MORITA EQUIVALENCES BETWEEN SOME BLOCKS
FOR \( p \)-SOLVABLE GROUPS

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ABSTRACT. We prove that any Morita equivalence between some blocks with
Abelian defect groups and cyclic inertia quotients for \( p \)-solvable groups is basic.

1. Let \( \mathcal{O} \) be a complete discrete valuation ring of characteristic zero with an
algebraically closed residue field \( k \) of characteristic \( p \); further we assume that the
quotient \( K \) of \( \mathcal{O} \) is a splitting field for all finite groups below. Throughout the paper,
all \( \mathcal{O} \)-modules are finitely generated and \( \mathcal{O} \)-algebras are \( \mathcal{O} \)-free; an
\( \mathcal{O} \)-subalgebra of any \( \mathcal{O} \)-algebra \( A \) does not necessarily contain the identity element \( 1 \)
of \( A \) and any homomorphism between two \( \mathcal{O} \)-algebras does not necessarily preserve their identity
elements. For an \( \mathcal{O} \)-algebra \( A \), we denote by \( J(A) \) the Jacobson radical of \( A \) and by \( A^* \) the multiplicative group of all invertible elements of \( A \).

For a ring \( A \), we denote by \( A^o \) its opposite ring; if \( A \) is an \( \mathcal{O} \)-algebra, so is \( A^o \). Let \( G \) be a finite group. Obviously the map \( G \to G, x \mapsto x^{-1} \) induces an
\( \mathcal{O} \)-algebra isomorphism \( o \) between \( \mathcal{O}G \) and \( (\mathcal{O}G)^o \); in particular, \( (\mathcal{O}G)^o \) becomes
an \( \mathcal{O} \)-interior algebra (see 3) through \( o \). Let \( b \) be a block idempotent of the
group algebra \( \mathcal{O}G \) and let \( b^o \) be the image of \( b \) in \( (\mathcal{O}G)^o \) through \( o \). Then the
algebra isomorphism \( o \) between \( \mathcal{O}G \) and \( (\mathcal{O}G)^o \) induces an \( \mathcal{O} \)-interior algebra
isomorphism (see 3) between \( \mathcal{O}Gb^o \) and \( (\mathcal{O}Gb)^o \), still denoted by \( o \) for convenience.

Let \( G' \) be another group and let \( b' \) be a block idempotent of the group algebra
\( \mathcal{O}G' \). The \( \mathcal{O} \)-algebra isomorphism \( o \) above between block algebras applied to \( G' \)
and \( b' \) always allows us to identify an \( (\mathcal{O}Gb, \mathcal{O}G'b') \)-bimodule \( M \) with an
\( \mathcal{O}(G \times G') \)-module associated to the block \( b \otimes b'^o \) \( (\mathcal{O}(G \times G') \cong \mathcal{O}G \otimes \mathcal{O}G') \). From now on,
we always use the identification without further notice.

An indecomposable \( \mathcal{O}(G \times G') \)-module \( M \) associated with \( b \otimes b'^o \) induces a Morita
equivalence between \( \mathcal{O}Gb \) and \( \mathcal{O}G'b' \) if

\[ M \otimes_{\mathcal{O}G'b'} M^* \cong \mathcal{O}Gb \]
as \( \mathcal{O}(G \times G) \)-modules and

\[ M^* \otimes_{\mathcal{O}Gb} M \cong \mathcal{O}G'b' \]
as \( \mathcal{O}(G' \times G) \)-modules, where \( M^* \) is the dual of \( M \). Let \( P'' \) be a vertex of the
\( \mathcal{O}(G \times G') \)-module \( M \) and let the \( \mathcal{O}P'' \)-module \( S'' \) be a source module of \( M \). If \( S'' \)
is an endo-permutation $\mathcal{O}P'$-module (see [12]), then we say that $M$ induces a basic Morita equivalence (see [8]) between $\mathcal{O}Gb$ and $\mathcal{O}G'b'$.

It is well known that all defect pointed groups of the block $b$ of $G$ over $\mathcal{O}$ exactly form a conjugacy class under the $G$-conjugation action (see [8]). For a defect pointed group $P_\gamma$ on $\mathcal{O}Gb$, we denote by $N_G(P_\gamma)$ the stabilizer of $P_\gamma$ under the $G$-conjugation action, by $E_G(P_\gamma)$ the usual inertia quotient $N_G(P_\gamma)/PC_G(P)$ of the block $b$, and by $(\mathcal{O}G)_\gamma$ a source algebra of $\mathcal{O}Gb$ corresponding to $P_\gamma$, where $C_G(P)$ is the centralizer of $P$ in $G$. Here we would like to remind readers to see Puig’s work [6] for more on defect pointed groups and source algebras, etc.

**Theorem 2.** Let $G$ and $G'$ be $p$-solvable groups and let $b$ and $b'$ be blocks of $G$ and $G'$, respectively, with Abelian defect groups and cyclic inertia quotients. Then any Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G'b'$ is basic.

3. Now we are ready to introduce some concepts for a remark on Theorem 2. Let $G$ be a finite group and let $\mathcal{O}G$ be the group algebra of $G$ over $\mathcal{O}$. An $\mathcal{O}G$-interior algebra $A$ is an $\mathcal{O}$-algebra $A$ endowed with a unitary $\mathcal{O}$-algebra homomorphism $\rho : \mathcal{O}G \rightarrow A$. For any $g \in G$ and $a \in A$, we define the left multiplication of $g$ on $A$ by $ga = \rho(g)a$ and the right multiplication of $g$ on $A$ by $ag = a\rho(g)$. Let $A'$ be another $\mathcal{O}G$-interior algebra. An $\mathcal{O}G$-algebra homomorphism $f : A \rightarrow A'$ is said to be $\mathcal{O}G$-interior if for any $g \in G$ and $a \in A$,

$$f(ga) = gf(a) \quad \text{and} \quad f(ag) = f(a)g.$$  

Furthermore $f$ is an embedding of $\mathcal{O}G$-interior algebras $A$ and $A'$ if $f$ is injective and $\text{Im}(f) = f(1_A)A'f(1_A)$. We can also endow the tensor product $A \otimes \mathcal{O}A'$ with an $\mathcal{O}G$-interior algebra structure through the homomorphism $\mathcal{O}G \rightarrow A \otimes \mathcal{O}A'$, $g \in G \rightarrow g1_A \otimes g1_A$.

**Remark 4.** Keep the same assumption as in Theorem 2. Let $M$ be an $\mathcal{O}(G \times G')$-module inducing a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G'b'$ in Theorem 2, let $P''$ be a vertex of it and let the $\mathcal{O}P''$-module $S''$ be a source of it. By Theorem 2, the $\mathcal{O}P''$-module $S''$ is an endo-permutation module. Let $\sigma$ and $\sigma'$ be the projections of $P''$ into $G$ and $G'$, respectively, and set $P = \sigma(P'')$ and $P' = \sigma'(P'')$. Then by [8] Cor. 7.4 and Th. 6.9, the group homomorphisms $\sigma : P'' \rightarrow P$ and $\sigma' : P'' \rightarrow P'$ are isomorphisms, $P$ and $P'$ are defect groups of $b$ and $b'$, respectively, and there exist defect pointed groups $P_\gamma$ of $b$ and $P'_{\gamma'}$ of $b'$ such that we have the following $\mathcal{O}P$-interior algebra embedding:

$$(\mathcal{O}G)_\gamma \rightarrow \text{Res}_{\sigma^{-1}}(\text{End}_\mathcal{O}(S'')) \otimes \mathcal{O} \text{Res}_{\sigma'\circ\sigma^{-1}}((\mathcal{O}G')_{\gamma'}) \quad .$$

5. Before beginning the proof of Theorem 2, we first start with the structure of source algebras of block algebras for $p$-solvable groups. It is well known that blocks of finite $p$-solvable groups were first systematically investigated by P. Fong [1], who determined their character theory; a complete characterization of their source algebras has been given in [8]; similar references also include [3] and [5].

Let $G$ be a finite $p$-solvable group and let $b$ be a block of the group algebra $\mathcal{O}G$ with a defect pointed group $P_\gamma$; suppose that $P$ is Abelian. Then from [8] Prop. 3.1 and Prop. 3.3], one can easily conclude that there exist an indecomposable endo-permutation $\mathcal{O}P$-module $N$, a $P'$-subgroup $L$ of $\text{Aut}(P)$ and an $\mathcal{O}^*$-group $\bar{L}$ with $\mathcal{O}^*$-quotient $L$ such that we have the following embedding of $\mathcal{O}P$-interior algebras:

$$(\mathcal{O}G)_\gamma \rightarrow \text{End}_\mathcal{O}(N) \otimes \mathcal{O}_*(P \rtimes \bar{L}),$$
where \( \hat{L} \) acts on \( P \) through the composition of the natural surjection \( \hat{L} \to L \) with the inclusion \( L \subset \text{Aut}(P) \). The embedding (5.1) is also Puig’s description \([7]\) of source algebras of block algebras for \( p \)-solvable groups in the abelian defect group case. Here a so-called \( O^* \)-group \( \hat{L} \) with \( O^* \)-quotient \( L \) is just a central extension of \( L \) by \( O^* \) (see \([7]\)); if \( O = k \), then we call \( \hat{L} \) a \( k^* \)-group with \( k^* \)-quotient \( L \) (see \([9]\)).

**Remark 6.** We can choose the \( O^* \)-group \( \hat{L} \) in (5.1) to be a \( k^* \)-group \( \hat{L}^k \) with \( k^* \)-quotient \( L \). Indeed, by \([11]\) Chap. III, Prop. 8], there exists a canonical decomposition

\[
O^* \cong (1 + f(O)) \times k^* ;
\]

so we can identify \( k^* \) as a subgroup of \( O^* \) and then the canonical surjective homomorphism \( \hat{L}/k^* \to \hat{L}/O^* \) induces the following short exact sequence:

\[
1 \to O^*/k^* \to \hat{L}/k^* \to L \to 1.
\]

Since \( L \) is a \( p' \)-group and \( O^*/k^* \cong 1 + f(O) \), by \([9]\) Prop. 4.6], the sequence splits and there exists a subgroup \( \hat{L}^k \) of \( \hat{L} \) which is a \( k^* \)-group with \( k^* \)-quotient \( L \).

**Remark 7.** Keep the same notation as in 5. Choose \( i \in \gamma \) and set \( (OG)_\gamma = iOGi \). Denote by \((OG)_\gamma^p \) the centralizer of \( Pi \) in \((OG)_\gamma \) and by \( N((OG)_\gamma; Pi) \) the normalizer of the group \( Pi \) in \((OG)_\gamma^* \). On the one hand, by \([3]\) 2.15.7, 2.15.9 and 2.16.3], there exists a canonical isomorphism between \( E_G(P_i) \) and \( N((OG)_\gamma; Pi)/P((OG)_\gamma^p)^* \). On the other hand, by \([11]\) Chap. III, Prop. 8], there exists a canonical decomposition

\[
((OG)_\gamma^p)^* \cong (i + f((OG)_\gamma^p)) \times k^* ;
\]

extending the unique decomposition \( O^* \cong (1 + f(O)) \times k^* \). In this case, \( k^* \) can be identified with a subgroup of \((OG)_\gamma^p)^* \) and then the group

\[
N((OG)_\gamma; Pi)/P(i + f((OG)_\gamma^p))
\]

is a central extension of \( E_G(P_i) \) by \( k^* \), denoted by \( \hat{E}_G(P_i)^\circ \).

In the embedding (5.1), if we choose \( \hat{L} \) to be a \( k^* \)-group, then by \([11]\) Prop. 5.11] and \([3]\) 2.12.4], \( \hat{E}_G(P_i)^\circ \) is isomorphic to \( \hat{L} \) as \( k^* \)-groups.

**Lemma 8.** Let \( P \) be a common \( p' \)-subgroup of finite groups \( G \) and \( G' \) and let \( b \) (resp. \( b' \)) be a block of \( G \) (resp. \( G' \)) and let \( P_{\gamma} \) (resp. \( P_{\gamma'} \)) be a defect pointed group of \( b \) (resp. \( b' \)). If there exist an indecomposable \( OP \)-module \( S \) and an embedding of \( OP \)-interior algebras \( (OG)_\gamma \to \text{End}_O(S) \otimes_O (OG')_{\gamma'} \), then there exists an \( O(G \times G') \)-bimodule \( M \) inducing a basic Morita equivalence between \( OGb \) and \( OGb' \).

**Proof.** Let \( \Delta(P) \) be the diagonal subgroup of \( P \times P \) and let \( \sigma \) and \( \sigma' \) be the projections of \( \Delta(P) \) into the first and second factor; obviously \( \sigma \) and \( \sigma' \) are group isomorphisms. Considering the \( OP \)-module \( S \) as an \( O_{\Delta(P)} \)-module through the isomorphism \( \sigma \), we can restate the \( OP \)-interior algebra embedding as the following:

\[
(OG)_\gamma \to \text{Res}_{\sigma^{-1}}\text{End}_O(S) \otimes_O \text{Res}_{\sigma'\circ\sigma^{-1}}(((OG')_{\gamma'})^* .
\]

Then \([3]\) 6.12.2 and Theorem 6.9 and Theorem 7.2] imply that there exists an indecomposable \( O(G \times G') \)-bimodule \( M \) inducing a basic Morita equivalence between \( OGb \) and \( OGb' \).

**Proposition 9.** If the \( O(G \times G') \)-module \( M \) (resp. \( O(G' \times G'') \)-module \( M' \)) induces a basic Morita equivalence between block algebras \( OGb \) and \( OGb' \) (resp. between
block algebras $OG'$ and $OG''$), then $M \otimes_{OG'} M'$ induces a basic Morita equivalence between $OGb$ and $OG''b$.

Proof. Let $OQ''$-module $S''$ and $OQ'''$-module $S'''$ be sources of the $O(G \times G')$-module $M$ and of the $O(G' \times G'')$-module $M'$, respectively. Let $\sigma$ and $\sigma'$ be the projections of $Q''$ into $G$ and $G'$ and let $\rho$ and $\rho'$ be the projections of $Q'''$ into $G'$ and $G''$; by [8, Cor. 7.4], all these projections are group isomorphisms. Since the images $\sigma'(Q'')$ and $\rho(Q''')$ are defect groups of $b'$ by [8, Theorem 6.9], with a suitable choice of $Q''$ and $Q'''$, we can assume that the images $\text{Im}(\sigma')$ and $\text{Im}(\rho)$ coincide. Set $P = \text{Im}(\sigma)$, $P' = \text{Im}(\rho)$ and $P'' = \text{Im}(\rho')$. Let $\pi$ and $\pi'$ be the projections of $P \times P''$ into $P$ and $P''$. It is easy to check that we have the $O(G \times G''')$-module isomorphism

$$
\text{Ind}^{G \times G'''}_{P \times P''}(OG'') \otimes \text{Res}_{\sigma'}(S'') \otimes \text{Res}_{\rho'}(S''') \cong \text{Ind}^{G \times G'''}_{P \times P''}(S'') \otimes \text{Res}_{\sigma'}(S'') \otimes \text{Res}_{\rho'}(S'''),
$$

mapping $(a \otimes b) \otimes (c \otimes s'' \otimes s''')$ onto $((a \otimes c) \otimes s'') \otimes ((1 \otimes b) \otimes s''')$, where $a \in OG$, $b \in OG''$, $c \in OG'$, $s'' \in S''$, $s''' \in S'''$, and $OG'$ as an $O(P \times P'')$-module is defined by

$$(x, y) d = ((\rho \circ \rho'^{-1} \circ \pi')(x, y))d((\sigma' \circ \sigma^{-1} \circ \pi)(x^{-1}, y^{-1}))$$

for $x \in P$, $y \in P''$ and $d \in OG'$. Moreover as an $O(P \times P'')$-module,

$$OG' \cong \bigoplus_{z \in P' \backslash G' / P'} \text{Ind}^{P \times P''}_{N_z}(O),$$

where $P' \backslash G' / P'$ denotes a set of all representatives of the double cosets of $P'$ in $G'$ and

$$N_z = \{(x \circ (\sigma \circ \sigma^{-1})(x), z((\sigma' \circ \sigma^{-1})(x))z^{-1}) | x \in (\sigma \circ \sigma^{-1})(z^{-1}P'z \cap P')\}$$

for any $z \in P' \backslash G' / P'$. So now we can conclude that the order of a vertex of any indecomposable direct summand of $\text{Ind}^{G \times G'''}_{Q''}(S'') \otimes_{OG'} \text{Ind}^{G \times G'''}_{Q''}(S''')$ is less than or equal to $|P| = |P'| = |P''|$. Obviously $M \otimes_{OG'} M'$ is an indecomposable direct summand of $\text{Ind}^{G \times G'''}_{Q''}(S'') \otimes_{OG'} \text{Ind}^{G \times G'''}_{Q''}(S''')$ as $O(G \times G'')$-modules, thus the order of its vertex is less than or equal to $|P| = |P'| = |P''|$. Then by [8, Theorem 6.9], the order of any vertex of $M \otimes_{OG'} M'$ is forced to be equal to $|P| = |P''|$. Finally [8, Cor. 7.4] implies that any source module of $M \otimes_{OG'} M'$ as $O(G \times G''')$ is an endo-permutation module.

10. Let $G$ and $G'$ be finite $p$-solvable groups and let $b$ and $b'$ be blocks of the group algebras $OG$ and $OG'$ with defect pointed groups $P_\gamma$ and $P_{\gamma'}$, respectively. We also assume that $P$ and $P'$ are Abelian and that $E_G(P_\gamma)$ and $E_G(P_{\gamma'})$ are cyclic. Since $G$ is $p$-solvable, by (5.1) and Remarks 6 and 7, there exists an indecomposable endo-permutation $OP$-module $S$, a $P'$-subgroup $L$ of Aut($P$) and a $k^*$-group $\hat{L}$ with $k^*$-quotient $\hat{L}$ such that there exists the following embedding of $OP$-interior algebras:

$$(10.1) \quad (OG)_\gamma \rightarrow \text{End}_O(S) \otimes_O O_*(P \times \hat{L}),$$

where the action of $\hat{L}$ on $P$ lifts that of $L$ on $P$ and $\hat{L}$ is isomorphic to $\hat{E}_G(P_\gamma)^\circ$ as $k^*$-groups. Since $E_G(P_\gamma)$ is cyclic, $\hat{E}_G(P_\gamma)^\circ$ is isomorphic to $k^* \times L$. Therefore the
embedding (10.1) can be restated as

\[(O\mathcal{G})_r \rightarrow \text{End}_\mathcal{O}(S) \otimes \mathcal{O}(P \times L)\]

Set \(H = P \times L\); by Lemma 8, there exists an \(O(G \times H)\)-module \(N\) inducing a basic Morita equivalence between \(O\mathcal{G}b\) and \(O\mathcal{H}\).

Similarly for \(G'\) and its block \(b'\), there also exists an \(O(G' \times H')\)-module \(N'\) inducing a basic Morita equivalence between \(O\mathcal{G}'b'\) and \(O\mathcal{H}'\), where \(H'\) is the semi-direct product \(P' \rtimes L'\) and \(L'\) is a \(p'\)-subgroup of \(\text{Aut}(P')\). Suppose that the \(O(G \times G')\)-module \(M\) induces a Morita equivalence between \(O\mathcal{G}b\) and \(O\mathcal{G}'b'\). Then the \(O(H \times H')\)-module \(N^* \otimes_{OG} M \otimes_{OG'} N'\) induces a Morita equivalence between \(O\mathcal{H}\) and \(O\mathcal{H}'\). Moreover by Proposition 9, the Morita equivalence induced by \(M\) between \(O\mathcal{G}b\) and \(O\mathcal{G}'b'\) being basic is equivalent to the Morita equivalence induced by \(N^* \otimes_{OG} M \otimes_{OG'} N'\) between \(O\mathcal{H}\) and \(O\mathcal{H}'\) being basic. Therefore Theorem 2 is reduced to prove that any Morita equivalence between \(O(P \times L)\) and \(O(P' \times L')\) is basic.

Note that if \(P\) is an Abelian group and \(E\) is an Abelian \(p'\)-subgroup of \(\text{Aut}(P)\), then \(P \times E\) is a \(p\)-constrained group, and the group algebra \(O(P \times E)\) itself is a block algebra with the unique defect pointed group \(P_{[1]}\). In greater generality, we will prove in Theorem 14 below that any Morita equivalence between such group algebras \(O(P \times E)\) is basic, thus Theorem 2 is proved. In order to do so, we also need [8, Theorem B] to hold over \(O\) for the group algebra \(O(P \times E)\).

**Lemma 11.** Assume that \(O\) contains a primitive \(p\)-th unity root \(\zeta\) and \(A\) is an \(O\)-algebra. Then the group \(1 + (\zeta - 1)^2A\) contains no nontrivial torsion \(p\)-element.

**Proof.** Assume that \(1 + a \in 1 + (\zeta - 1)^2A\) is an element of order \(p\). Then we have that \(1 = (1 + a)^p = 1 + pa + pa^2b + a^n\) for some \(b \in A\). Since \(pO = (\zeta - 1)^{p-1}O\), denoting by \(h\) the highest number such that \(a \in (\zeta - 1)^hA\), then we have that \(pa\) does not belong to \((\zeta - 1)^{p+h}A\). But since \(h \geq 2\), \(pa^2b + a^n\) belongs to \((\zeta - 1)^{p+h}A\). So it is a contradiction.

For a finite group \(G\), by \(I(O\mathcal{G})\) we always denote the augmentation ideal of the group algebra \(O\mathcal{G}\).

**Lemma 12.** Assume that \(P\) is a \(p\)-group and \(E\) is a \(p'\)-subgroup of \(\text{Aut}(P)\). Then the image of any \(p\)-subgroup \(Q\) of \(1 + I(O(P \times E))\) centralizing \(P\) in

\[O(P \times E)/I(O\mathcal{P})O(P \times E) \cong OE\]

is trivial.

**Proof.** Assume that \(O\) contains a primitive \(p\)-th unity root \(\zeta\); then \(pO = (\zeta - 1)^{p-1}O\).

Obviously \(P\) acts on the group \(P \times E\) by conjugation and \(P \times E\) is divided into a disjoint of \(P\)-orbits on \(P \times E\). Let \(I\) be a set of representatives of all these \(P\)-orbits and let \(J_g\) be a set of representatives of all left cosets of \(C_P(g)\) in \(P\) for any \(g \in I\); then any element \(a \in C_{O(P \times E)}(P)\) can be written in the sum

\[\sum_{g \in I} a_g \sum_{n \in J_g} ngn^{-1}, \quad \text{where} \; a_g \in O\]
Assume that \( p \) is odd and consider the sum \( \sum_{n \in P/C_P(g)} ngn^{-1} \). If \( P = C_P(g) \), then by our assumption, \( g \in Z(P) \), and if \( |P : C_P(g)| = p^\alpha > 1 \), then

\[
\sum_{n \in I_g} ngn^{-1} \in I(O\mathcal{P})\mathcal{O}(P \times E) + p^\alpha \mathcal{O}(P \times E).
\]

Since \( Q \) is contained in \( 1 + I(O(P \times E)) \), the image of \( Q \) in

\[
\mathcal{O}(P \times E) / I(O\mathcal{P})\mathcal{O}(P \times E) \cong \mathcal{O}E
\]

is contained in

\[
1 + p\mathcal{O}E \subset 1 + (\zeta - 1)^2 \mathcal{O}E.
\]

Thus by Lemma 11, the image of \( Q \) in \( \mathcal{O}E \) is trivial.

Assume that \( p \) is equal to 2. If \( |P : C_P(g)| = 2^\alpha > 1 \), where \( \alpha \neq 1 \), then as in the case of \( p \) being odd, it is easy to check that the class sum \( \sum_{n \in I_g} ngn^{-1} \in I(O\mathcal{P})\mathcal{O}(P \times E) + 4\mathcal{O}(P \times E) \). Now suppose that \( |P : C_P(g)| = 2 \). Consider the quotient group \( P/\Phi(P) \) of \( P \) by the Frattini subgroup and let \( g_{p'} \) be the \( p' \)-part of \( g \).

Suppose \( C_P(g_{p'}) = C_P(g) \). In that case, \( g_{p'} \) induces a nontrivial \( p' \)-automorphism of \( P/\Phi(P) \) and we have a \( g_{p'} \)-stable short exact sequence as the following:

\[
1 \longrightarrow C_P(g)/\Phi(P) \longrightarrow P/\Phi(P) \longrightarrow P/C_P(g) \longrightarrow 1.
\]

Then by Maschke's Theorem,

\[
P/\Phi(P) \cong C_P(g)/\Phi(P) \bigoplus P/C_P(g)
\]

as a \( \langle g_{p'} \rangle \)-module. Since \( g_{p'} \) centralizes \( C_P(g)/\Phi(P) \) and \( P/C_P(g) \), so does \( g_{p'} \) on \( P/\Phi(P) \) and further on \( P \). By the assumption on \( P \times E \), we have that \( g_{p'} = 1 \). This contradicts the equality \( C_P(g_{p'}) = C_P(g) \). Thus \( C_P(g_{p'}) = P \) since \( |P : C_P(g)| = 2 \).

By the assumption on \( P \times E \) again, \( g_{p'} \in P \). \( g_{p'} = 1 \) and \( g \in P \). Therefore the class sum \( \sum_{n \in I_g} ngn^{-1} \in \mathcal{O} + I(O\mathcal{P})\mathcal{O}(P \times E) \). Now it is clear that the image of \( Q \) in \( \mathcal{O}(P \times E) / I(O\mathcal{P})\mathcal{O}(P \times E) \cong \mathcal{O}E \) is contained in

\[
1 + 4\mathcal{O}E = 1 + (-2 - 1)^2 \mathcal{O}E.
\]

Then by Lemma 11 again, the image of \( Q \) in \( \mathcal{O}E \) is trivial.

Finally suppose that \( \mathcal{O} \) does not contain a \( p \)-th primitive unity root and let \( \hat{\mathcal{O}} = \mathcal{O}[\zeta] \), where \( \zeta \) is a \( p \)-th primitive unity root. By the hypothesis, \( Q \) is contained in \( 1 + I(\hat{\mathcal{O}}(P \times E)) \) and the image of \( Q \) in

\[
\hat{\mathcal{O}}(P \times E) / I(\hat{\mathcal{O}}P)\hat{\mathcal{O}}(P \times E) \cong \hat{\mathcal{O}} \otimes \mathcal{O}(P \times E) / I(O\mathcal{P})\mathcal{O}(P \times E)
\]

is trivial; thus the image of \( Q \) in \( \mathcal{O}(P \times E) / I(O\mathcal{P})\mathcal{O}(P \times E) \) is trivial.

**Proposition 13.** Assume that \( P \) is a \( p \)-group and \( E \) is a \( p' \)-subgroup of \( \text{Aut}(P) \). Then any \( p \)-subgroup \( Q \) of \( 1 + I(\mathcal{O}(P \times E)) \) centralizing \( P \) is a subgroup of \( P \).

**Proof.** Without loss of generality, we assume that \( Q = \langle c \rangle \).

Consider \( \mathcal{O}(P \times E) \) as an \( \mathcal{O}(P \times E \times P \times Q) \)-module defined by \( (x, y, z)a = xay^{-1}z^{-1} \) for \( x \in P \times E, y \in P, z \in Q \) and \( a \in \mathcal{O}(P \times E) \). By Lemma 12, the image of \( Q \) in \( \mathcal{O}(P \times E) / I(O\mathcal{P})\mathcal{O}(P \times E) \cong \mathcal{O}E \) is trivial and \( P \times P \times Q / P \times 1 \times 1 \)
acts trivially on $OE$. By [2] Theorem A1.2, $\text{Res}_{P \times P \times Q}(O(P \times E))$ is a permutation module and thus

$$\text{Res}_{P \times P \times Q}(O(P \times E)) \cong \bigoplus_{j=1}^{\infty} \text{Ind}_{\mathcal{U}_j}^{P \times P \times Q} O_{\mathcal{U}_j},$$

where $U_j$ is a subgroup of $P \times P \times Q$ and $O_{\mathcal{U}_j}$ is the trivial $O_{\mathcal{U}_j}$-module. Since $O(P \times E)$ is projective as left and right $OP$-modules, it is easily concluded that

$$U_j \cap P \times 1 \times 1 = 1 \quad \text{and} \quad U_j \cap 1 \times P \times 1 = 1$$

for any $j$ and that

$$|U_j| = |P||Q|$$

for any $j$.

We claim that for any $j$, there exists some $z_j \in P$ and some $\theta_j \in \text{Aut}(P)$ such that

$$U_j = \{(x, \theta_j(x)z_j^{-1}, c)|x \in P, i \in \mathbb{Z}\}.$$

Let $\{a_i\}_{1 \leq i \leq |P|}$ be a $P \times P \times Q$-stable $O$-basis of $\text{Ind}_{\mathcal{U}_j}^{P \times P \times Q} O_{\mathcal{U}_j}$ and assume that the stabilizer of $a_1$ is $U_j$. On the one hand, since $U_j \cap 1 \times P \times 1 = 1$, for any $x \in P$, there exists a unique $x' \in P$ such that $x_1 = a_1x'$; so we can define a map

$$\theta : P \rightarrow P, \quad x \mapsto x'.$$

Since $U_j \cap 1 \times 1 \times 1 = 1$, it is easily checked that the map $\theta$ is a group isomorphism and that $\{(x, \theta(x), 1)|x \in P\}$ is a subgroup of $U_j$. On the other hand, for any $i \in \mathbb{Z}$, there exists a unique $z_j \in P$ such that we have that $a_1z_j^i = a_1c^i$; so $\{1, z_j^{-1}, c^i|i \in \mathbb{Z}\}$ is a subgroup of $U_j$. Therefore

$$U_j \geq \{(x, \theta_j(x)z_j^{-1}, c)|x \in P, i \in \mathbb{Z}\}.$$

Obviously the cardinality of the latter set is equal to $|P||Q|$. So

$$U_j = \{(x, \theta_j(x)z_j^{-1}, c)|x \in P, i \in \mathbb{Z}\}.$$

Obviously $O(P \times E)$ as an $O(P \times E \times P \times Q)$-module is indecomposable and relatively projective to $P \times P \times Q$. Then by Higman’s Criterion on relatively projective modules, $O(P \times E)$ is a direct summand of

$$\text{Ind}_{P \times P \times Q}^{P \times P \times P \times Q} \left(\text{Res}_{P \times P \times Q}(O(P \times E))\right) \cong \text{Ind}_{P \times P \times P \times Q}^{P \times P \times P \times Q} \left(\bigoplus_{j=1}^{\infty} \text{Ind}_{\mathcal{U}_j}^{P \times P \times Q} O_{\mathcal{U}_j}\right)$$

as an $O(P \times E \times P \times Q)$-module. Therefore there exists some $j$ such that

$$O(P \times E) \cong \text{Ind}_{\mathcal{U}_j}^{P \times P \times P \times Q} O_{\mathcal{U}_j}.$$

This means that there exists $u \in O(P \times E)$ such that $O(P \times E)u = O(P \times E)$ and $u$ is stable under the action of $U_j$; that is, $u$ is a unit of $O(P \times E)$ and $u = uz_j^{-1}c$; thus $c \in P$.

**Theorem 14.** Let $P$ (resp. $P'$) be an Abelian $p$-group and $E$ (resp. $E'$) be an Abelian $p'$-subgroup of $\text{Aut}(P)$ (resp. $\text{Aut}(P')$). Set $G = P \times E$ and $G' = P' \times E'$. If the $O(G \times G')$-module $M$ induces a Morita equivalence between group algebras $OG$ and $OG'$, then the following hold:

1. $\dim_K(V') = \dim_K((K \otimes M) \otimes_{KG} V')$ for any simple $KG'$-module $V'$; in particular, there exists an $O$-algebra isomorphism $\rho : OG' \rightarrow OG$ such that $M$
as an $\mathcal{O}(G \times G')$-module is isomorphic to $\mathcal{O}G$, where $\mathcal{O}G$ as an $\mathcal{O}(G \times G')$-module is defined by $(x, y)a = x\rho(y^{-1})$ for $x \in G$, $y \in G'$ and $a \in \mathcal{O}G$.

(14.2) Let $P''$ be a vertex of the $\mathcal{O}(G \times G')$-module $M$ and let the $\mathcal{O}P''$-module $S$ be a source module of $M$. Then $\text{Rank}_\mathcal{O}(S) = 1$. That is to say, the Morita equivalence induced by $M$ between $\mathcal{O}G'$ and $\mathcal{O}G$ is basic. In particular, $G'$ is isomorphic to $G$.

**Proof.** Let $V'$ be a $KG'$-module. Then it is well known that there is a full $\mathcal{O}G'$-lattice $W'$ in $V'$. Since all simple modules of $kG'$ have dimension 1,

$$\dim_k(V') = \text{Rank}_\mathcal{O}(W') = \dim_k(k \otimes_\mathcal{O} W')$$

is equal to the number of the composition factors (counting multiplicities) in a composition chain of $k \otimes_\mathcal{O} W'$. Since the $\mathcal{O}(G \times G')$-module $M$ induces a Morita equivalence between $\mathcal{O}G$ and $\mathcal{O}G'$, so does $\mathcal{K} \otimes_\mathcal{O} M$ between $kG'$ and $kG$; thus $V = (\mathcal{K} \otimes_\mathcal{O} M) \otimes_{kG'} V'$ is a simple module of $kG$ and $M \otimes_{\mathcal{O}G} W'$ is a full $\mathcal{O}G$-lattice in $V$. Moreover we have

$$\dim_k((\mathcal{K} \otimes_\mathcal{O} M) \otimes_{kG'} V') = \dim_k((k \otimes_\mathcal{O} M) \otimes_{kG'} (k \otimes_\mathcal{O} W')) = \dim_k(k \otimes_\mathcal{O} W') = \dim_k(V')$$

since $k \otimes_\mathcal{O} M$ induces a Morita equivalence between $kG$ and $kG'$, which preserves composition chains of $kG$-modules. It is more or less known that

$$\mathcal{K} \otimes_\mathcal{O} M \cong \bigoplus_{V' \in \text{Irr}(G')} ((\mathcal{K} \otimes_\mathcal{O} M) \otimes_{kG'} V') \otimes_{\mathcal{K}} V'^*$$

as a $\mathcal{K}(G \times G')$-module, where $V'^*$ denotes the dual of $V'$ as a $KG'$-module and $\text{Irr}(G')$ denotes the set of all simple $KG'$-modules. Since

$$\dim_k((\mathcal{K} \otimes_\mathcal{O} M) \otimes_{kG'} V') = \dim_k((\mathcal{K} \otimes_\mathcal{O} M) \otimes_{kG'} V'^*)$$

identifying $G \times 1$ with $G$,

$$\text{Res}_{G}(\mathcal{K} \otimes_\mathcal{O} M) \cong \bigoplus_{V' \in \text{Irr}(G')} \dim_k(V')(\mathcal{K} \otimes_\mathcal{O} M) \otimes_{kG'} V' \cong kG.$$

Since $M$ is projective as a left $\mathcal{O}G$-module, $M$ is isomorphic to $\mathcal{O}G$ as left $\mathcal{O}G$-modules. Similarly, $M$ is isomorphic to $\mathcal{O}G'$ as right $\mathcal{O}G'$-modules. Therefore there exists $m \in M$ such that $M = \mathcal{O}Gm$ and for any $x' \in \mathcal{O}G'$, there exists $x \in \mathcal{O}G$ such that $xm = mx'$. Moreover if another $y \in \mathcal{O}G$ also fulfills $ym = mx'$, then $(x - y)m = 0$ and thus $x = y$. Now we can define an $\mathcal{O}$-algebra homomorphism

$$\rho : \mathcal{O}G' \to \mathcal{O}G, \quad x' \mapsto x \quad \text{such that} \quad xm = mx'. $$

If $\rho(x') = 0$, then $mx' = 0$ and thus $Mx' = 0$. Since $M$ is isomorphic to $\mathcal{O}G'$ as right $\mathcal{O}G'$-modules, $x' = 0$ and so $\rho$ is injective; in particular, $1 \otimes \rho : k \otimes_\mathcal{O} \mathcal{O}G' \to k \otimes_\mathcal{O} \mathcal{O}G$ is injective. Since $\dim_k((\mathcal{K} \otimes_\mathcal{O} M) \otimes_{kG'} V') = \dim_k(V')$ for any simple $kG'$-module $V'$, we have $|G'| = |G'|$. Therefore $1 \otimes \rho$ is an isomorphism and further so is $\rho$. Consider the $\mathcal{O}(G \times G')$-module $(\mathcal{O}G)_{\rho} = \mathcal{O}G$ defined by $(x, y)a = x\rho(y^{-1})$ for $x \in G, a \in \mathcal{O}G$ and $y \in G'$. Then it is easy to check that

$$(\mathcal{O}G)_{\rho} \cong M \quad \text{as $\mathcal{O}(G \times G')$-modules}.$$ 

Up to now, the proof of (14.1) is done.
Set $Z = C_p(E)$ and $Z' = C_{p'}(E')$; obviously $Z$ and $Z'$ are maximal central $p$-subgroups of $G$ and $G'$, respectively. Set $\bar{G} = G/Z$ and $\bar{G}' = G'/Z'$. For $x \in G$ and $H \leq G$, denote by $\bar{x}$ and $\bar{H}$ the images of $x$ and $H$ in $\bar{G}$, respectively. Then $\bar{E}$ is isomorphic to a $p'$-subgroup of $\text{Aut}(\bar{P})$, $C_{p'}(\bar{E})$ is trivial and $\bar{G}$ is equal to $\bar{P} \times \bar{E}$; similar results also hold for $G'$, $P'$ and $E'$.

Suppose that $Z'$ is nontrivial. Let $\varphi$ be the character of $G'$ determined by the pullback of the trivial $OG$-module through $\rho$ and define an $O$-algebra homomorphism

$$\rho' : OG' \longrightarrow OG, \quad x \in G' \longmapsto \varphi(x^{-1})\rho(x).$$

Then it is easily checked that $\rho'$ is an isomorphism, that $\rho'$ maps $I(OG')$ onto $I(OG)$, and that $\rho'(Z')$ is a $p'$-subgroup of $1 + I(OG)$ centralized by $G$. By Proposition 13, $\rho'(Z') \subset Z$. Similarly, $\rho'^{-1}(Z) \subset Z'$ and thus $\rho'(Z') = Z$. Now it is clear that $\rho'$ induces an $O$-algebra isomorphism

$$\rho' : OG' \cong OG'/I(1 + OZ)OG' \longrightarrow OG/I(1 + OZ)OG \cong OG',$$

Consider an $O(\bar{G} \times \bar{G}')$-module $(\bar{O}\bar{G})_{\varphi'} = O\bar{G}$ defined by $(\bar{x}, \bar{y})a = \bar{x}\varphi(\bar{y}^{-1})$ for $\bar{x} \in \bar{G}, a \in O\bar{G}$ and $\bar{y} \in \bar{G}'$. We claim that $\text{Res}_{\bar{P} \times \bar{P}'}((\bar{O}\bar{G})_{\varphi'})$ is a permutation module. It is clear that $I(\bar{O}\bar{P})(\bar{O}\bar{G})_{\varphi'}$ is a submodule of the $O(\bar{G} \times \bar{G}')$-module $(\bar{O}\bar{G})_{\varphi'}$ and that the quotient module $(\bar{O}\bar{G})_{\varphi'}/(I(\bar{O}\bar{P})(\bar{O}\bar{G})_{\varphi'})$ is $O$-free. Let the $\bar{K}\bar{G}'$-module $\bar{V}'$ be an irreducible direct summand of

$$\bar{K} \otimes_O \text{Res}_{\bar{P} \times \bar{P}'}((\bar{O}\bar{G})_{\varphi'}/(I(\bar{O}\bar{P})(\bar{O}\bar{G})_{\varphi'})).$$

Since $\bar{G}/\bar{P}$ is Abelian, $\bar{K} \otimes_O \text{Res}_{\bar{P} \times \bar{P}'}((\bar{O}\bar{G})_{\varphi'}/(I(\bar{O}\bar{P})(\bar{O}\bar{G})_{\varphi'}))$ is a direct sum of irreducible $\bar{K}\bar{G}'$-modules with dimension 1. Therefore $\text{Dim}_{\bar{K}}(\bar{V}') = 1$. Then the restriction of $\bar{V}'$ to $P'$ determines an $E'$-stable linear character $\chi'$ of $P'$. Since $C_{p'}(E')$ is trivial, by the well-known Glauben theorem on characters, $\chi'$ has to be trivial. Therefore, $P'$ acts trivially on $\bar{V}'$. Since $\bar{V}'$ is any direct summand of

$$\bar{K} \otimes_O \text{Res}_{\bar{P} \times \bar{P}'}((\bar{O}\bar{G})_{\varphi'}/(I(\bar{O}\bar{P})(\bar{O}\bar{G})_{\varphi'}))$$

as $\bar{K}\bar{G}'$-modules, $P'$ acts trivially on $\bar{K} \otimes_O \text{Res}_{\bar{P} \times \bar{P}'}((\bar{O}\bar{G})_{\varphi'}/(I(\bar{O}\bar{P})(\bar{O}\bar{G})_{\varphi'}))$ and further trivially on $(\bar{O}\bar{G})_{\varphi'}/I(\bar{O}\bar{P})(\bar{O}\bar{G})_{\varphi'}$. Equivalently, $\bar{K}(\bar{P}') \subseteq 1 + I(\bar{O}\bar{P})O\bar{G}$. Since $\text{Res}_{\bar{P} \times \bar{P}'}((\bar{O}\bar{G})_{\varphi'})$ is free and $\text{Res}_{\bar{P} \times \bar{P}'}((\bar{O}\bar{G})_{\varphi'}/I(\bar{O}\bar{P})(\bar{O}\bar{G})_{\varphi'})$ is the direct sum of trivial modules, by [3] Theorem A1.2, $\text{Res}_{\bar{P} \times \bar{P}'}((\bar{O}\bar{G})_{\varphi'})$ is a permutation module.

Consider the $O(G \times G')$-module $(\bar{O}\bar{G})_{\varphi'} = O\bar{G}$ defined by $(x, y)a = x\varphi(y^{-1})$ for any $x \in G, a \in O\bar{G}$ and $y \in G'$. Obviously $I(O\bar{G})(\bar{O}\bar{G})_{\varphi'}$ is a submodule of $(\bar{O}\bar{G})_{\varphi'}$. Since $\rho'(Z') = Z$, the quotient module $(\bar{O}\bar{G})_{\varphi'}/I(O\bar{G})(\bar{O}\bar{G})_{\varphi'}$ becomes an $O(\bar{G} \times G')$-module and is isomorphic to $(O\bar{G})_{\varphi'}$. Since $O\bar{G}$ is free as left $O\bar{G}$-module, by [3] Theorem A1.2 again, we have that $\text{Res}_{\bar{P} \times \bar{P}'}((\bar{O}\bar{G})_{\varphi'})$ is a permutation module. Denote by $\bar{O}\varphi$ the $O\bar{G}'$-module $O$ obtained through the homomorphism $\varphi$ and define $(O\bar{G})_{\varphi'} \otimes_O \bar{O}\varphi$ as an $O(G \times G')$-module by $G$ acting on the left of $(O\bar{G})_{\varphi'}$ and trivially on the left of $\bar{O}\varphi$ and $G'$ acting diagonally on the right of $(O\bar{G})_{\varphi'} \otimes_O \bar{O}\varphi$. Then it is easy to check that $(O\bar{G})_{\varphi'} \otimes_O \bar{O}\varphi \cong (O\bar{G})_{\varphi'}$ as $O(G \times G')$-modules and therefore that any source module of $M = (O\bar{G})_{\varphi'}$ as an $O(G \times G')$-module has $O$-rank 1.

Suppose that $Z'$ is trivial. Then it follows by a proof similar to the nontrivial case of $Z'$ that any source module of $M$ is of $O$-rank 1. Now by [3] Cor. 7.4, we reach the conclusion that the Morita equivalence between $OG$ and $OG'$ induced by $M$ is basic.
Finally we prove that $G$ is isomorphic to $G'$. Let $\sigma$ and $\sigma'$ be projections of $P''$ into $G$ and $G'$, respectively. Then by [8, Cor. 7.4 and Th. 6.9], $P = \sigma(P'')$, $P' = \sigma'(P'')$, the group homomorphisms $\sigma : P'' \rightarrow P$ and $\sigma' : P'' \rightarrow P'$ are isomorphisms; moreover there exists the $\mathcal{O}P$-interior algebra embedding

$$\mathcal{O}G \rightarrow \text{Res}_{\sigma^{-1}}(\text{End}_{\mathcal{O}}(S)) \otimes \text{Res}_{\sigma'\circ\sigma^{-1}}(\mathcal{O}G'),$$

which actually is an isomorphism. Without loss of the generality, we can assume that $S$ is the trivial $\mathcal{O}P''$-module. Then we obtain an isomorphism between $\mathcal{O}G$ and $\mathcal{O}G'$, mapping $P$ onto $P'$. Now $G$ being isomorphic to $G'$ follows from [9, 2.15.7, 2.16.1 and 2.16.3].

15. Finally Theorem 2 follows from (14.2) and the second paragraph in 10.

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