PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 11, Pages 3143–3152
S 0002-9939(05)07917-7

SUMS OF SQUARES IN OCTONION ALGEBRAS

S. PUMPLÜN

(Communicated by Lance W. Small)

Abstract. Sums of squares in composition algebras are investigated using methods from the theory of quadratic forms. For any integer $m \geq 1$ octonion algebras of level 2^m and of level $2^m + 1$ are constructed.

Introduction

The investigation of sums of squares is a classical number-theoretic problem and goes back to Diophantes, Fermat, Lagrange and Gauss who studied how to express integers as sums of squares. The notion of level of a field seems to have been introduced by Artin and Schreier [AS]. It was later generalized to commutative rings (see Pfister [Pf] and Dai, Lam and Peng [DLP] for lists of references) and then to noncommutative rings, in particular to division rings and hence quaternion algebras over fields, for instance by Leep [Le] and Lewis [L3].

As mentioned already by Lewis [L1], the definition of level makes sense not just for associative unital rings. However, there seems to be nothing in the literature about this problem in a nonassociative setting. It turns out that much of the existing theory on sums of squares in noncommutative rings can be effortlessly transferred to quadratic algebras with a scalar involution. The best known among these are certainly the octonion algebras. We investigate the level of composition algebras over arbitrary rings, extending results on sums of squares in finite-dimensional division algebras (which are finite-dimensional over the center) by Leep, Shapiro and Wadsworth [LSW], and on the level of quaternion algebras over fields of characteristic not two by Koprowski [Ko], and Lewis [L2], [L3]. Furthermore, we construct octonion algebras of level 2^m (indeed, even octonion algebras of level 2^m, where -1 is not a sum of 2^m squares of pure octonions), and of level $2^m + 1$, for any integer $m \geq 1$ using arguments relying on function fields of quadratic forms as in Laghribi and Mammone [LM]. We do not know if other integers can also appear as a level of an octonion algebra (this seems to be still an open question for quaternion algebras as well). The aim of this paper is to give a first insight in how easily many, by now well-known results on sums of squares and levels, can be transferred to a nonassociative setting.

The author would like to thank T. Unger, who pointed out Lewis' work to her and gave her the idea to study sums of squares in this context.

Received by the editors December 9, 2002 and, in revised form, June 14, 2004.
2000 *Mathematics Subject Classification.* Primary 17A75; Secondary 11E25.
Key words and phrases. Composition algebras, quadratic algebras, level, sums of squares.

1. Preliminaries

For the convenience of the reader, we summarize the main facts about the algebras needed in this paper:

Let R be a unital commutative associative ring, and A a unital nonassociative R-algebra. The term "R-algebra" always refers to unital nonassociative algebras which are finitely generated projective as R-modules. We write A^2 for the set of squares of elements in A and ΣA^2 for the set of all sums of squares of elements in A. The smallest positive integer m such that -1 is a sum of m squares in A is called the *level* of A, denoted $s(A)$. If there is no such integer, we set $s(A) = \infty$.

Associativity in A is measured by the *associator* $[x, y, z] = (xy)z - x(yz)$ and commutativity by the *commutator* $[x, y] = xy - yx$. Define the *nucleus* of A by $\mathrm{Nuc}(A) = \{x \in A | [x, A, A] = [A, x, A] = [A, A, x]\}$ and the *commuter* by $\mathrm{Comm}(A) = \{x \in A | [x, A] = 0\}$. A map τ is called an *involution* on A if it is an anti-automorphism of period 2. If 2 is an invertible element in R, we have $A = \mathrm{Sym}(A, \tau) \oplus \mathrm{Skew}(A, \tau)$ with $\mathrm{Skew}(A, \tau) = \{x \in A \mid \tau(x) = -x\}$ the set of skew-symmetric elements and $\mathrm{Sym}(A, \tau) = \{x \in A \mid \tau(x) = x\}$ the set of symmetric elements in A with respect to τ. An involution is called *scalar* if all *norms* $\tau(x)x$ and all *traces* $\tau(x)+x$ are scalars in R. For a scalar involution, $n_A(x) = \tau(x)x$ (resp. $t_A(x) = \tau(x) + x$) is a quadratic (resp. a linear form) on A, whenever $a1_A = 0$ implies $a = 0$, for every $a \in R$ [M, p. 86]. Thus we will assume this whenever we refer to an algebra A with a scalar involution.

An R-algebra A is called *quadratic* in case there exists a quadratic form $n\colon A \to R$ such that $n(1_A) = 1$ and $x^2 - n(1_A, x)x + n(x)1_A = 0$ for all $x \in A$, where $n(x, y)$ denotes the induced symmetric bilinear form $n(x, y) = n(x + y) - n(x) - n(y)$. The form n is uniquely determined, usually denoted by n_A, and is called the *norm* of the quadratic algebra A.

Let A be a quadratic R-algebra with a scalar involution σ and norm form $n_A(x) = x\sigma(x)$ of rank greater than 2. Put $F = \mathrm{Skew}(A, \tau)$. If 2 is an invertible element in R, then $A = R \oplus F$ and $n_A = \langle 1\rangle \perp n_0$ with $n_0 = n_A|_F$. The multiplication in A can be described by

$$(a, u)(b, v) = (ab - B(u, v), av + bu + u \times v),$$

for $a, b \in R$ and $u, v \in F$. Here, $\times : F \times F \to F$ is a skew-symmetric R-bilinear map, and $B : F \times F \to R$ the symmetric bilinear form defined by $B(u, v) = \frac{1}{2} n_A(u, v)$. The scalar involution on A is given by $\sigma : A \to A$, $\sigma(a, u) = (a, -u)$.

An R-algebra C is called a *composition algebra* if it carries a quadratic form $n\colon C \to R$ satisfying the following two conditions: (i) its induced symmetric bilinear form $n(x, y) = n(x + y) - n(x) - n(y)$ is nondegenerate, i.e. determines an R-module isomorphism $C \xrightarrow{\sim} C^\vee = \mathrm{Hom}_R(C, R)$; (ii) n permits composition; that is, $n(xy) = n(x)n(y)$ for all $x, y \in C$.

Composition algebras are quadratic alternative algebras. More precisely, a quadratic form n of the composition algebra satisfying (i) and (ii) above agrees with its norm as a quadratic algebra and thus is unique. It is called the *norm* of the composition algebra C and is often denoted by n_C. Composition algebras only exist in ranks 1, 2, 4 or 8. Those of rank 2 are exactly quadratic étale R-algebras, those of rank 4 exactly the quaternion algebras. The ones of rank 8 are called *octonion algebras.*

A composition algebra over R is called *split* if it contains a composition subalgebra isomorphic to $R \oplus R$ (see [P] for an explicit description of all possible split composition algebras). A composition algebra C has a *canonical involution* $^-$ given by $\overline{x} = t_C(x)1_C - x$, where $t_C\colon C \to R$ is the *trace* given by $t_C(x) := n(1_C, x)$. This involution is scalar.

Let A be a quadratic R-algebra with scalar involution $*$ and let $\mu \in R$ be invertible. Then the R-module $A \oplus A$ becomes a quadratic R-algebra via the multiplication

$$(u,v)(u',v') = (uu' + \mu v'^* v, v'u + vu'^*)$$

for $u, u', v, v' \in A$, with involution $(u,v)^* = (u^*, -v)$. It is called the *(classical) Cayley-Dickson doubling* of A, and is denoted by $\mathrm{Cay}(A,\mu)$. The new involution $*$ is a scalar involution on $\mathrm{Cay}(A,\mu)$ with norm $n_{\mathrm{Cay}(A,\mu)}((u,v)) = n_A(u) - \mu n_A(v)$. The Cayley-Dickson doubling process depends on the scalar μ only up to an invertible square. By repeated application of the Cayley-Dickson doubling process starting from a composition algebra C over R we obtain either again a composition algebra (if the rank of the new algebra is less than or equal to 8), or a *generalized Cayley-Dickson algebra* of rank ≥ 16. The latter are no longer alternative, but still flexible (i.e., $x(yx) = (xy)x$, for all elements $x, y \in A$) with a scalar involution [M].

Over fields, the classical Cayley-Dickson process generates all possible composition algebras. Over rings, a more general version is required, which yields all those composition algebras containing a composition subalgebra of half their rank. This *generalized Cayley-Dickson doubling process* is due to Petersson [P]: Let D be a composition algebra of rank ≤ 4 over R with canonical involution $^-$. Let P be a finitely generated projective right D-module of rank one, with a nondegenerate $^-$-hermitian form $h\colon P \times P \to D$ (i.e., a biadditive map $h\colon P \times P \to D$ with $h(ws, w't) = \bar{s}h(w,w')t$ and $h(w,w') = \overline{h(w',w)}$ for all $s, t \in D$, $w, w' \in P$, and where $P \to \overline{P}^{\vee}$, $w \mapsto h(w,\cdot)$ is an isomorphism of right D-modules). The R-module $C = D \oplus P$ is an R-algebra via the multiplication

$$(u,w)(u',w') = (uu' + h(w',w), w' \cdot u + w \cdot \overline{u}')$$

for $u, u' \in D$, $w, w' \in P$, with $\cdot$ denoting the right D-module structure of P. It is called $\mathrm{Cay}(D,P,h)$. Its norm is given by $n((u,w)) = n_D(u) - h(w,w)$. D itself is canonically a (free) right D-module of rank one, equipped with a nondegenerate hermitian form $h_0\colon D \times D \to D$, $(w,w') \mapsto \overline{w}'w$. For any $\mu \in R^{\times}$, we obtain in this special case the "classical" doubling $\mathrm{Cay}(D,\mu) = \mathrm{Cay}(D,D,\mu h_0)$.

2. Some classical results in a nonassociative setting

As a first step we consider some elementary cases where every element in the algebra is a sum of squares. Of course, rings of characteristic 2 will always play a special role; for instance, let A be an R-algebra with a scalar involution. Then

$$\Sigma A^2 \subset \{x \in A \mid tr_A(x) \in k^2\},$$

for any ring R of characteristic 2, since in that case $t_A(x)^2 = t_A(x^2)$ holds for the trace map t_A. From now on, we will exclusively deal with rings where 2 is an invertible element. The proof of [LSW, 1.1] easily generalizes as follows:

2.1. **Lemma.** *Let A be an algebra over R where R can be viewed as a subring of A, and where $R \subset \mathrm{Nuc}(A) \cap \mathrm{Comm}(A) = \mathrm{Center}(A)$. If $-1 \in \Sigma R^2$ (e.g. if R is a* *field which is not formally real), then*

$$A = \Sigma A^2.$$

Proof. The proof is exactly as given in [LSW, 1.1]: Let $-1 = \sum_{i=1}^m x_i^2$ in R, with $x_i \in R$. For every $a \in A$ we have

$$a = (\frac{a+1}{2})^2 - (\frac{a-1}{2})^2 = (\frac{a+1}{2})^2 + \sum_{i=1}^m (\frac{x_i(a-1)}{2})^2 \in \Sigma A^2.$$

□

2.2. **Examples.** (i) ([LSW, 1.2]) Every element in the split quaternion algebra $D = \mathrm{Mat}_2(R)$ is a sum of 3 squares.

(ii) Let $a \in R^\times$. Every element in the split octonion algebra $C = \mathrm{Cay}(D, a)$ is a sum of 6 squares and $s(C) \leq 3$. In particular, every element in Zorn's algebra of vector matrices $\mathrm{Zor}(R)$ is a sum of 6 squares. (This follows directly from (i): Each element x in C can be written as $x = (u, v)$ with $u, v \in D = \mathrm{Mat}_2(R)$. Since both u and v are sums of 3 squares in D this implies the assertion.)

(iii) Let C be a composition algebra over R (resp., any R-algebra A with a scalar involution such that $R \subset \mathrm{Center}(A)$). If there exists an invertible element $u \in \mathrm{Skew}(C, \bar{\ })$ such that $n_C(u) \in R^{\times 2}$, then $s(C) = 1$. (Since $n_C(u) = -u^2$ write $-1 = \frac{1}{n_C(u)} u^2 = (a^{-1}u)^2$ if $n_C(u) = a^2$ with $a \in R^\times$.) This condition is equivalent to $C = \mathrm{Cay}(T, P, N)$ with $T = \mathrm{Cay}(R, -1)$ if C is a quaternion algebra. It is satisfied for any octonion algebra C containing a quadratic étale algebra isomorphic to $T = \mathrm{Cay}(R, -1)$.

2.3. **Lemma.** *Let k be a field of characteristic not 2. Then any split composition algebra C over k of dimension greater than 2 has $s(C) = 1$.*

Proof. If $(a, b)_F$ is a split quaternion algebra, then the form $\langle a, b, -ab \rangle$ is isotropic, and thus there are elements $x_i \in F$, not all zero, such that $-1 = ax_1^2 + bx_2^2 - abx_3^2$. Hence $-1 = (x_1 i + x_2 j + x_3 k)^2$ with $1, i, j, k$ a standard basis for $(a, b)_F$. This implies the assertion for split octonions. □

We call a quadratic form q over a ring R *isotropic* if there exists an element x such that $q(x) = 0$, and *weakly isotropic* if its multiple $m \times q = q \perp \ldots \perp q$ is isotropic, for some integer m. It is well known that zero is a nontrivial sum of squares in a central simple algebra over a field of characteristic not 2 if and only if the trace form of the algebra is weakly isotropic [L2]. This turns out to be true in a more general context. Again the *trace form* is defined to be the quadratic form $tr_A : A \to R$, $x \to t_A(x^2)$, where t_A is the trace $t_A(x) = x + \bar{x}$ of an algebra A with scalar involution $\bar{\ }$.

2.4. **Proposition.** (i) *Let A be any R-algebra with a scalar involution (e.g. a composition algebra). Then 0 is a nontrivial sum of squares in A if and only if the trace form tr_A is a weakly isotropic quadratic form.*

(ii) (*cf.* [LSW, 2.4]) *Let k be a formally real SAP field (e.g. a formally real algebraic extension of $\mathbb{Q}$, or a field of transcendence degree ≤ 1 over a real closed field). Then 0 is a nontrivial sum of squares in every composition algebra over k of dimension greater than 2.*

Proof. (i) If $0 = \sum_{i=1}^{m} x_i^2$ with $x_i \in A$ not all zero, then $0 = \sum_{i=1}^{m} tr_A(x_i^2)$ and hence tr_A is weakly isotropic. Conversely, if tr_A is weakly isotropic, then there are $x_i \in A$ not all zero such that $0 = \sum_{i=1}^{m} tr_A(x_i^2) = x_1^2 + \overline{x_1}^2 + ... + x_m^2 + \overline{x_m}^2$, and thus 0 is a nontrivial sum of squares in A.

(ii) This is straightforward, since in the above situation, every trace form of a composition algebra of dimension greater than 2 is weakly isotropic [LSW, 2.3]. □

2.5. **Example** (cf. [LSW, 2.5])**.** Let k_0 be a formally real field, and let $k = k_0(x_1, x_2, x_3)$ be a purely transcendental field extension of k. Then $C = \mathrm{Cay}\,(k, x_1, x_2, x_3)$ is a composition division algebra over k and by Springer's theorem, t_C is strongly anisotropic; hence 0 is not a nontrivial sum of squares in C by 2.4 (i).

There is a hermitian analogue of 2.4 (i) (cf. [Se] and [U] for corresponding results for central simple algebras with involutions, [PU] for results on the hermitian level of composition algebras): Define the *involution trace form* of an algebra A with scalar involution by $t_\tau : C \to R$, $x \to t_A(\tau(x)x)$ whenever τ is any involution on A. Instead of sums of squares, we now look at sums of hermitian squares, i.e. sums of elements of the type $\tau(x)x$ with $x \in A$.

2.6. **Proposition.** *Let C be a composition algebra over a ring R and let τ be any involution on C. Then 0 is a nontrivial sum of hermitian squares $\tau(x)x$ in C if and only if the involution trace form t_τ is a weakly isotropic quadratic form.*

Proof. If $0 = \sum_{i=1}^{m} \tau(x_i)x_i$ with $x_i \in C$ not all zero, then it follows that $0 = \sum_{i=1}^{m} tr_C(\tau(x_i)x_i)$ and hence t_τ is weakly isotropic. Conversely, we know that $\tau \circ {}^{-} = {}^{-} \circ \tau$, for any involution τ on C [Pu1]. Hence $t_\tau(x) = \tau(x)x + \bar{x}\tau(\bar{x})$. If t_τ is weakly isotropic, then there are $x_i \in C$ not all zero such that $0 = \sum_{i=1}^{m} tr_\tau(x_i) = \sum_{i=1}^{m}(\tau(x_i)x_i + \bar{x}_i\tau(\bar{x}_i))$. Put $y_i = \tau(\bar{x}_i)$. Then $0 = \sum_{i=1}^{m}(\tau(x_i)x_i + \tau(y_i)y_i)$ is a nontrivial sum of hermitian squares in C. □

This proof works for any quadratic R-algebra with a scalar involution as long as τ commutes with it.

The next result is well known for fields and is proved in [LSW, Theorem D] for division algebras which are finite-dimensional over their center (and thus in particular for quaternion algebras as well). The proof given there easily generalizes to octonion algebras.

2.7. **Theorem.** *Let k be a field of characteristic not 2, and let C be a composition division algebra over k of dimension greater than 2. The following are equivalent:*

(i) *Zero is a nontrivial sum of squares in C.*
(ii) $-1 \in \Sigma C^2$.
(iii) $C = \Sigma C^2$.

Proof. The only nontrivial step is to prove that (i) implies (iii). Suppose that 0 is a nontrivial sum of squares in C. Without loss of generality assume that k is formally real (otherwise 2.1 applies and we are done). Thus k has characteristic zero. Put $V = \{x \in C | -x^2 \in \Sigma C^2\}$. Then V is a k-subspace of C which is invariant under all the automorphisms of C. Thus V must be C, 0, $k1$ or $\mathrm{Skew}(C, {}^{-})$ by [J, Theorem 7]. By assumption there are $y_i \in C$ such that $0 = y_1^2 + ... + y_m^2$. These elements cannot all lie in k, since k is formally real; thus assume $y_1 \notin k$. Moreover, $y_1 \in V$ since $-y_1^2 = y_2^2 + ... + y_m^2$. Therefore $V \not\subset k$. Assume that $V = \mathrm{Skew}(C, {}^{-})$. Then $1 \notin V$ since $t_C(1) \neq 0$ and hence also $-1 \notin \Sigma C^2$. If two elements $a, b \in V$

commute, then $ab = \frac{1}{2}((a+b)^2 + (-a^2) + (-b^2)) \in \Sigma C^2$. Therefore we look at a subspace of V whose elements commute: Let T be a maximal subfield of C, i.e. $C = \mathrm{Cay}(T, d, e)$. Then $t_C(x) = t_{T/k}(x)$ for all elements $x \in T$, where $t_{T/k}$ is the field trace of the field extension T/k. Define $W = T \cap V = \{x \in T | t_{T/k}(x) = 0\}$. Since $T = k(\sqrt{c})$ for some element $c \in k$ it follows that $W = k \cdot \sqrt{c}$, implying that $-1 = \sqrt{c}(-\frac{1}{c}\sqrt{c}) \in WW \subset \Sigma C^2$, a contradiction. Thus V must be C. This, however, means that $a = \frac{1}{2}((a+1)^2 + (-a)^2 + (-1)) \in \Sigma C^2$, for all $a \in C$. $\square$

Obviously, the proof of the above theorem generalizes as follows to algebras over rings:

2.8. **Theorem.** *Let C be a composition algebra over R of rank greater than 2, satisfying the following two conditions:*

(1) *0, C, $R1$ and* $\mathrm{Skew}(C, ^{-})$ *are the only invariant submodules relative to* $\mathrm{Aut}(C)$.

(2) *C contains a quadratic étale R-algebra isomorphic to a classical Cayley-Dickson doubling* $\mathrm{Cay}(R, a)$.

Then the following are equivalent:

(i) *Zero is a nontrivial sum of squares in C.*

(ii) $-1 \in \Sigma C^2$.

(iii) $C = \Sigma C^2$.

The following two statements generalize [LSW, Theorem A] and a result in [Ko].

2.9. **Corollary.** *For a composition algebra C over a field k of characteristic not 2 the following are equivalent:*

(i) *The trace form t_C is weakly isotropic.*

(ii) $-1 \in \Sigma C^2$.

(iii) $C = \Sigma C^2$.

Let A be a quadratic R-algebra with a scalar involution σ and norm form $n_A(x) = x\sigma(x)$ of rank greater than 2. Recall that $A = R \oplus F$ and $n_A = \langle 1 \rangle \perp n_0$ with $n_0 = n_A|_F$ for $F = \mathrm{Skew}(A, \sigma)$, if $2 \in R^\times$. The multiplication is given by $(a, u)(b, v) = (ab - B(u, v), av + bu + u \times v)$, for $a, b \in R$ and $u, v \in F$ with $\times : F \times F \to F$ a skew-symmetric R-bilinear map, and $B : F \times F \to R$ is given by $B(u, v) = \frac{1}{2} n_A(u, v)$.

2.10. **Lemma.** *Let A be a quadratic R-algebra with scalar involution σ (e.g. a composition algebra) of rank greater than 2. Then -1 is a sum of m squares of "pure" elements in C, i.e. elements in* $\mathrm{Skew}(A, \sigma)$*, if and only if $-1 \in D(m \times (-n_0))$. If R is a field, this is equivalent to the form $\langle 1 \rangle \perp m \times (-n_0)$ being isotropic.*

Proof. If $-1 = u_1^2 + ... + u_m^2$ with $u_i \in \mathrm{Skew}(A, \sigma)$, then $-1 = -n_0(u_1) - ... - n_0(u_m)$ and it follows that -1 is represented by $m \times (-n_0)$. Conversely, $-1 = -n_0(u_1) - ... - n_0(u_m)$ implies that $-1 = u_1^2 + ... + u_m^2$. $\square$

2.11. **Lemma.** *Let A be a quadratic R-algebra with a scalar involution σ (e.g. a composition algebra) of rank greater than 2. Then $s(A) \le m$ implies that $-1 \in D(m \times (\langle 1 \rangle \perp (-n_0)))$. In particular, if R is a field, then $s(C) \le m$ implies that $(m+1) \times \langle 1 \rangle \perp m \times (-n_0)$ is isotropic.*

Proof. Obviously, $(a, u)^2 = (a^2 - B(u, u), 2au)$ for all $a \in R, u \in F$. Hence $s(A) \le m$ implies $-1 = \sum_{i=1}^m (a_i, u_i)^2 = (\sum_{i=1}^m a_i^2 - \sum_{i=1}^m B(u_i, u_i),\ 2\sum_{i=1}^m a_i u_i)$. Thus

$\sum_{i=1}^m a_i u_i = 0$ and $\sum_{i=1}^m a_i^2 - \sum_{i=1}^m B(u_i, u_i) = -1$. In particular, the quadratic form $m \times (\langle 1\rangle \perp (-n_0))$ over R represents -1. □

We easily rephrase [Le, Theorem 2.2] for generalized Cayley-Dickson algebras and octonion algebras:

2.12. **Proposition.** *Let k_0 be a field of characteristic not 2, and let $A = \mathrm{Cay}(k_0, a_1, ..., a_d)$ with $d \geq 2$ be a composition algebra or a generalized Cayley-Dickson algebra, with norm $n_A = \langle 1\rangle \perp n_0$. If the quadratic form $(2^m+1) \times \langle 1\rangle \perp (2^m - 1) \times (-n_0)$ is isotropic over k_0, then $s(A) \leq 2^m$.*

The proof is analogous to the one given in [Le], since all arguments use quadratic forms only and rely on the fact that the forms $2^m \times \langle 1\rangle$ and n_A are Pfister forms.

3. Octonion algebras of level $2^m + 1$ and of level 2^m

Laghribi and Mammone [LM] presented examples of quaternion division algebras of level 2^m and 2^m+1 for any integer $m \geq 1$. (The existence of such quaternion division algebras was already proved by Lewis [L3].) Their method of proof can be generalized to obtain examples of octonion division algebras of levels 2^m and 2^m+1 for any integer $m \geq 1$.

Let $s \geq 1$ be an integer, k_0 a formally real field, and $k = k_0(x,y,z)$ the rational function field in three variables over k_0. Define $C = \mathrm{Cay}(k,x,y,z)$ and $\widetilde{\psi_s} = \langle 1\rangle \perp s \times (-n_0)$. Since $\widetilde{\psi_s}$ is isotropic over its function field $k(\widetilde{\psi_s})$, we know that -1 is a sum of s squares of pure octonions in $C \otimes_k k(\widetilde{\psi_s})$, and in particular that $s(C \otimes_k k(\widetilde{\psi_s})) \leq s$ by 2.10. As in the analogous situation for quaternion algebras considered in [LM], we are able to show more when $s = 2^m + 1$.

3.1. **Theorem.** *Let $m \geq 1$ be an integer, and let $k = k_0(x,y,z)$ be the rational function field in three variables over a formally real field k_0. Let $C = \mathrm{Cay}(k,x,y,z)$ with $n_C = \langle 1\rangle \perp n_0$ and put*

$$\widetilde{\psi_m} = \langle 1\rangle \perp (2^m+1) \times (-n_0).$$

Then $C = \mathrm{Cay}(k,x,y,z) \otimes_k k(\widetilde{\psi_m})$ is an octonion division algebra of level 2^m+1.

For the proof we need two results which are analogous to [LM, 1.2, 1.4]:

3.2. **Proposition.** *Let $k = k_0(x_1, ..., x_d)$ be the rational function field in d variables over a formally real field k_0, $d \geq 2$. Consider the d-fold Pfister form $n = \langle\langle x_1, ..., x_d\rangle\rangle = \langle 1\rangle \perp n_0$ over k, with n_0 denoting its pure part.*

(i) *For any integers $m, l \geq 1$ the quadratic form $m \times \langle 1\rangle \perp l \times (-n_0)$ is anisotropic over k.*

(ii) *Let φ be a quadratic form over k of dimension greater than or equal to 2^d+1 or of dimension 2^d with $\det\varphi \neq 1$. Then $n \otimes_k k(\varphi)$ stays anisotropic over $k(\varphi)$.*

Proof. (i) We use induction on d. The induction beginning ($d = 2$) is given by [LM, 1.2]. Now let $d \geq 3$ and assume that the quadratic form $m \times \langle 1\rangle \perp l \times (-n_0) = m \times \langle 1\rangle \perp l \times (-n_0') \perp x_d(l \times \langle\langle x_1, ..., x_{d-1}\rangle\rangle)$ is isotropic over k, with n_0' the pure part of $\langle\langle x_1, ..., x_{d-1}\rangle\rangle$. By Springer's theorem, this means that the form $m \times \langle 1\rangle \perp l \times (-n_0')$ must be isotropic over $k(x_1, ..., x_{d-1})$ (since the form $l \times \langle\langle x_1, ..., x_{d-1}\rangle\rangle$ is anisotropic), contradicting our induction hypothesis.

(ii) The proof is completely analogous to the one given in [LM, 1.4] and will be omitted here. □

3.3. **Proposition** (cf. [LM, 2.3]). *Under the above assumptions, the quadratic form* $\widetilde{\varphi_m} = (2^m+1) \times \langle 1\rangle \perp 2^m \times \langle x, y, -xy, z, -xz, -yz, xyz\rangle$ *stays anisotropic over* $k(\widetilde{\psi_m})$.

Proof. Using the notation from [LM], put $\psi_m = \langle 1\rangle \perp (2^m+1)\times\langle x, y, -xy\rangle$. Then ψ_m is a subform of $\widetilde{\psi_m}$. Since ψ_m is isotropic over its function field $k(\psi_m)$, so is $\widetilde{\psi_m}$; thus there exists a k-place from $k(\widetilde{\psi_m})$ to $k(\psi_m)$ by Knebusch [K, Theorem 3.3]. Now $\widetilde{\varphi_m} = (2^m+1)\times\langle 1\rangle \perp 2^m \times \langle x, y, -xy, z, -xz, -yz, xyz\rangle = \varphi_m \perp z(2^m \times \langle 1, -x, -y, xy\rangle)$ over $k_0(x, y, z)$, where $\varphi_m = (2^m+1)\times\langle 1\rangle \perp 2^m \times \langle x, y, -xy\rangle$ as in [LM, 2.3]. If $\widetilde{\varphi_m}$ is isotropic over $k(\widetilde{\psi_m})$, then $\widetilde{\varphi_m}$ is isotropic over $k(\psi_m)$. It follows that φ_m or $2^m \times \langle 1, -x, -y, xy\rangle$ is isotropic over $k_0(x, y)(\psi_m)$. However, by [LM, 2.3], φ_m never is. Put $\alpha_m = (2^m+1)\times\langle 1, -x\rangle$. Then $y\alpha_m$ is a subform of ψ_m. If $2^m \times \langle 1, -x, -y, xy\rangle$ is isotropic over $k_0(x, y)(\psi_m)$, it must then also be isotropic over $k_0(x, y)(\alpha_m)$ [K, Theorem 3.3]. This in turn implies that the quadratic form $2^m \times \langle 1, -x\rangle$ is isotropic over $k_0(x)(\alpha_m)$, a contradiction to [LM, 2.2]. □

Proof of Theorem 3.1. Let $C_m = \mathrm{Cay}(k, x, y, z) \otimes_k k(\widetilde{\psi_m})$ with $m \geq 1$. By 3.2, C_m is a division algebra and $s(C_m) \leq 2^m + 1$. If $s(C_m) < 2^m+1$, then the form $\widetilde{\varphi_m} = (2^m+1)\times\langle 1\rangle \perp 2^m \times \langle x, y, -xy, z, -xz, -yz, xyz\rangle$ becomes isotropic over $k(\widetilde{\psi_m})$, contradicting 3.3. □

Note that the following remark made in [LM] applies here as well: Let $\widetilde{\varphi_s} = s \times \langle 1\rangle \perp (s-1)\times\langle x, y, -xy, z, -xz, -yz, xyz\rangle$ and let $\widetilde{\psi_s} = \langle 1\rangle \perp s\times\langle x, y, -xy, z, -xz, -yz, xyz\rangle$. For each integer s for which the quadratic form

$$\widetilde{\varphi_s} = s \times \langle 1\rangle \perp (s-1) \times \langle x, y, -xy, z, -xz, -yz, xyz\rangle$$

stays anisotropic over $k(\widetilde{\psi_s})$, whenever $\widetilde{\varphi_s}$ and $\widetilde{\psi_s}$ are anisotropic, we are able to construct an octonion algebra of level s in a similar way as before in 3.1. Again, there indeed are integers s for which the quadratic form $\widetilde{\varphi_s}$ becomes isotropic over $k(\widetilde{\psi_s})$, for instance $s = 2^m$ with $m \geq 2$ (since [LM, 2.5] can be generalized to our situation accordingly).

If we take the generalized Cayley-Dickson algebra $A = \mathrm{Cay}(k, x_1, ..., x_d)$, $d \geq 4$, over the rational function field $k = k_0(x_1, ..., x_d)$, then this is a quadratic algebra with scalar involution. Its norm is exactly the form $n_A = \langle\langle x_1, ..., x_d\rangle\rangle$ in 3.2. We know that A is a division algebra if and only if n_A is anisotropic, and A contains no subalgebra of dimension 3 [B, Satz 5]. If again $\widetilde{\psi_s} = \langle 1\rangle \perp s \times (-n_0)$, the same argument as used above shows that -1 is a sum of s squares of elements in $\mathrm{Skew}(A \otimes_k k(\widetilde{\psi_s}), \sigma)$. In particular, $s(A \otimes_k k(\widetilde{\psi_s})) \leq s$. Moreover, if desired, the proof of 3.1 can be adapted accordingly to show that $s(A \otimes_k k(\widetilde{\psi_s}))$ with $s = 2^m+1$ is a generalized Cayley-Dickson algebra of level 2^m+1.

3.4. **Theorem.** *Let* $m \geq 1$ *be an integer, let* $k = k_0(x, y, z)$ *be the rational function field in three variables over a formally real field* k_0, *and put*

$$\widetilde{\lambda_m} = (2^m+1) \times \langle 1\rangle \perp (2^m-1) \times (-n_0).$$

Then $C_m = \mathrm{Cay}(k, x, y, z) \otimes_k k(\widetilde{\lambda_m})$ *is an octonion division algebra of level* 2^m.

This shows the existence of octonion algebras of level 2^m. For the proof, we need the equivalent of [LM, 3.4] as follows.

3.5. **Proposition.** *The quadratic form* $\widetilde{\gamma_m} = 2^m \times \langle 1\rangle \perp (2^m-1) \times (-n_0)$ *stays anisotropic over* $k(\widetilde{\lambda_m})$.

Proof. The quadratic form $\lambda_m = (2^m+1) \times \langle 1\rangle \perp (2^m-1) \times \langle x, y, -xy\rangle$ is a subform of $\widetilde{\lambda_m}$. Hence there exists a k-place from $k(\widetilde{\lambda_m})$ to $k(\lambda_m)$. Now $\widetilde{\gamma_m} = \gamma_m \perp z((2^m-1) \times \langle 1, -x, -y, xy\rangle)$ with $\gamma_m = 2^m \times \langle 1\rangle \perp (2^m-1) \times \langle x, y, -xy\rangle$. Assume that $\widetilde{\gamma_m}$ is isotropic over $k(\widetilde{\lambda_m})$. Then it must also be isotropic over $k(\lambda_m)$. Hence γ_m or $(2^m-1) \times \langle 1, -x, -y, xy\rangle$ is isotropic over $k_0(x, y)(\lambda_m)$. However, γ_m never is [LM, 3.4]. Put $\mu_m = (2^m+1) \times \langle 1\rangle$. Then μ_m is a subform of λ_m, and thus there exists a $k_0(x, y)$-place from $k_0(x, y)(\lambda_m)$ to $k_0(x, y)(\mu_m)$. This implies that the quadratic form $(2^m-1) \times \langle 1, -x, -y, xy\rangle$ is isotropic over $k_0(x, y)(\mu_m)$, and in turn that the form $(2^m-1) \times \langle 1, -x\rangle$ is isotropic over $k_0(x)(\mu_m)$, contradicting [LM, 3.3]. □

Proof of Theorem 3.4. Let $C_m = \mathrm{Cay}(k, x, y, z) \otimes_k k(\widetilde{\lambda_m})$ with $m \geq 1$. This is a division algebra by 3.2 (ii). Moreover, $s(C_m) \leq 2^m$ (2.12). In case $s(C_m) < 2^m$ it follows that the form $\widetilde{\gamma_m}$ becomes isotropic over $k(\widetilde{\lambda_m})$, a contradiction to 3.5. □

Again, the same idea can be used to construct examples of generalized Cayley-Dickson algebras of level 2^m; for example, taking $A = \mathrm{Cay}(k, x_1, ..., x_d)$, then $s(A \otimes_k k(\widetilde{\lambda_m})) = 2^m$ where $\widetilde{\lambda_m} = (2^m+1) \times \langle 1\rangle \perp (2^m-1) \times (-n_0)$ is a generalized Cayley-Dickson algebra of level 2^m.

We end with the analogue of [LM, 3.5], which reproves [Ko]:

3.6. **Proposition.** *Under the same assumptions as in* 3.4, -1 *is not a sum of* 2^m *squares of pure octonions in* $C_m = \mathrm{Cay}(k, x, y, z) \otimes_k k(\widetilde{\lambda_m})$.

Proof. Put $\widetilde{\theta} = \langle 1\rangle \perp 2^m \times (-n_0) = \theta \perp z(2^m \times \langle 1, -x, -y, xy\rangle)$ with $\theta = \langle 1\rangle \perp 2^m \times \langle x, y, -xy\rangle$ as in [LM]. Assume that $\widetilde{\theta}$ is isotropic over $k(\widetilde{\lambda_m})$. By the same argument as in the proof of 3.5 this implies that $\widetilde{\theta}$ is isotropic over $k(\lambda_m)$, which in turn means that the forms θ or $2^m \times \langle 1, -x, -y, xy\rangle$ are isotropic over $k_0(x, y)(\lambda_m)$. However, this is a contradiction as seen in the proof of 3.5, since θ is anisotropic over $k_0(x, y)(\lambda_m)$ by [LM, 3.5]. □

We thus have even constructed examples of octonion algebras of level 2^m, where -1 is not a sum of squares of pure octonions. Of course, the same argument can be applied to generalized Cayley-Dickson algebras, implying that in the algebra $\mathrm{Cay}(k, x_1, ..., x_d) \otimes_k k(\widetilde{\lambda_m})$ constructed above, -1 is not a sum of 2^m squares of pure elements as well.

REFERENCES

[AS] Artin, E., Schreier, O., *Algebraische Konstruktion reeller Koerper*, Abh. Sem. Hamburg 5 (1927), 85-99.

[B] Becker, E., *Über eine Klasse flexibler quadratischer Divisionsalgebren*, J. für Reine u. Angew. Math. 256 (1972), 25-57. MR0320099 (47:8640)

[DLP] Dai, Z.D., Lam, T.-Y., Peng, C. K., *Levels in algebra and topology*, Bull. Amer. Math. Soc. 3 (1980), 845-848. MR0578376 (81j:10072)

[J] Jacobson, N., *Composition algebras and their automorphisms*, Rend. Circ. Mat. Palermo **7**, 55–80 (1958). MR0101253 (21:66)

[K] Knebusch, M., *Generic splitting of quadratic forms*, Proc. London Math. Soc. 33 (1976), 65-93. MR0412101 (54:230)

[Ko] Koprowski, P., *Sums of squares of pure quaternions*, Proc. Royal Irish Acad. 98A (1) (1998), 63-65. MR1760207 (2001f:11056)

[LM] Laghribi, A., Mammone, P., *On the level of a quaternion algebra*, Comm. Alg. 29(4) (2001), 1821-1828. MR1853127 (2002f:16043)

[Le] Leep, D.B., *Levels of Division Algebras*, Glasgow Math. Journal 32 (1990), 365-370. MR1073676 (92g:11036)

[LSW] Leep, D.B., Shapiro, D.B., Wadsworth, A.R., *Sums of Squares in Division Algebras*, Math. Z. 190 (1985), 151-162. MR0797536 (87i:11047)

[L1] Lewis, D., *On the Level*, Irish Math. Soc. Bulletin 19 (1987), 33-48. MR0936518 (89f:11063)

[L2] Lewis, D., *Sums of Squares in Central Simple Algebras*, Math. Z. 190 (1985), 497-498. MR0808916 (87a:11031)

[L3] Lewis, D., *Levels of quaternion algebras*, Rocky Mountain J. 19 (1989), 787-792. MR1043248 (91d:11033)

[M] McCrimmon, K., *Nonassociative algebras with scalar involution*, Pacific J. of Math. 116(1) (1985), 85-108. MR0769825 (86d:17003)

[P] Petersson, H., *Composition algebras over algebraic curves of genus 0*, Trans. Amer. Math. Soc. 337 (1993), 473-493. MR1108613 (93g:17006)

[PR] Petersson, H., Racine, M., *Reduced models of Albert algebras*, Math. Z. 223 (3) (1996), 367-385. MR1417849 (98c:17028)

[Pf] Pfister, A., *Quadratic forms with applications to algebraic geometry and topology*, London Math. Soc. Lecture Notes Series 217, Cambridge University Press, Cambridge (1995). MR1366652 (97c:11046)

[Pu1] Pumplün, S., *Involutions on composition algebras*, Indag. Math. 14 (2) (2003), 241-248. MR2026818 (2004k:17006)

[Pu2] Pumplün, S., *Quaternion algebras over elliptic curves*, Comm. Algebra **26** (12), 4357–4373 (1998). MR1661277 (2000d:16027)

[PU] Pumplün, S., Unger, T., *The Hermitian level of composition algebras*, Manuscripta Math. 109 (4) (2002), 511-525. MR1946716 (2003j:17005)

[Se] Serhir, A., *Sur les niveaux hermitiens de certains algèbres simples centrales à involution*, Ph.D. Thesis, Université Catholique de Louvain (1998).

[U] Unger, T., *Quadratic Forms and Central Simple Algebras with Involution*, Ph.D. Thesis, University College Dublin (2000).

School of Mathematics, The University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom

E-mail address: `susanne.pumpluen@nottingham.ac.uk`