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ON SUCCESSIVE COEFFICIENTS OF ODD UNIVALENT FUNCTIONS

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ABSTRACT. The relative growth of successive coefficients of odd univalent functions is investigated. We prove that a conjecture of Hayman is true.

1. INTRODUCTION

Let S be the class of functions$f(z) = z + a_2z^2 + \dots$ regular and univalent in $|z| < 1$ and let S_2 be the subclass of odd functions $f(z) = \sum_{k=0}^{\infty} b_k z^{2k+1}$ in S. In the investigation of the relative growth of successive coefficients of functions in S_2, Goluzin [1] has established the inequality

(1) $$||b_n| - |b_{n-1}|| \le An^{-1/4} \ln n \ \ (n = 2, 3, \dots),$$

where A is an absolute constant. In 1963, W. Hayman [2] formulated the conjecture that

(2) $$||b_n| - |b_{n-1}|| \le A(\varepsilon)n^{-1/2+\varepsilon}$$

for every $\varepsilon > 0$. Lucas [3] and Huke [4], coming close to this conjecture, proved respectively the estimates

$$||b_n| - |b_{n-1}|| = O(n^{-b}),$$

where $b = \sqrt{2} - 1$ and $b = 0.42667$. In this paper we shall prove Hayman's conjecture. If $f_2 \in S_2$, $f_2(z) = \sqrt{f(z^2)} = \sum_{k=0}^{\infty} b_k z^{2k+1}$, where f is in S. Thus we only need to study the successive coefficients of the function $(f(z)/z)^{\frac{1}{2}} = \sum_{k=0}^{\infty} b_k z^k$, where f is in S. Our main result is

Theorem 1.1. *Let $f \in S$ and let the coefficients b_k $(k = 0, 1, 2, \dots)$ be defined by the expansion*

(3) $$(f(z)/z)^{1/2} = \sum_{k=0}^{\infty} b_k z^k.$$

Then for $n = 2, 3, \dots$,

(4) $$||b_n| - |b_{n-1}|| \le An^{-1/2} \log n,$$

where A is an absolute constant.

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The exponent $-\frac{1}{2}$ is sharp in Theorem 1.1, since

$$f(z) = z(1-z^4)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} b_{2k+1} z^{2k+1}$$

is univalent in $|z| < 1$ and

$$b_{4k+1} \sim k^{-\frac{1}{2}}/\sqrt{\pi}, k \to \infty,$$

while

$$b_{4k-1} = 0.$$

2. An auxiliary result

Assume, as usual, that the logarithmic coefficients $2\gamma_k$ $(k = 1, 2, \ldots)$ of the function $f(z) \in S$ are defined by the expansion

$$\log(f(z)/z) = 2\sum_{k=1}^{\infty} \gamma_k z^k. \tag{5}$$

Theorem 2.1. *Let $f \in S$. Let $r_1 = (1-1/n)^{\frac{1}{2}}, r_2 = (1-1/2n)^{\frac{1}{2}}$ $(n = 2, 3, \ldots)$. If t is chosen such that $|f(t)| = \max\{|f(z)| : |z| = r_2\}$ and if $z = r_1 e^{i\theta}$, then*

$$\int_0^{2\pi} |\frac{f(z)}{z}(1-t^{-1}z)| d\theta \leq A \log n, \tag{6}$$

where A is an absolute constant.

Proof. We consider the function $g(z) = 1/f(z^{-1})$ $(|z| > 1)$ in $\sum$ which satisfies

$$1/g(z) = f(z^{-1}) = z^{-1} + \sum_{n=2}^{\infty} a_n z^{-n}. \tag{7}$$

Let b_{kl} $(k, l = 1, 2, ...)$ be the Grunsky coefficients of $g(z)$. We define

$$\alpha_k = \alpha_k(s) = \sum_{l=1}^{\infty} b_{kl} s^{-l} \quad (|s| > 1, k = 1, 2, \ldots), \tag{8}$$

$$h = h(z, s) = \frac{z-s}{g(z)-g(s)} = \sum_{\nu=0}^{\infty} \beta_\nu(s) z^{-\nu} \quad (|z|, |s| > 1). \tag{9}$$

It follows that for a fixed s (see [7], p. 82),

$$\frac{d}{dz}\log h = \frac{1}{z-s} - \frac{g'(z)}{g(z)-g(s)} = -\sum_{k=0}^{\infty} k\alpha_k(s) z^{-k-1} \tag{10}$$

and

$$\frac{d}{dz} h = \frac{1}{g(z)-g(s)} - \frac{(z-s)g'(z)}{(g(z)-g(s))^2} = -\sum_{\nu} \nu\beta_\nu(s) z^{-\nu-1}. \tag{11}$$

The Grunsky inequality shows that (see [7], p. 83)

$$\sum_{k=1}^{\infty} k|\alpha_k(r)|^2 \leq \sum_{k=1}^{\infty} \frac{r^{-2k}}{k} = \log\frac{1}{1-r^{-2}}. \tag{12}$$

Write $\beta_\nu = \beta_\nu(s)$. Let $|s| = \rho = r_2^{-1}$. Applying Milin's theorem (see [7], p. 80), we obtain from (12) that for $n = 1, 2, ...,$

$$|\beta_n(r)|^2 \le \exp\{\sum_{k=1}^{n} k|\alpha_k|^2\} \le n^{-1}\exp\{\sum_{k=1}^{\infty} k|\alpha_k|^2\} \le n^{-1}\frac{1}{1-\rho^{-2}} \le A, \tag{13}$$

where A is an absolute constant. We obtain from (9), (10) and (11) that (see [7], p. 82)

$$\frac{d}{dz}\frac{z-s}{g(z)} = (1-\frac{g(s)}{g(z)})h\frac{d}{dz}\log h - g(s)h\frac{d}{dz}\frac{1}{g(z)}. \tag{14}$$

We choose a fixed s such that

$$|g(s)| = \min\{|g(z)| : |z| = \rho\} = \frac{1}{M(\frac{1}{\rho})}, \tag{15}$$

where

$$M(\frac{1}{\rho}) = \max\{|f(z)| : |z| = \frac{1}{\rho}\}. \tag{16}$$

Write $z = re^{i\theta}$, $r \in [\rho, 2]$, $\theta \in [0, 2\pi]$ and $\lambda = r_1^{-1}$ $(n = 1, 2, ...)$. We obtain from (14) that

$$\begin{aligned} &\int_0^{2\pi}\int_\lambda^2 |\frac{d}{dz}\frac{z-s}{g(z)}|drd\theta \\ &\le \int_0^{2\pi}\int_\lambda^2 |1-\frac{g(s)}{g(z)}||h\frac{d}{dz}\log h|drd\theta + \int_0^{2\pi}\int_\lambda^2 |g(s)||h\frac{d}{dz}\frac{1}{g(z)}|drd\theta. \end{aligned} \tag{17}$$

Now we estimate the two terms I_1 and I_2 on the right-hand side of (17).

(a) Since $|g(s)| \le |g(z)|$, by (15) and (16), we obtain from Schwarz's inequality and (9) and (10) that

$$\begin{aligned} |I_1| &\le 2[\int_0^{2\pi}\int_\lambda^2 |h|^2 d\theta dr]^{\frac{1}{2}}[\int_0^{2\pi}\int_\lambda^2 |\frac{d}{dz}\log h|^2 d\theta dr]^{\frac{1}{2}} \\ &\le 4\pi[\int_\lambda^2 \sum_{\nu=0}^{\infty} |\beta_\nu|^2 r_{-2\nu} dr]^{\frac{1}{2}}[\int_\lambda^2 \sum_{\nu=1}^{\infty} \nu^2|\alpha_\nu|^2 r^{-2\nu-2}]^{\frac{1}{2}} \\ &\le 4\pi[2+\sum_{\nu=1}^{\infty}\frac{1}{2\nu-1}|\beta_\nu|^2\lambda^{-2\nu+1}]^{\frac{1}{2}}[\sum_{\nu=1}^{\infty}\frac{\nu}{2\nu+1}\nu|\alpha_\nu|^2\lambda^{-2\nu-1}]^{\frac{1}{2}}. \end{aligned} \tag{18}$$

It follows from (12), (13) and (18) that

$$|I_1| \le A\log\frac{1}{1-\lambda^{-2}} \le A\log n, \tag{19}$$

where A is an absolute constant.

(b) We apply Schwarz's inequality again to estimate I_2. It follows from (7) and (13) that

$$
\begin{aligned}
|I_2| &\le |g(s)|[\int_\lambda^2\int_0^{2\pi}|h|^2 d\theta dr]^{\frac{1}{2}}[\int_\lambda^2\int_0^{2\pi}|\frac{d}{dz}\frac{1}{g(z)}|^2 d\theta dr]^{\frac{1}{2}} \\
&\le 2\pi|g(s)|[2+2+\sum_{\nu=1}^{\infty}\frac{1}{2\nu-1}|\beta_\nu|^2\lambda^{-2\nu+1}]^{\frac{1}{2}}[\int_\lambda^2\sum_{\nu=1}^{\infty}\nu^2|\alpha_\nu|^2 r^{-2\nu}dr]^{\frac{1}{2}} \\
&\le A|g(s)|[\log\frac{1}{1-\rho^{-2}}]^{\frac{1}{2}}[\sum_{\nu=1}^{\infty}\frac{\nu}{2\nu-1}\nu|\alpha_\nu|^2\lambda^{-2\nu+1}]^{\frac{1}{2}},
\end{aligned}
\tag{20}
$$

where A is an absolute constant. Because $f(z)$ is univalent in $|z|<1$, we obtain from (15) and (16) that

$$
\sum_{\nu=1}^{\infty}\frac{\nu}{2\nu-1}\nu|\alpha_\nu|^2\lambda^{-2\nu} \le \frac{1}{\pi}\int\int_{|z|<1/\rho}|f'(z)|^2 d\sigma \le (M(\frac{1}{\rho}))^2 = |g(s)|^2. \tag{21}
$$

We obtain from (20) and (21) that

$$
|I_2| \le A\log\frac{1}{1-\rho^{-2}} \le A\log n, \tag{22}
$$

where A is an absolute constant. The estimate obtained in (a) and (b) for the terms on the right-hand side of (17) shows that there exists an absolute constant A such that

$$
\int_0^{2\pi}\int_\lambda^2|\frac{d}{dz}\frac{z-s}{g(z)}|dr d\theta \le A\log n. \tag{23}
$$

Let $z_1 = 2e^{i\theta}$, $z_2 = \lambda e^{i\theta}$ and $z = re^{i\theta}$, $r\in[\rho,2]$. We obtain from (23) that

$$
\begin{aligned}
\int_0^{2\pi}|\frac{z_2-s}{g(z_2)}|d\theta - \int_0^{2\pi}|\frac{z_1-s}{g(z_1)}|d\theta &\le \int_0^{2\pi}\int_\lambda^2|\frac{\partial}{\partial r}|\frac{z-s}{g(z)}||dr d\theta \\
&\le \int_0^{2\pi}\int_\lambda^2|\frac{d}{dz}\frac{z-s}{g(z)}|dr d\theta \le A\log n.
\end{aligned}
\tag{24}
$$

It is clear that

$$
\int_0^{2\pi}|\frac{z_1-s}{g(z_1)}|d\theta \le 2\pi M(\frac{1}{2})(|z_1|+|s|) \le A, \tag{25}
$$

where A is an absolute constant. We obtain from (24) and (25) that

$$
\int_0^{2\pi}|f(z_2^{-1})||z_2^{-1}-s^{-1}|d\theta = \frac{1}{|z_2 s|}\int_0^{2\pi}|\frac{z_2-s}{g(z_2)}|d\theta \le A\log n, \tag{26}
$$

where A is an absolute constant. The inequality (26) implies for $z = r_1 e^{i\theta}$, $|t| = r_2$ and $|f(t)| = \max\{|f(z)| : |z| = r_2\}$ that

$$
\int_0^{2\pi}|f(z)||z-t|d\theta \le A\log n. \tag{27}
$$

□

3. Proof of Theorem 1.1

Proof. We define two functions and give their power series expansions (we choose single-valued brunch with value 1 at $z = 0$) when $|z| < r_2$:

$$u(z) = (f(z)/z)^{1/2}(1 - t^{-1}z)^{1/2} = \sum_{k=0}^{\infty} d_k z^k, \tag{28}$$

$$v(z) = u(z)(1 - t^{-1}z)^{1/2} = (f(z)/z)^{1/2}(1 - t^{-1}z) = \sum_{k=0}^{\infty} c_k z^k, \tag{29}$$

where $|t| = r_2 = [1 - 1/(2n)]^{\frac{1}{2}}$ and $|f(t)| = \max\{|f(z)| : |z| = r_2\}$. Theorem 2.1 shows that

$$\sum_{k=0}^{\infty} |d_k|^2 r_1^{2k} = \frac{1}{2\pi}\int_0^{2\pi} |u(r_1 e^{i\theta})|^2 d\theta \le A \log n. \tag{30}$$

Hence, we obtain from (30) that

$$\sum_{k=0}^{\infty} |c_k|^2 r_1^{2k} = \frac{1}{2\pi}\int_0^{2\pi} |v(r_1 e^{i\theta})|^2 d\theta \le \frac{2}{2\pi}\int_0^{2\pi} |u(r_1 e^{i\theta})|^2 d\theta \le A \log n, \tag{31}$$

where A is an absolute constant. Let

$$\log v(z) = \sum_{k=1}^{\infty} A_k z^k \tag{32}$$

for $|z| < r_2$. Then it follows from (29) that

$$A_k = \gamma_k - t^{-k}/k \ \ (k = 1, 2, \ldots). \tag{33}$$

It is clear that $v'(z) = v(z)[\log v(z)]'$. Comparing coefficients, we obtain the recursion formula

$$nc_n = \sum_{k=1}^{n} kA_k c_{n-k}. \tag{34}$$

Applying Schwarz's inequality to the recursion formula (34), we obtain that

$$n^2|c_n|^2 r_1^{2n} \le \sum_{k=1}^{n} k^2|A_k|^2 r_1^{2k} \sum_{k=0}^{n} |c_k|^2 r_1^{2k} \le n \sum_{k=1}^{n} k|A_k|^2 r_1^{2k} \sum_{k=0}^{n} |c_k|^2 r_1^{2k}$$

$$\le n \sum_{k=1}^{\infty} k|A_k|^2 r_1^{2k} \sum_{k=0}^{\infty} |c_k|^2 r_1^{2k}. \tag{35}$$

It follows by (33) that

$$\sum_{k=1}^{\infty} k|A_k|^2 r_1^{2k} \le 2\Big(\sum_{k=1}^{\infty} k|\gamma_k|^2 r_1^{2k} + \sum_{k=1}^{\infty} \frac{1}{k}\big(\frac{r_1}{r_2}\big)^{2k}\Big). \tag{36}$$

Since $r_1/r_2 < r_2$, it follows that

$$\sum_{k=1}^{\infty} k^{-1}\big(\frac{r_1}{r_2}\big)^{2k} < \sum_{k=1}^{\infty} k^{-1} r_2^{2k} = \log \frac{1}{1 - r_2^2} < \log 2n. \tag{37}$$

On the other hand, we have (see [6], pp. 283-290)

$$\sum_{k=1}^{\infty} k|\gamma_k|^2 r_1^{2k} \leq \max\{\log|f(z)/z| : |z| = r_1\} \leq \log\frac{1}{(1-r_1)^2} < 2\log n. \tag{38}$$

Combining (37) and (38), we obtain from (36) that

$$\sum_{k=1}^{\infty} k|A_k|^2 r_1^{2k} \leq 6\log 2n. \tag{39}$$

Thus, it follows from (31), (39) and (35) that

$$|c_n| \leq An^{-1/2}\log n, \tag{40}$$

where A is an absolute constant. It follows from (29) that $c_n = b_n - t^{-1}b_{n-1}$. Hence, we have

$$||b_n| - |b_{n-1}|| \leq |c_n| + ([1 - \frac{1}{2n}]^{-\frac{1}{2}} - 1)|b_{n-1}|. \tag{41}$$

It is well known that $|b_n| < 1.17$ $(n = 1, 2, \ldots)$ (see [5]). We obtain from (40) and (41) that

$$||b_n| - |b_{n-1}|| \leq An^{-\frac{1}{2}}\log n + O(\frac{1}{n}). \tag{42}$$

Finally, Theorem 1 follows from (42). □

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