

## GRÜSS TYPE INEQUALITIES IN INNER PRODUCT MODULES

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(Communicated by David R. Larson)

ABSTRACT. In this paper we give some properties of a generalized inner product in modules over  $H^*$ -algebras and  $C^*$ -algebras and we obtain inequalities of Grüss type.

### 1. INTRODUCTION

One of the earliest results which gave estimates for the Chebyshev difference

$$T(f, g) = \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx, \quad f, g \in L[a, b],$$

was the Grüss inequality given in the following theorem ([5], see also [9, pp. 295-310]).

**Theorem 1.1.** *Let  $f$  and  $g$  be real functions defined and integrable on  $[a, b] \subset \mathbb{R}$ . If there exist real constants  $\varphi, \phi, \gamma, \Gamma$  such that*

$$\varphi \leq f(x) \leq \phi, \quad \gamma \leq g(x) \leq \Gamma, \quad \text{for } x \in [a, b],$$

then

$$T(f, g) \leq \frac{1}{4}(\phi - \varphi)(\Gamma - \gamma).$$

The constant  $\frac{1}{4}$  is the best possible.

This inequality was investigated and generalized by a number of mathematicians. These investigations of the Grüss and related inequalities have been stimulated by their applications in different areas of applied mathematics, such as in perturbed quadrature rules ([1], [2]) and in the approximation of integral transforms ([4], [6] and references therein). For further reading about related results we refer to the monograph [9, pp. 295-310]. In the last decade some papers about Grüss type inequalities in real and complex inner product spaces have appeared. Here we mention papers such as [3] and [15].

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Received by the editors June 11, 2004.

2000 *Mathematics Subject Classification.* Primary 46L08, 46H25; Secondary 46CXX, 26D99.

*Key words and phrases.* Generalized inner product, inner product  $H^*$ -module, inner product  $C^*$ -module, Cauchy-Schwarz inequality, Grüss inequality.

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The basic result from these papers is the following theorem ([3]):

**Theorem 1.2.** *Let  $(H, (\cdot, \cdot))$  be an inner product space over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $e \in H$ ,  $\|e\| = 1$ . If  $\varphi, \phi, \gamma, \Gamma \in \mathbb{K}$  and  $x, y$  are vectors in  $H$  such that conditions*

$$\left\| x - \frac{\phi + \varphi}{2}e \right\| \leq \frac{1}{2}|\phi - \varphi|, \quad \left\| y - \frac{\Gamma + \gamma}{2}e \right\| \leq \frac{1}{2}|\Gamma - \gamma|$$

*hold, then we have the inequality*

$$|(x, y) - (x, e)(e, y)| \leq \frac{1}{4}|\phi - \varphi||\Gamma - \gamma|.$$

*The constant  $\frac{1}{4}$  is the best possible in the sense that it cannot be replaced by a smaller one.*

In this paper we give inequalities of the Grüss type in certain abstract structures called inner product modules. The second section contains preliminaries. In the third section we describe a new generalized semi-inner product generated by the generalized inner product which is the first-defined product in an inner product module. In the fourth and the fifth sections Grüss type inequalities and their refinement in inner product  $H^*$ -modules and inner product  $C^*$ -modules are proved.

## 2. PRELIMINARIES

An inner product module arises as a generalization of a complex inner product space when the complex field is replaced by a  $C^*$ -algebra or a proper  $H^*$ -algebra. The idea of replacing the complex numbers by the elements of a  $C^*$ -algebra first appeared in the work of Kaplansky ([7]) and by the elements of a proper  $H^*$ -algebra in the work of Saworotnow ([12]).

A proper  $H^*$ -algebra is a complex Banach  $*$ -algebra  $(A, \|\cdot\|)$  whose underlying Banach space is a Hilbert space with respect to the inner product  $\langle \cdot, \cdot \rangle$  satisfying  $\langle ab, c \rangle = \langle b, a^*c \rangle$  and  $\langle ba, c \rangle = \langle b, ca^* \rangle$  for all  $a, b, c \in A$ . A  $C^*$ -algebra is a complex Banach  $*$ -algebra  $(A, \|\cdot\|)$  such that  $\|a^*a\| = \|a\|^2$  for every  $a \in A$ .

If  $A$  is a proper  $H^*$ -algebra or a  $C^*$ -algebra and  $a \in A$  is such that  $Aa = 0$  or  $aA = 0$ , then  $a = 0$ .

An element  $a$  in a proper  $H^*$ -algebra  $A$  is called positive ( $a \geq 0$ ) if  $\langle ax, x \rangle \geq 0$  for every  $x \in A$ . Every positive element  $a$  in a proper  $H^*$ -algebra is self-adjoint (that is,  $a^* = a$ ). An element  $a$  in a  $C^*$ -algebra  $A$  is called positive ( $a \geq 0$ ) if it is self-adjoint and has positive spectrum. An element  $a^*a$  is positive for every  $a \in A$ , in both structures.

The trace-class associated with a proper  $H^*$ -algebra  $A$  is defined as the set  $\tau(A) = \{ab : a, b \in A\}$ . It is a self-adjoint two-sided ideal of  $A$  which is dense in  $A$ . There is a positive linear functional  $\text{tr}$  on  $\tau(A)$ , called trace, such that  $\text{tr}(ab) = \text{tr}(ba) = \langle a, b^* \rangle = \langle b, a^* \rangle$  for all  $a, b \in A$ . In particular,  $\text{tr}(a^*a) = \|a\|^2$  for every  $a \in A$ .

If  $A$  is a proper  $H^*$ -algebra, for every positive  $a \in \tau(A)$  there exists a unique positive  $b \in A$  such that  $b^2 = a$ . Such a  $b$  is denoted by  $a^{1/2}$ , and it is called the square root of  $a$ . If  $A$  is a  $C^*$ -algebra, the square root of  $a$  exists for every positive  $a \in A$ . Especially, in both cases, for every  $a \in A$  there exists the square root of  $a^*a$  which is denoted by  $|a|$  and called the absolute value of  $a$ .

The trace-class is a Banach  $*$ -algebra with respect to the norm  $\tau(\cdot)$  defined by  $\tau(a) = \text{tr}(|a|)$ . Let us mention that  $|\text{tr}(a)| \leq \tau(a)$  and  $\|a\| \leq \tau(a)$  for every  $a \in \tau(A)$ . More details concerning the trace-class can be found in [13] and [14].

Let  $A$  be a proper  $H^*$ -algebra or a  $C^*$ -algebra. A semi-inner product module over  $A$  is a right module  $H$  over  $A$  together with a generalized semi-inner product, that is, with a mapping  $[\cdot, \cdot]$  on  $H \times H$ , which is  $\tau(A)$ -valued if  $A$  is a proper  $H^*$ -algebra, or  $A$ -valued if  $A$  is a  $C^*$ -algebra, having the following properties:

- (i)  $[f, g + h] = [f, g] + [f, h]$  for all  $f, g, h \in H$ ,
- (ii)  $[f, ga] = [f, g]a$  for all  $f, g \in H$  and  $a \in A$ ,
- (iii)  $[f, g]^* = [g, f]$  for all  $f, g \in H$ ,
- (iv)  $[f, f] \geq 0$  for every  $f \in H$ .

Since  $[f, f]$  is a positive element in  $\tau(A)$  if  $A$  is a proper  $H^*$ -algebra, and in  $A$  if  $A$  is a  $C^*$ -algebra, there is the square root of  $[f, f]$  which is denoted by  $|f|$  and called the absolute value of  $f$ .

If, in addition,  $[f, f] = 0$  implies  $f = 0$ , then  $[\cdot, \cdot]$  is called a generalized inner product and  $H$  is called an inner product module over  $A$ . We shall say that  $H$  is a (semi-)inner product  $H^*$ -module if it is a (semi-)inner product module over a proper  $H^*$ -algebra, and that  $H$  is a (semi-)inner product  $C^*$ -module if it is a (semi-)inner product module over a  $C^*$ -algebra.

As we can see, an inner product module obeys the same axioms as an ordinary inner product space, except that the inner product takes values in a more general structure than in the field of complex numbers.

If  $H$  is a semi-inner product  $C^*$ -module, then the following Cauchy-Schwarz inequality holds:

$$\|[f, g]\|^2 \leq \| [f, f] \| \cdot \| [g, g] \| \quad (f, g \in H)$$

(e.g. [8, Proposition 1.1] together with [11, Theorem 2.2.5.(3)]).

If  $H$  is a semi-inner product  $H^*$ -module, then there are two forms of the Cauchy-Schwarz inequality:

$$|\operatorname{tr}([f, g])|^2 \leq \operatorname{tr}([f, f]) \cdot \operatorname{tr}([g, g]) \quad (f, g \in H)$$

(the weak Cauchy-Schwarz inequality), and

$$\tau([f, g])^2 \leq \operatorname{tr}([f, f]) \cdot \operatorname{tr}([g, g]) \quad (f, g \in H)$$

(the strong Cauchy-Schwarz inequality). In his pioneering paper [12], Saworotnow proved the strong Cauchy-Schwarz inequality, but the set of axioms he was dealing with was richer than in the definition of a semi-inner product  $H^*$ -module. The direct proof of the strong Cauchy-Schwarz inequality for a semi-inner product  $H^*$ -module can be found in [10].

**Lemma 2.1.** *Let  $A$  be a proper  $H^*$ -algebra or a  $C^*$ -algebra, and let  $H$  be a semi-inner product module over  $A$ . For all  $f, g \in H$ ,*

$$[f, g] + [g, f] \leq \frac{1}{2}|f + g|^2.$$

*Proof.*

$$|f + g|^2 - 2([f, g] + [g, f]) = [f + g, f + g] - 2([f, g] + [g, f]) = [f - g, f - g] \geq 0. \quad \square$$

Every proper  $H^*$ -algebra  $A$  and every  $C^*$ -algebra  $A$  is an inner product module over itself with respect to the generalized inner product given by  $[a, b] = a^*b$  for all  $a, b \in A$ . Therefore, an immediate consequence of Lemma 2.1 is:

**Corollary 2.2.** *Let  $A$  be a proper  $H^*$ -algebra or a  $C^*$ -algebra. For all  $a, b \in A$ ,*

$$a^*b + b^*a \leq \frac{1}{2}|a + b|^2.$$

If  $A$  is a proper  $H^*$ -algebra or a  $C^*$ -algebra, then  $p \in A$  is called a projection if it is non-zero and  $p^* = p = p^2$ . Let us emphasize that  $\|p\| = 1$  if  $A$  is a  $C^*$ -algebra, but if  $A$  is a proper  $H^*$ -algebra, then  $\|p\|^2 = \text{tr}(p)$ , which is not equal to 1 in general.

**Definition 2.3.** Let  $A$  be a proper  $H^*$ -algebra or a  $C^*$ -algebra, and let  $(H, [\cdot, \cdot])$  be a semi-inner product module over  $A$ . An element  $h \in H$  is called a lifted projection if  $|h|$  is a projection in  $A$ .

In the following lemma a useful characterization of lifted projections in inner product modules is given.

**Lemma 2.4.** *Let  $(H, [\cdot, \cdot])$  be an inner product  $H^*$ -module or an inner product  $C^*$ -module and let  $h \in H$ . Then  $h$  is a lifted projection if and only if  $h|h| = h$ .*

*Proof.* The equality

$$\begin{aligned} |h|h| - h|^2 &= [h|h| - h, h|h| - h] \\ &= [h|h|, h|h|] - [h, h|h|] - [h|h|, h] + [h, h] \\ &= |h|[h, h]|h| - [h, h]|h| - |h|[h, h] + [h, h] \\ &= |h| \cdot |h|^2 \cdot |h| - |h|^2 \cdot |h| - |h| \cdot |h|^2 + |h|^2 = ||h|^2 - |h||^2 \end{aligned}$$

implies that  $h|h| - h = 0$  if and only if  $|h|^2 = |h|$ . □

Although there are inner product  $H^*$ -modules and inner product  $C^*$ -modules which cannot be equipped with the structure of a complex vector space compatible with the structure of  $A$ , Lemma 2.4 enables us to define  $\alpha h := h(\alpha|h|)$  for every  $\alpha \in \mathbb{C}$  and every lifted projection  $h \in H$ .

The following technical lemma will be needed later on.

**Lemma 2.5.** *Let  $A$  be a proper  $H^*$ -algebra or a  $C^*$ -algebra. Let  $(H, [\cdot, \cdot])$  be an inner product module over  $A$ , and let  $h \in H$  be a lifted projection. For all  $f \in H$  and  $\alpha, \beta \in \mathbb{C}$ ,*

$$0 \leq |f|^2 - [f, h][h, f] \leq \left| f - \frac{\alpha + \beta}{2}h \right|^2.$$

*Proof.* First we have

$$\begin{aligned} |f|^2 - [f, h][h, f] &= [f, f] - [f, h][h, f] \\ &= [f, f] - [h, f]^*[h, f] - [f, h][h, f] + [f, h][h, f] \\ &= [f, f] - [h, f]^*[h, f] - [f, h][h, f] + [f, h|h|][h|h|, f] \\ &= [f, f] - [h, f]^*[h, f] - [f, h][h, f] + [f, h]|h|^2[h, f] \\ &= [f, f] - [h, f]^*[h, f] - [f, h][h, f] + [h, f]^*[h, h][h, f] \\ &= [f - h[h, f], f - h[h, f]] \geq 0. \end{aligned}$$

If we denote

$$x = [\alpha h - f, f - \beta h], \quad y = (\alpha|h| - [h, f])^* ([h, f] - \beta|h|),$$

then

$$\begin{aligned} y - x &= (\overline{\alpha}|h| - [f, h]) ([h, f] - \beta|h|) - [\alpha h - f, f - \beta h] \\ &= \overline{\alpha}|h|[h, f] - [f, h][h, f] - \overline{\alpha}\beta|h|^2 + \beta[f, h]|h| - [\alpha h, f] + [f, f] \\ &\quad + [\alpha h, \beta h] - [f, \beta h] \\ &= [\alpha h, f] - [f, h][h, f] - \overline{\alpha}\beta|h|^2 + [f, \beta h] - [\alpha h, f] + |f|^2 + [\alpha h, \beta h] - [f, \beta h] \\ &= |f|^2 - [f, h][h, f]. \end{aligned}$$

Applying Corollary 2.2 on  $\alpha|h| - [h, f]$  and  $[h, f] - \beta|h|$ , we get

$$\begin{aligned} &(\alpha|h| - [h, f])^* ([h, f] - \beta|h|) + ([h, f] - \beta|h|)^* (\alpha|h| - [h, f]) \\ &\leq \frac{1}{2} |(\alpha|h| - [h, f]) + ([h, f] - \beta|h|)|^2 = \frac{1}{2} |\alpha - \beta|^2 |h|, \end{aligned}$$

that is,

$$y + y^* \leq \frac{1}{2} |\alpha - \beta|^2 |h|.$$

Therefore,

$$\begin{aligned} |f|^2 - [f, h][h, f] &= \frac{1}{2} ((y - x) + (y - x)^*) = \frac{1}{2} (y + y^*) - \frac{1}{2} (x + x^*) \\ &\leq \frac{1}{2} \left( \frac{1}{2} |\alpha - \beta|^2 |h| \right) - \frac{1}{2} (x + x^*) = \frac{1}{4} |\alpha - \beta|^2 |h| - \frac{1}{2} (x + x^*) \\ &= \frac{1}{4} |\alpha - \beta|^2 |h| - \frac{1}{2} ([\alpha h - f, f - \beta h] + [f - \beta h, \alpha h - f]) \\ &= \frac{1}{4} |\alpha - \beta|^2 |h|^2 - \left( \frac{\overline{\alpha + \beta}}{2} [h, f] + \frac{\alpha + \beta}{2} [f, h] - |f|^2 - \frac{\overline{\alpha}\beta + \alpha\overline{\beta}}{2} |h|^2 \right) \\ &= |f|^2 - \frac{\alpha + \beta}{2} [f, h] - \frac{\overline{\alpha + \beta}}{2} [h, f] + \frac{|\alpha + \beta|^2}{4} |h|^2 \\ &= \left[ f - \frac{\alpha + \beta}{2} h, f - \frac{\alpha + \beta}{2} h \right] = \left| f - \frac{\alpha + \beta}{2} h \right|^2. \end{aligned}$$

□

### 3. ON A GENERALIZED SEMI-INNER PRODUCT GENERATED BY $[\cdot, \cdot]$

**Definition 3.1.** Let  $(H, [\cdot, \cdot])$  be an inner product  $H^*$ -module or an inner product  $C^*$ -module over  $A$  and let  $h \in H$  be a lifted projection. The mapping  $[\cdot, \cdot]_h$  is defined on  $H \times H$  by

$$[f, g]_h = [f, g] - [f, h][h, g].$$

Note that the mapping  $[\cdot, \cdot]_h$  is  $\tau(A)$ -valued if  $A$  is a proper  $H^*$ -algebra, and  $A$ -valued if  $A$  is a  $C^*$ -algebra.

As simple consequences of the properties of the generalized inner product  $[\cdot, \cdot]$  the following properties are valid:

- (i)  $[f, g + k]_h = [f, g]_h + [f, k]_h$  for all  $f, g, k \in H$ ,
- (ii)  $[f, ga]_h = [f, g]_h a$  for all  $f, g \in H$  and  $a \in A$ ,
- (iii)  $[f, g]_h^* = [g, f]_h$  for all  $f, g \in H$ .

From Lemma 2.5 we obtain the positivity of  $[\cdot, \cdot]_h$ :

$$(iv) [f, f]_h \geq 0 \text{ for every } f \in H.$$

Hence, we have proved the following proposition.

**Proposition 3.2.** *Let  $(H, [\cdot, \cdot])$  be an inner product  $H^*$ -module or an inner product  $C^*$ -module and let  $h$  be a lifted projection in  $H$ . The function  $[\cdot, \cdot]_h$  defined by*

$$[f, g]_h = [f, g] - [f, h][h, g]$$

*is a generalized semi-inner product on  $H$ .*

**Theorem 3.3.** *Let  $(H, [\cdot, \cdot])$  be an inner product  $H^*$ -module and let  $h$  be a lifted projection in  $H$ . Then for any  $f, g \in H$  the following inequality holds:*

$$(3.1) \quad \tau([f, g] - [f, h][h, g])^2 \leq (\|f\|^2 - \|[h, f]\|^2) \cdot (\|g\|^2 - \|[h, g]\|^2).$$

*Proof.* Using the strong Cauchy-Schwarz inequality for  $[\cdot, \cdot]_h$  we have

$$\begin{aligned} \tau([f, g] - [f, h][h, g])^2 &= \tau([f, g]_h)^2 \leq \tau([f, f]_h) \cdot \tau([g, g]_h) \\ &= (\tau(|f|^2) - \tau([h, f]^*[h, f])) \cdot (\tau(|g|^2) - \tau([h, g]^*[h, g])) \\ &= (\|f\|^2 - \|[h, f]\|^2) \cdot (\|g\|^2 - \|[h, g]\|^2). \end{aligned}$$

□

**Theorem 3.4.** *Let  $(H, [\cdot, \cdot])$  be an inner product  $C^*$ -module and let  $h$  be a lifted projection in  $H$ . Then for any  $f, g \in H$  the following inequality holds:*

$$(3.2) \quad \|[f, g] - [f, h][h, g]\|^2 \leq \|f\|^2 - [f, h][h, f] \cdot \|g\|^2 - [g, h][h, g].$$

*Proof.* Using Cauchy-Schwarz inequality for  $[\cdot, \cdot]_h$  we have

$$\begin{aligned} \|[f, g] - [f, h][h, g]\|^2 &= \|[f, g]_h\|^2 \leq \|[f, f]_h\| \cdot \|[g, g]_h\| \\ &= (\|f\|^2 - [f, h][h, f]) \cdot (\|g\|^2 - [g, h][h, g]). \end{aligned}$$

□

#### 4. GRÜSS INEQUALITY IN INNER PRODUCT $H^*$ -MODULES

**Theorem 4.1.** *Let  $A$  be a proper  $H^*$ -algebra. Let  $(H, [\cdot, \cdot])$  be an inner product module over  $A$  and let  $h \in H$  be a lifted projection. If  $\varphi, \phi, \gamma, \Gamma \in \mathbb{C}$  and  $f, g \in H$  are such that the assumptions*

$$(4.1) \quad \left\| f - \frac{\phi + \varphi}{2} h \right\| \leq \frac{1}{2} |\phi - \varphi| \cdot \|h\|, \quad \left\| g - \frac{\Gamma + \gamma}{2} h \right\| \leq \frac{1}{2} |\Gamma - \gamma| \cdot \|h\|$$

*hold, then we have the inequality*

$$(4.2) \quad \begin{aligned} \tau([f, g] - [f, h][h, g]) &\leq \frac{1}{4} |\phi - \varphi| |\Gamma - \gamma| \|h\|^2 \\ &\quad - \left( \frac{1}{4} |\phi - \varphi|^2 \cdot \|h\|^2 - \left\| f - \frac{\phi + \varphi}{2} h \right\|^2 \right)^{\frac{1}{2}} \\ &\quad \times \left( \frac{1}{4} |\Gamma - \gamma|^2 \cdot \|h\|^2 - \left\| g - \frac{\Gamma + \gamma}{2} h \right\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

*The constant  $\frac{1}{4}$  is the best possible.*

*Proof.* From Lemma 2.5 we have

$$|f|^2 - [f, h][h, f] \leq \left| f - \frac{\phi + \varphi}{2} h \right|^2.$$

This implies

$$\operatorname{tr}(|f|^2 - [f, h][h, f]) \leq \operatorname{tr} \left( \left| f - \frac{\phi + \varphi}{2} h \right|^2 \right),$$

that is,

$$(4.3) \quad \| |f| \|^2 - \| [h, f] \|^2 \leq \left\| \left| f - \frac{\phi + \varphi}{2} h \right| \right\|^2.$$

Analogously,

$$(4.4) \quad \| |g| \|^2 - \| [h, g] \|^2 \leq \left\| \left| g - \frac{\Gamma + \gamma}{2} h \right| \right\|^2.$$

After multiplying (4.3) and (4.4), we obtain

$$(4.5) \quad \left( \| |f| \|^2 - \| [h, f] \|^2 \right) \cdot \left( \| |g| \|^2 - \| [h, g] \|^2 \right) \leq \left\| \left| f - \frac{\phi + \varphi}{2} h \right| \right\|^2 \cdot \left\| \left| g - \frac{\Gamma + \gamma}{2} h \right| \right\|^2.$$

Applying the inequality  $(m^2 - n^2)(p^2 - q^2) \leq (mp - nq)^2$  ([9, p. 117]) on

$$m = \frac{1}{2} |\phi - \varphi| \| |h| \|, \quad n = \left( \frac{1}{4} |\phi - \varphi|^2 \cdot \| |h| \|^2 - \left\| \left| f - \frac{\phi + \varphi}{2} h \right| \right\|^2 \right)^{\frac{1}{2}},$$

$$p = \frac{1}{2} |\Gamma - \gamma| \| |h| \|, \quad q = \left( \frac{1}{4} |\Gamma - \gamma|^2 \cdot \| |h| \|^2 - \left\| \left| g - \frac{\Gamma + \gamma}{2} h \right| \right\|^2 \right)^{\frac{1}{2}},$$

we get

$$(4.6) \quad \left\| \left| f - \frac{\phi + \varphi}{2} h \right| \right\|^2 \cdot \left\| \left| g - \frac{\Gamma + \gamma}{2} h \right| \right\|^2 \leq \left( \frac{1}{4} |\phi - \varphi| |\Gamma - \gamma| \| |h| \|^2 \right. \\ \left. - \left( \frac{1}{4} |\phi - \varphi|^2 \cdot \| |h| \|^2 - \left\| \left| f - \frac{\phi + \varphi}{2} h \right| \right\|^2 \right)^{\frac{1}{2}} \right. \\ \left. \times \left( \frac{1}{4} |\Gamma - \gamma|^2 \cdot \| |h| \|^2 - \left\| \left| g - \frac{\Gamma + \gamma}{2} h \right| \right\|^2 \right)^{\frac{1}{2}} \right)^2.$$

It remains to compare the inequalities (3.1), (4.5) and (4.6).

If  $l$  is a non-zero element in  $H$  such that  $[l, h] = 0$ , then there exists  $a \in A$  satisfying  $|l|a \neq 0$ . If we define  $\lambda = \frac{\| |h| \|}{\| |l|a \|}$  and  $k = l(\lambda a)$ , then  $[k, h] = 0$  and  $\| |k| \| = \| |h| \|$ . For

$$(4.7) \quad f = \frac{|\phi - \varphi|}{2} k + \frac{\phi + \varphi}{2} h, \quad g = \frac{|\Gamma - \gamma|}{2} k + \frac{\Gamma + \gamma}{2} h$$

the assumptions (4.1) are fulfilled and (4.2) becomes an equality. Note that the second summand on the right side in (4.2) vanishes for chosen  $f$  and  $g$ .  $\square$

**Corollary 4.2.** *Let  $A$  be a proper  $H^*$ -algebra. Let  $(H, [\cdot, \cdot])$  be an inner product module over  $A$  and let  $h \in H$  be a lifted projection. If  $\varphi, \phi, \gamma, \Gamma \in \mathbb{C}$  and  $f, g \in H$  are such that the assumptions (4.1) hold, then*

$$\tau([f, g] - [f, h][h, g]) \leq \frac{1}{4}|\phi - \varphi||\Gamma - \gamma| \|h\|^2.$$

The constant  $\frac{1}{4}$  is the best possible.

## 5. GRÜSS INEQUALITY IN INNER PRODUCT $C^*$ -MODULES

**Theorem 5.1.** *Let  $A$  be a  $C^*$ -algebra. Let  $(H, [\cdot, \cdot])$  be an inner product module over  $A$  and let  $h \in H$  be a lifted projection. If  $\varphi, \phi, \gamma, \Gamma \in \mathbb{C}$  and  $f, g \in H$  are such that the assumptions*

$$(5.1) \quad \left\| f - \frac{\phi + \varphi}{2}h \right\| \leq \frac{1}{2}|\phi - \varphi|, \quad \left\| g - \frac{\Gamma + \gamma}{2}h \right\| \leq \frac{1}{2}|\Gamma - \gamma|$$

hold, then we have the inequality

$$(5.2) \quad \begin{aligned} & \|[f, g] - [f, h][h, g]\| \leq \frac{1}{4}|\phi - \varphi||\Gamma - \gamma| \\ & - \left( \frac{1}{4}|\phi - \varphi|^2 - \left\| f - \frac{\phi + \varphi}{2}h \right\|^2 \right)^{\frac{1}{2}} \left( \frac{1}{4}|\Gamma - \gamma|^2 - \left\| g - \frac{\Gamma + \gamma}{2}h \right\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

The constant  $\frac{1}{4}$  is the best possible.

*Proof.* According to Lemma 2.5,

$$0 \leq |f|^2 - [f, h][h, f] \leq \left| f - \frac{\phi + \varphi}{2}h \right|^2.$$

If we apply [11, Theorem 2.2.5.(3)], we get

$$\||f|^2 - [f, h][h, f]\| \leq \left\| f - \frac{\phi + \varphi}{2}h \right\|^2,$$

that is,

$$(5.3) \quad \||f|^2 - [f, h][h, f]\| \leq \left\| f - \frac{\phi + \varphi}{2}h \right\|^2.$$

In a similar way we conclude that

$$(5.4) \quad \||g|^2 - [g, h][h, g]\| \leq \left\| g - \frac{\Gamma + \gamma}{2}h \right\|^2.$$

From (5.3) and (5.4), the inequality

$$(5.5) \quad \||f|^2 - [f, h][h, f]\| \cdot \||g|^2 - [g, h][h, g]\| \leq \left\| f - \frac{\phi + \varphi}{2}h \right\|^2 \cdot \left\| g - \frac{\Gamma + \gamma}{2}h \right\|^2$$

follows. As in the proof of Theorem 4.1, we get (4.6). Let us emphasize that  $\|h\| = 1$  in our case. Finally, compare the inequalities (3.2), (5.5) and (4.6), taking into account that  $\|h\| = 1$ .

The equality in (5.2) holds for the same  $f, g \in H$  for which we get the equality in (4.2). Again the second summand on the right side vanishes for such  $f$  and  $g$ .  $\square$

**Corollary 5.2.** *Let  $A$  be a  $C^*$ -algebra. Let  $(H, [\cdot, \cdot])$  be an inner product module over  $A$  and let  $h \in H$  be a lifted projection. If  $\varphi, \phi, \gamma, \Gamma \in \mathbb{C}$  and  $f, g \in H$  are such that the assumptions (5.1) hold, then*

$$\| [f, g] - [f, h][h, g] \| \leq \frac{1}{4} |\phi - \varphi| |\Gamma - \gamma|.$$

The constant  $\frac{1}{4}$  is the best possible.

*Remark 5.3.* If  $A$  is the  $C^*$ -algebra of complex numbers, the related result can be found in [3].

*Remark.* It would be interesting, if possible, to obtain Grüss type inequalities in inner product  $H^*$ -modules and inner product  $C^*$ -modules inside the positive cone of the underlying algebra, that is, to obtain analogous results dropping the norms.

#### ACKNOWLEDGEMENTS

We express our thanks to the referee for his useful comments.

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