  and  $Y^\perp$  corresponds to a least upper bound  $I_Y \vee I_{Y^\perp} (= I_Y \vee (I_Y)^\perp = I_Y \oplus (I_Y)^\perp = A)$  of  $I_Y$  and  $I_{Y^\perp}$ . Since  $A$  and  $X$  correspond by the Rieffel correspondence, we conclude that  $X = Y \oplus Y^\perp$ .  $\square$

We denote by  $\widehat{A}$  the spectrum of  $A$ , that is, the set of (unitary) equivalence classes of nonzero irreducible representations of  $A$  equipped with the Jacobson topology. We note that  $\widehat{A}$  is a locally compact space, not necessarily a  $T_0$ -space. The reader is referred to [1] for the spectrum of a  $C^*$ -algebra.

**Corollary 2.4.** *Let two  $C^*$ -algebras  $A$  and  $B$  be Morita equivalent and let  $X$  be an  $A$ - $B$ -imprimitivity bimodule. Consider the following conditions:*

- (1) *The spectrum  $\widehat{A}$  of  $A$  is discrete in the Jacobson topology.*
- (2) *The spectrum  $\widehat{B}$  of  $B$  is discrete in the Jacobson topology.*
- (3) *Every closed  $A$ - $B$ -submodule of  $X$  is complemented in  $X$ .*
- (4) *Every closed  $A$ - $B$ -submodule of  $X$  is orthogonally closed in  $X$ .*

*Then we have (1)  $\iff$  (2)  $\implies$  (3)  $\iff$  (4). If either  $\widehat{A}$  or  $\widehat{B}$  is a  $T_1$ -space, then conditions (1) – (4) are equivalent.*

*Proof.* This easily follows from [3, Theorem 2.3] and Theorem 2.3 above.  $\square$

Note that in the implication (4)  $\implies$  (1) above, the assumption that either  $\widehat{A}$  or  $\widehat{B}$  be a  $T_1$ -space is necessary (see [3, Remark 2.4 (2)]). We end this paper by stating a remark.

**Remark 2.5.** Let  $A$  and  $B$  be  $C^*$ -algebras and let  $X$  be an  $A$ - $B$ -imprimitivity bimodule. Consider the following conditions:

- (1) *Every closed right  $B$ -submodule of  $X$  is orthogonally closed in  $X$ .*
- (2) *Every closed right  $B$ -submodule of  $X$  is orthogonally complemented in  $X$ .*
- (3) *Every closed  $A$ - $B$ -submodule of  $X$  is orthogonally closed in  $X$ .*

(4) Every closed  $A$ - $B$ -submodule of  $X$  is orthogonally complemented in  $X$ .

Then the equivalence of (1) and (2) follows from [7], the implication (2)  $\implies$  (4) is obvious, and the equivalence of (3) and (4) is nothing but Theorem 2.3. When  $B$  is simple, it follows from an easy application of the Rieffel correspondence that condition (4) is true. In this case, if  $B$  is not an elementary  $C^*$ -algebra, both conditions (1) and (2) are false (see [5] and [7]), that is, condition (4) does not necessarily imply condition (2). Thus we see that (1)  $\iff$  (2)  $\not\iff$  (3)  $\iff$  (4).

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